NST Part III Experimental and Theoretical Physics — Lent Term 2025

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GAUGE FIELD THEORY

Examples Sheet 1

An uppercase A, B, or C after each question (or part question) number indicates the rough degree of difficulty.

The Klein-Gordon and Dirac equations

1. (a A) By applying the minimal substitution prescription $p^\mu \to p^\mu - qA^\mu$ to the Klein-Gordon equation, obtain a covariant equation describing a spin zero particle of charge q in the presence of an electromagnetic potential $A^\mu(x)$. Show that the resulting equation is invariant under the charge conjugation operation $\phi \to \phi^c = \phi^*$, $q \to -q$.

   (b A) Apply the minimal substitution prescription to the Dirac equation, and show that the resulting equation is invariant under the charge conjugation operation $\psi \to \psi^c = C(\overline{\psi})^T$, $q \to -q$, provided the 4×4 matrix C satisfies the relation $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$.

   (c A) Show that, in the Pauli-Dirac representation, a suitable choice for C is $C = i\gamma^2\gamma^0$.

   (d B) Find the result of applying the charge conjugation operation to the plane-wave solutions $u_1(p)e^{-ip.x}$ and $u_2(p)e^{-ip.x}$ defined in the lecture notes.

Dirac gamma matrix algebra and trace theorems

2. Defining $\not{a} \equiv \gamma^\mu a_\mu$ and using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, prove the following results:

   (a A) The trace of the product of an odd number of γ-matrices is zero

   (b A) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$, $\quad \mathrm{Tr}(\not{a}\not{b}) = 4(a \cdot b)$

   (c B) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda) = 4[g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda}]$
   $\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)]$
   $\mathrm{Tr}(\not{a}\gamma^\mu\not{b}\gamma_\mu) = -8(a \cdot b)$

   (d A) $\gamma_\mu\not{a}\gamma^\mu = -2\not{a}$

   (e A) $\gamma_\mu\not{a}\not{b}\gamma^\mu = 4(a \cdot b)$

(f B) $\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$.

(g A) $\mathrm{Tr}(\gamma^5) = 0$

(h B) $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0, \quad \mathrm{Tr}(\gamma^5\not{a}\not{b}) = 0$

(i C) $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}, \quad \mathrm{Tr}(\gamma^5\not{a}\not{b}\not{c}\not{d}) = 4i\epsilon^{\mu\nu\lambda\sigma}a_\mu b_\nu c_\lambda d_\sigma$

[Hint: Useful tricks are to use $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$, and to insert $(\gamma^5)^2 = 1$ into a trace and then use $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$.]

[$\epsilon^{\mu\nu\lambda\sigma}$ is the totally antisymmetric Levi-Civita tensor defined as

$$\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} -1 & \text{if } \mu\nu\lambda\sigma \text{ is an } \textit{even} \text{ permutation of 0123,} \\ +1 & \text{if } \mu\nu\lambda\sigma \text{ is an } \textit{odd} \text{ permutation of 0123,} \\ 0 & \text{if any two indices are the same.} \end{cases}$$

Particle decay

3. The Lorentz-invariant matrix element for the decay $\pi^- \to \mu^-\overline{\nu}_\mu$ is given by

$$M_{\mathrm{fi}} = \frac{G_{\mathrm{F}}}{\sqrt{2}} f_\pi \, p_{1\mu} \overline{u}(p_3)\gamma^\mu(1-\gamma^5)v(p_4) \ ,$$

where p_1 is the four-momentum of the π^-, of spin zero, p_3 and p_4 are the four-momenta of the μ^- and $\overline{\nu}_\mu$, respectively, and f_π is a constant.

(a B) Show that, summed over all possible spin states of the muon, the $\pi^- \to \mu^-\overline{\nu}_\mu$ decay rate is proportional to

$$2(p_1\cdot p_3)(p_1\cdot p_4) - m_\pi^2(p_3\cdot p_4) + m_\nu\left[2(p_1\cdot p_3)(p_1\cdot s_4) - m_\pi^2(p_3\cdot s_4)\right] \ ,$$

where s_4 is a four-vector describing the anti-neutrino spin state, defined by

$$v(p_4)\overline{v}(p_4) \equiv (\not{p}_4 - m_\nu)\tfrac{1}{2}(1+\gamma^5\not{s}_4).$$

For a positive-helicity antineutrino, the spin four-vector s^μ is given by

$$s_4^\mu = \frac{1}{m_\nu}(p^*, 0, 0, E_\nu^*) \ ,$$

while for a negative-helicity neutrino it has opposite sign. Hence show that the decay rate is proportional to

$$m_\mu^2(m_\pi^2 - m_\mu^2) + m_\nu^2(m_\pi^2 + 2m_\mu^2) - m_\nu^4 \pm 2(m_\mu^2 - m_\nu^2)m_\pi p^* \ ,$$

where p^* is the momentum of the antineutrino in the π rest frame, and the + (−) sign corresponds to decay into positive-helicity (negative-helicity) antineutrinos.

(b C) Assuming $m_\nu \ll m_\pi$ and $m_\nu \ll m_\mu$, show that, to order m_ν^2, the fraction of negative-helicity antineutrinos produced in π^- decay is given by

$$R_\downarrow \approx \frac{m_\nu^2}{m_\mu^2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^{-2} .$$

Evaluate $R_\downarrow$ for an assumed antineutrino mass of 0.1 eV .

Quantum fields

4 A. Show that if Ψ and Ψ^* are taken as independent classical fields, the Lagrangian density

$$\mathcal{L} = \frac{\hbar}{2i}\left(\Psi\frac{\partial\Psi^*}{\partial t} - \Psi^*\frac{\partial\Psi}{\partial t}\right) - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}\Psi^* - V(\mathbf{r})\Psi^*\Psi$$

leads to the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\Psi + V(\mathbf{r})\Psi$$

and its complex conjugate. What are the momentum densities conjugate to Ψ and Ψ^* ? Deduce the Hamiltonian density, and verify that integrating it over all space gives the usual expression for the energy.

The Dirac field

5 A. The Fourier representation of the Dirac field operator is

$$\hat{\psi}(\mathbf{r},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}\sum_s \left[\hat{c}_s(\mathbf{k})u_s(\mathbf{k})e^{-ik\cdot x} + \hat{d}_s^\dagger(\mathbf{k})v_s(\mathbf{k})e^{+ik\cdot x}\right]$$

where $k^\mu = (\omega, \mathbf{k})$ and $\omega = \sqrt{\mathbf{k}^2 + m^2}$. The creation and annihilation operators satisfy the anticommutation relations

$$\{\hat{c}_s(\mathbf{k}), \hat{c}_{s'}^\dagger(\mathbf{k}')\} = \{\hat{d}_s(\mathbf{k}), \hat{d}_{s'}^\dagger(\mathbf{k}')\} = (2\pi)^3\, 2\omega\, \delta_{ss'}\delta^3(\mathbf{k}-\mathbf{k}') \;,$$

all other anticommutators being zero. Show that this implies that the field and its conjugate momentum density satisfy the anticommutation relation

$$\{\hat{\psi}_\alpha(\mathbf{r},t), \hat{\pi}_\beta(\mathbf{r}',t)\} = i\,\delta_{\alpha\beta}\delta^3(\mathbf{r}-\mathbf{r}')$$

where $\hat{\pi} = i\hat{\psi}^\dagger$, and α and β are spinor component labels.

Show also that the normal-ordering properties of the creation and annihilation operators lead to the relation

$$:\hat{\overline{\psi}}_\beta(y)\hat{\psi}_\alpha(x): = -:\hat{\psi}_\alpha(x)\hat{\overline{\psi}}_\beta(y):$$

for the normal-ordered products of field operator components at spacetime points x and y.

The charge operator for a Dirac field

6 B. Show that the standard particle and antiparticle spinors $u_s(\boldsymbol{k})$, $v_s(\boldsymbol{k})$ in the Pauli-Dirac representation satisfy the relations

$$u_s^\dagger(\mathbf{k})u_{s'}(\mathbf{k}) = v_s^\dagger(\mathbf{k})v_{s'}(\mathbf{k}) = 2\omega\delta_{ss'}$$

$$u_s^\dagger(\mathbf{k})v_{s'}(-\mathbf{k}) = v_s^\dagger(\mathbf{k})u_{s'}(-\mathbf{k}) = 0$$

For a free spin-half particle described by a field operator $\hat{\psi}(x)$, verify that the current

$$\hat{J}^\mu(x) = \hat{\overline{\psi}}(x)\gamma^\mu\hat{\psi}(x)$$

is conserved, and show that it leads to a conserved (normal-ordered) charge operator

$$:\hat{Q}: = \int \mathrm{d}^3\boldsymbol{k}\, N(\boldsymbol{k}) \sum_s \left[\hat{c}_s^\dagger(\boldsymbol{k})\hat{c}_s(\boldsymbol{k}) - \hat{d}_s^\dagger(\boldsymbol{k})\hat{d}_s(\boldsymbol{k})\right] \ .$$

Show that the single-particle states $\hat{c}_s^\dagger(\boldsymbol{k})\,|0\rangle$ and $\hat{d}_s^\dagger(\boldsymbol{k})\,|0\rangle$ are eigenstates of $:\hat{Q}:$ and obtain the corresponding eigenvalues.