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GAUGE FIELD THEORY

Examples Sheet 2

Compton scattering

7. (a A) Show that the gauge transformation $A^\mu \to A^\mu + \partial^\mu \chi$ shifts the polarization vector ϵ^μ by an amount proportional to the vector k^μ.

(b B) Show that the contributions to the amplitude of the two leading-order Feynman diagrams for Compton scattering, $\gamma(k) + \mathrm{e}^-(p) \to \gamma(k') + \mathrm{e}^-(p')$, are not separately gauge invariant, but that their sum is.

[Hint: use the Dirac equations $\bar{u}'(\not{p}' - m) = (\not{p} - m)u = 0$.]

Compton scattering

8. When summed over the spin states of all initial and final state particles, the leading-order matrix element squared for Compton scattering, $\gamma(k) + \mathrm{e}^-(p) \to \gamma(k') + \mathrm{e}^-(p')$, is

$$\sum_{\text{spins}} |M_{\text{fi}}|^2 = 8e^4 \left[\frac{A_{11}}{(k \cdot p)^2} + \frac{A_{22}}{(k' \cdot p)^2} - \frac{A_{12}}{(k \cdot p)(k' \cdot p)} \right]$$

.

where the modulus-squared of the amplitude for each leading-order Feynman diagram contributes the terms containing A_{11} and A_{22}, while interference between the two leading-order diagrams contributes the term containing A_{12}.

(a B) Show that

$$A_{11} = \frac{1}{32} \mathrm{Tr}\left[(\not{p}' + m)\gamma^\nu (\not{p} + \not{k} + m)\gamma^\mu (\not{p} + m)\gamma_\mu (\not{p} + \not{k} + m)\gamma_\nu \right].$$

(b B) Given that

$$\begin{aligned} A_{11} &= (k \cdot p)(k' \cdot p) + m^2 (k \cdot p) + m^4 \\ A_{22} &= -m^2 (k' \cdot p) + (k \cdot p)(k' \cdot p) + m^4 \ , \\ A_{12} &= m^2 (k \cdot p) - m^2 (k \cdot p') + 2m^4 \ , \end{aligned}$$

show that the differential cross section for unpolarised Compton scattering in the laboratory frame is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 1 + \cos^2\theta\right]$$

where ω and ω' are the energies of the incoming and outgoing photons, and θ is the angle between their directions.

(c B) Show that, in the low energy limit $\omega \ll m$, the total cross section is given by the Thomson cross section

$$\sigma_{\mathrm{T}} = \frac{8\pi\alpha^2}{3m^2} \ .$$

(d C) (optional) Show that, in the high energy limit $\omega \gg m$, the total cross section is given approximately by

$$\sigma(s) \approx \frac{2\pi\alpha^2}{s}\ln\frac{s}{m^2} \ .$$

Interacting fields: W decay

9. The decay of a $W^{\pm}$ boson into a charged lepton $\ell^{\pm}$ and a neutrino (or antineutrino) is governed by interaction terms in the Lagrangian density of the form

$$-\frac{g}{2\sqrt{2}}\,\hat{\bar{\psi}}_\nu\gamma^\mu(1-\gamma^5)\hat{\psi}_\ell\hat{W}_\mu - \frac{g}{2\sqrt{2}}\,\hat{\bar{\psi}}_\ell\gamma^\mu(1-\gamma^5)\hat{\psi}_\nu\hat{W}^\dagger_\mu \ .$$

(a A) For the case that the (anti)neutrino, of mass m_ν, is described by a Dirac field with a Fourier representation of the form

$$\hat{\psi}(\mathbf{r},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}\sum_s\left[\hat{c}_s(\mathbf{k})u_s(\mathbf{k})e^{-ik\cdot x} + \hat{d}^\dagger_s(\mathbf{k})v_s(\mathbf{k})e^{+ik\cdot x}\right] \ ,$$

where $\omega = \sqrt{\mathbf{k}^2 + m^2}$, show that the decays $W^+ \to \ell^+\nu_\ell$ and $W^- \to \ell^-\overline{\nu}_\ell$ are permitted, while the decays $W^+ \to \ell^+\overline{\nu}_\ell$ and $W^- \to \ell^-\nu_\ell$ are not.

(b B) Show that the leading order transition amplitude $S_{\mathrm{fi}} = -i\int\langle f|\hat{\mathcal{H}}_I|i\rangle\,\mathrm{d}^4x$ for the decay $W^-(p_1) \to \ell^-(p_2)\overline{\nu}_\ell(p_3)$ is given by

$$S_{\mathrm{fi}} = -i(2\pi)^4\frac{g}{2\sqrt{2}}\bar{u}(\mathbf{p}_2)\gamma^\mu(1-\gamma^5)v(\mathbf{p}_3)\epsilon_\mu(\mathbf{p}_1)\delta^4(p_2+p_3-p_1) \ .$$

Obtain the corresponding transition amplitude for the decay $W^+ \to \ell^+\nu_\ell$.

Scalar QED

10 B. Write down the Lagrangian density $\mathcal{L}$ which results from applying the minimal substitution prescription to the free particle Lagrangian

$$\mathcal{L}_0 = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi \ ,$$

where $\phi(x)$ is a non-Hermitian scalar field. Show that the Lagrangian $\mathcal{L}$ is gauge invariant. Identify the interaction terms contained in $\mathcal{L}$ and their corresponding Feynman diagram vertices.

Non-Abelian gauge symmetry

11. A non-Abelian gauge theory containing a particle multiplet Ψ possesses a local phase invariance $\Psi \to U(x)\Psi = \exp(ig\omega_j(x)T_j)\Psi$, with covariant derivative $D_\mu \Psi = (\partial_\mu + igA_\mu)\Psi$, where

$$A^\mu(x) = A^\mu_j(x) T_j$$

is a matrix of gauge fields and the T_j are generators of the symmetry group in the representation carried by the multiplet Ψ.

(a B) Show that, for infinitesimal transformations, the gauge fields A^μ_l transform as

$$A^\mu_l \to A'^\mu_l = A^\mu_l - \partial^\mu \omega_l - g f_{jkl} \omega_j A^\mu_k \ ,$$

independent of the representation carried by the multiplet Ψ.

(b C) (optional) By directly transforming the matrix field A^μ, show that the matrix field strength tensor

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

transforms under a general gauge transformation as

$$F_{\mu\nu} \to U F_{\mu\nu} U^\dagger \ .$$

Hence show that the term $-\frac{1}{4} F^{\mu\nu}_j F_{j\mu\nu}$ is gauge-invariant, where the field strength tensor $F^{\mu\nu}_j$ is defined as

$$F^{\mu\nu}_j \equiv \partial^\mu A^\nu_j - \partial^\nu A^\mu_j - g f_{jkl} A^\mu_k A^\nu_l \ .$$