University of Cambridge Part III Natural Sciences Tripos

Gauge Field Theory

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1 Avant propos

These lectures are intended to take you from something that you (hopefully) know very well – the Schrödinger equation of non-relativistic quantum mechanics – to the current state-of-the-art in our understanding of the fundamental particles of Nature and their interactions. That state-of-the-art is described by a gauge field theory (hence the title of these lectures) called the "Standard Model" of particle physics, of which the Higgs boson, recently discovered at the CERN LHC, is a key part. All other physics (except gravity) and indeed every phenomenon in the Universe, from consciousness to chemistry, is but a convoluted application of it. Going further, it turns out that (despite what you may have read in the newspapers) even quantum gravity (in its general relativistic incarnation) makes perfect sense as a gauge field theory, provided we don't ask what happens at energy scales beyond the Planck scale of 10^{19} GeV. So rather a lot is known. As the late Sidney Coleman put it at the beginning of *his* lecture course, "Not only God knows, but I know, and by the end of this semester, you will know too."

A gauge field theory is a special type of quantum field theory, in which matter fields (like electrons and quarks, which make up protons and neutrons) interact with each other via forces that are mediated by the exchange of vector bosons (like photons and gluons, which bind quarks together in nucleons). The Standard Model provides a consistent theoretical description of all of the known forces except gravity. Perhaps more pertinently, it has been spectacularly successful in describing essentially all experiments performed so far, including the most precise measurements in the history of science. The recent discovery of the Higgs boson, at CERN's Large Hadron Collider, constitutes the final piece in the jigsaw of its experimental verification.

As well as learning about of all of this, we hope to resolve, along the way, a number of issues that must have appeared mysterious to you in your previous studies. We shall see *why* a relativistic generalization of the Schrödinger equation is not possible and hence why you have been stuck with the non-relativistic version until now, even though you have known all about relativity for years. We shall learn *why* electrons have spin half, *why* their gyromagnetic ratio is (about) two, and *why* identical electrons cannot occupy the same quantum state. More to the point, we shall see how it is even conceivable that two electrons can be *exactly* identical. We shall see *why* it is not possible to write down a Schrödinger equation for the photon and hence why your lecturers, up until now, have taken great pains to avoid discussing electromagnetism and quantum mechanics at the same time. We shall understand *why* it is possible that three forces of nature (the strong and weak nuclear forces, together with electromagnetism) which appear to be so different in their nature, have essentially the same underlying theoretical structure. We shall learn what rôle the Higgs boson plays in the theory and *why* it was expected to appear at the LHC. Finally, we shall learn about tantalizing hints that we need a theory that goes beyond the Standard Model – gravity, neutrino masses, grand unification, and the hierarchy problem.

That is the good news. The bad news is that all this is rather a lot to learn in only twelve lectures, given that I assume only that the reader has a working knowledge of non-relativistic quantum mechanics, special relativity, and Maxwell's equations.[1] Our coverage of the material will necessarily be brief. Many important derivations and details will be left out. It goes without saying that any student who wants more than just a glimpse of this subject will need to devote rather more time to its proper study. For that, the books recommended below are as good as any place to begin.

2 Bedtime Reading

- Quantum Field Theory, Mandl F and Shaw G (2nd edn Wiley 2009) [1].

  This short book makes for a good companion to this course, covering most of the material using the same (canonical quantization) approach.

- Quantum Field Theory in a Nutshell, Zee A (2nd edn Princeton University Press 2010) [2].

  This is a wonderful book, full of charming insights and doing (in not so many pages) a great job of conveying the ubiquity of quantum field theory in modern particle and condensed matter physics research. Written mostly using the path integral aproach, but don't let that put you off.

- An Introduction to Quantum Field Theory, Peskin M E and Schroeder D V (Addison-Wesley 1995) [3].

  The title claims it is an introduction, but don't be misled – this book will take you a lot further than that. Suffice to say, this is where most budding particle theorists learn field theory these days.

[1] For those in Cambridge, there are no formal prerequisites, though it surely can do no harm to have taken the Part III 'Particle Physics' or 'Quantum Field Theory' Major Options.

- Gauge Theories in Particle Physics, Aitchison I J R and Hey A J G (4th edn 2 vols IoP 2012) [4, 5].

  These two volumes are designed for experimental particle physicists and offer a gentler (if longer) introduction to the ideas of gauge theory. The canonical quantization approach is followed and both volumes are needed to cover this course.

- An Invitation to Quantum Field Theory, Alvarez-Gaume L and Vazquez-Mozo M A (Springer Lecture Notes in Physics vol 839 2011)[6].

  At a similar level to these notes, but discusses other interesting aspects not covered here. An earlier version can be found at [7].

  The necessary group theory aspects of the course are covered in the above books, but to learn it properly I would read

- Lie Algebras in Particle Physics, Georgi H (2nd edn Frontiers in Physics vol 54 1999) [8].

3 Notation and conventions

To make the formulæ as streamlined as possible, we use a system of units in which there is only one dimensionful quantity (so that we may still do dimensional analysis) – energy – and in which $\hbar = c = 1$. Thus $E = mc^2$ becomes $E = m$, and so on. Additionally, we typically use eV (electron-Volt) units to measure energy. In such units, the electron mass $m_e = 0.91 \times 10^{-27}$g takes the value $m_e = 0.51$ MeV.

For relativity, we set $x^0 = t, x^1 = x, x^2 = y, x^3 = z$ and denote the components of the position 4-vector by x^μ, with a Greek index. The components of spatial 3-vectors will be denoted by Latin indices, *e.g.* $x^i = (x, y, z)$. We define Lorentz transformations as those transformations which leave the metric $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ invariant (they are said to form the group $SO(3, 1)$). Thus, under a Lorentz transformation, $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, we must have that $\eta^{\mu\nu} \to \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho \eta^{\sigma\rho} = \eta^{\mu\nu}$. The reader may check, for example, that a boost along the x axis, given by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

with $\gamma^2 = (1 - \beta^2)^{-1}$, has just this property.

Any set of four components transforming in the same way as x^μ is called a *contravariant 4-vector*. The derivative $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ (which we denote by ∂_μ), transforms as the (matrix) inverse of x^μ. Thus we define, $\partial_\mu \to \partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu$, with $\Lambda_\mu{}^\nu \Lambda^\mu{}_\rho = \delta^\nu_\rho$, where $\delta = \text{diag}(1, 1, 1, 1)$. Any set of four components transforming in the same way as ∂_μ is called a *covariant 4-vector*. We now make the rule that indices may be raised or lowered using the metric tensor $\eta^{\mu\nu}$ or its inverse, which we write as $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Thus, $x_\mu = \eta_{\mu\nu} x^\nu = (t, -x, -y, -z)$. With this rule, any expression in which all indices are contracted pairwise with one index of each pair upstairs and one downstairs is manifestly Lorentz invariant. For example,[2] $x_\mu x^\mu = t^2 - x^2 - y^2 - z^2 \to x'_\mu x'^\mu = x_\mu x^\mu$.

When we come to spinors, we shall need the *gamma matrices*, γ^μ, which are a set of four, 4 x 4 matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \cdot 1$, where 1 denotes the 4 x 4 unit matrix. In these lecture notes, we shall use two different representations, both of which are common in the literature. The first is the *chiral representation*, given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \tag{3.2}$$

where $\sigma^\mu = (1, \sigma^i)$, $\overline{\sigma}^\mu = (1, -\sigma^i)$, and σ^i are the usual 2 x 2 Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

For this representation,

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

The other representation for gamma matrices is the *Pauli-Dirac representation*, in which we replace

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5}$$

and hence

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

We shall often employ Feynman's *slash* notation, where, *e.g.*, $\not{a} \equiv a_\mu \gamma^\mu$ and we shall often write an identity matrix as 1, or indeed omit it altogether. Its presence should always be clear from the context.[3]

Finally, it is to be greatly regretted that the electron was discovered before the positron and hence the *particle* has negative charge. We therefore set $e < 0$.

4 Relativistic quantum mechanics

4.1 Why QM does and doesn't work

As promised, we begin with the Schrödinger equation of non-relativistic quantum mechanics. Here it is:

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + V\psi. \tag{4.1}$$

[2]We employ the usual Einstein summation convention, $x_\mu x^\mu \equiv \sum_{\mu=0}^{3} x_\mu x^\mu$.
[3]All this cryptic notation may seem obtuse to you now, but most people grow to love it.

For free particles, with $V(x) = 0$, the equation admits plane wave solutions of the form $\psi \propto e^{i(p \cdot x - Et)}$, provided that $E = \frac{p^2}{2m}$, corresponding to the usual Energy-momentum dispersion relation for free, non-relativistic particles.

No doubt all of this, together with the usual stuff about $|\psi(x)|^2$ being interpreted as the probability to find a particle at x, is old hat to you. By now, you have solved countless complicated problems in quantum mechanics with spinning electrons orbiting protons, bouncing off potential steps, being perturbed by hyperfine interactions, and so on. But at the risk of boring you, and before we leap into the weird and wonderful world of relativistic quantum mechanics and quantum field theory, I would like to spend a little time dwelling on what quantum mechanics really is.

The reason I do so is because the teaching of quantum mechanics these days usually follows the same dogma: firstly, the student is told about the failure of classical physics at the beginning of the last century; secondly, the heroic confusions of the founding fathers are described and the student is given to understand that no humble undergraduate student could hope to actually *understand* quantum mechanics for himself; thirdly, a *deus ex machina* arrives in the form of a set of postulates (the Schrödinger equation, the collapse of the wavefunction, *etc*); fourthly, a bombardment of experimental verifications is given, so that the student cannot doubt that QM is correct; fifthly, the student learns how to solve the problems that will appear on the exam paper, hopefully with as little thought as possible.

The problem with this approach is that it does not leave much opportunity to wonder exactly in what regimes quantum mechanics does and does not work, or indeed why it has a chance of working at all. This, unfortunately, risks leaving the student high and dry when it turns out that QM (in its non-relativistic, undergraduate incarnation) is not a panacea and that it too needs to be superseded.

To give an example, every student knows that $\int dx |\psi(x,t)|^2$ gives the total probability to find the particle and that this should be normalized to one. But *a priori*, this integral could be a function of t, in which case either the total probability to find the particle would change with time (when it should be fixed at unity) or (if we let the normalization constant be time-dependent) the normalized wavefunction would no longer satisfy the Schrödinger equation. Neither of these is palatable. What every student does not know, perhaps, is that this calamity is automatically avoided in the following way. It turns out that the current

$$j^\mu = (\rho, \mathbf{j}) = (\psi^* \psi, -\frac{i}{2m}(\psi^* \nabla \psi - \psi \nabla \psi^*)) \tag{4.2}$$

is conserved, satisfying $\partial_\mu j^\mu = 0$. (For now, you can show this directly using the Schrödinger equation, but soon we shall see how such conserved currents can be identified just by looking at the Lagrangian; in this case, the current conservation follows because a phase rotated wavenfunction $\psi' = e^{i\alpha}\psi$ also satisfies the Schrödinger equation.) Why *conserved*? Well, integrating $\partial_\mu j^\mu = 0$ we get that the rate of change of the time component of the current in a given volume is equal to (minus) the flux of the spatial component of the current out of that volume:

$$\frac{d}{dt}\int \rho dV = -\int_{\partial V} \mathbf{j}\cdot d\mathbf{S}. \tag{4.3}$$

In particular, $\psi^*\psi$ integrated over *all* space, is constant in time. This is a notion which is probably familiar to you from classical mechanics and electromagnetism. It says that $\psi^*\psi$, which we interpret as the probability density in QM, is conserved, meaning that the probability interpretation is a consistent one.

This conservation of the total probability to find a particle in QM is both its salvation and its downfall. Not only does it tell us that QM is consistent in the sense above, but it also tells that QM cannot hope to describe a theory in which the number of particles present changes with time. This is easy to see: if a particle disappears, then the total probability to find it beforehand should be unity and the total probability to find it afterwards should be zero. Note that in QM we are not forced to consider states with a single particle (like a single electron in the Coulomb potential of a hydrogen atom), but we are forced to consider states in which the number of particles is fixed for all time. Another way to see this is that the wavefunction for a many-particle state is given by $\psi(x_1, x_2, \dots)$, where $x_1, x_2, \dots$ are the positions of the different particles. But there is no conceivable way for this wavefunction to describe a process in which a particle at x_1 disappears and a different particle appears at some other x_3.

Unfortunately, it happens to be the case in Nature that particles do appear and disappear. An obvious example is one that (amusingly enough) is usually introduced at the beginning of a QM course, namely the photoelectric effect, in which photons are annihilated at a surface. It is important to note that it is not the relativistic nature of the photons which prevents their description using QM, it is the fact that their number is not conserved. Indeed, phonons arise in condensed matter physics as the quanta of lattice vibrations. They are non-relativistic, but they cannot be described using QM either.

Ultimately, this is the reason why our attempts to construct a relativistic version of QM will fail: in the relativistic regime, there is sufficient energy to create new particles and such processes cannot be described by QM. This particle creation is perhaps not such a surprise. You already know that in relativity, a particle receives a contribution to its energy from its mass via $E = mc^2$. This suggests (but certainly does not prove) that if there is enough E, then we may be able to create new sources of m, in the form of particles. It turns out that this does indeed happen and indeed much of current research in particle physics is based on it: by building colliders (such as the Large Hadron Collider) producing ever-higher energies, we are able to create new particles, previously unknown to science and to study their properties.

Even though our imminent attempt to build a relativistic version of QM will eventually fail, it will turn out to be enormously useful in finding a theory that does work. That theory is called Quantum Field Theory and it will be the subject of the next section. For now, we will press ahead with relativistic QM.

4.2 The Klein-Gordon equation

To write down a relativistic version of the Schrödinger equation is easy - so easy, in fact, that Schrödinger himself wrote it down *before* he wrote down the equation that made him famous. Starting from the expectation that the free theory should have plane wave solutions (just as in the non-relativistic case), of the form $\phi \propto e^{-iEt+i\mathbf{p}\cdot\mathbf{x}} = e^{-ip_\mu x^\mu}$ and noting that the relativistic dispersion relation $p^\mu p_\mu = m^2$ should be reproduced, we infer the *Klein-Gordon equation*

$$(\partial_\mu \partial^\mu + m^2)\phi = 0. \tag{4.4}$$

If we assume that ϕ is a single complex number, then it must be a Lorentz scalar, being invariant under a Lorentz transformation: $\phi(x^\mu) \to \phi'(x'^\mu)$. The Klein-Gordon equation is then manifestly invariant under Lorentz transformations. The problems with this equation quickly become apparent. Firstly, the probability density cannot be $|\phi|^2$ as it is in the non-relativistic case, because $|\phi|^2$ transforms as a Lorentz scalar (*i.e.* it is invariant), rather than as the time component of a 4-vector (the probability density transforms like the inverse of a volume, which is Lorentz contracted). Moreover, $|\phi|^2$ is not conserved in time. To find the correct probability density, we must find a conserved quantity. Again, we shall soon have the tools in hand to do so ourselves, but for now we pull another rabbit out of the hat, claiming that the 4-current

$$j^\mu = i(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \tag{4.5}$$

satisfies $\partial_\mu j^\mu = 0$ (exercise), meaning that its time component integrated over space, $\int dxi(\phi\frac{\partial}{\partial t}\phi^* - \phi^*\frac{\partial}{\partial t}\phi)$ is a conserved quantity. So far so good, but note that $\int dxi(\phi\frac{\partial}{\partial t}\phi^* - \phi^*\frac{\partial}{\partial t}\phi)$ is not necessarily positive. Indeed, for plane waves of the form $\phi = Ae^{\mp ip_\mu x^\mu}$, we obtain $\rho = \pm 2E|A|^2$. There is a related problem, which is that the solutions $\phi = Ae^{\pm ip_\mu x^\mu}$, correspond to both positive and negative energy solutions of the relativistic dispersion relation: $E = \pm\sqrt{p^2 + m^2}$. Negative energy states are problematic, because there is nothing to stop the vacuum decaying into these states. In classical relativistic mechanics, the problem of these negative energy solutions never reared its ugly head, because we could simply throw them away, declaring that all particles (or rockets or whatever) have positive energy. But when we solve a wave equation (as we do in QM), completeness requires us to include both positive and negative energy solutions in order to be able to find a general solution.

4.3 The Dirac equation

In 1928, Dirac tried to solve the problem of negative-energy solutions by looking for a wave equation that was first order in time-derivatives, the hope being that one could then obtain a dispersion relation of the form $E = +\sqrt{p^2 + m^2}$ directly, without encountering negative-energy states. Dirac realised that one could write an equation that was linear in both time and space derivatives of the form

$$(i\gamma^\nu \partial_\nu - m)\psi = 0 \tag{4.6}$$

that implied the Klein-Gordon equation for ψ, provided that the 4 constants γ^ν were matrices. To wit, acting on the left with $(i\gamma^\mu\partial_\mu + m)$, we obtain

$$(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = 0. \tag{4.7}$$

Since $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, we may symmetrize to get

$$(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu - m^2)\psi = 0. \tag{4.8}$$

Thus, (minus) the Klein-Gordon equation is recovered if the anticommutator is such that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{4.9}$$

The γ^ν evidently cannot be simply numbers, since, for example, $\gamma^0\gamma^1 = -\gamma^1\gamma^0$. In fact, the smallest possible matrices that implement this relation are 4x4, as you may show by trial and error. Any set of matrices satisfying the algebra will do, but some are more convenient than others, depending on the problem at hand. We will either use the *chiral representation*

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \tag{4.10}$$

where $\sigma^\mu = (1, \sigma^i)$, $\overline{\sigma}^\mu = (1, -\sigma^i)$, and σ^i are the usual 2 x 2 Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.11}$$

or we will use the *Pauli-Dirac* representation in which we replace

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.12}$$

Note that γ^0 is Hermitian in either representation, whereas γ^i are anti-Hermitian. This can be conveniently written as $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, but note that this equation (and the hermiticity properties) are not basis-independent. Since the γ^ν are 4 x 4 matrices, the wavefunction ψ must have 4 components. It is not a 4-vector (and nor are the γ^ν, despite the suggestive notation, since they are constants and do not transform). It transforms in a special way under Lorentz transformations (which we don't have time to go through here, sadly) and we call it a 4-component *spinor*.

In the rest of these lectures we will avoid component notation but let's spell it out at least once. A Dirac spinor ψ has four components, ψ_α, traditionally indexed by Greek letters $\alpha, \beta, \cdots = 1, 2, 3, 4$. In component notation a gamma matrix γ^μ reads $(\gamma^\mu)_{\alpha,\beta}$. Thus, a product like $\gamma^\mu\psi$ reads $(\gamma^\mu)_{\alpha,\beta}\psi_\beta$, etc.

It can be shown that Dirac's equation has a conserved current given by

$$j^\mu = (\psi^\dagger\psi, \psi^\dagger\gamma^0\gamma^i\psi), \tag{4.13}$$

where $\psi^\dagger$ is the Hermitian conjugate (transpose conjugate) of ψ.

We can rewrite the above result in a much more instructive form. Let's introduce the spinor (sometimes called *Dirac adjoint*; very important for the rest of these lectures!)

$$\overline{\psi} = \psi^\dagger \gamma^0 \,. \tag{4.14}$$

Its utility lies in the fact that $\overline{\psi}\psi$ is a Lorentz invariant, whereas $\psi^\dagger\psi$ is not. [4] Indeed, as it should be clear from eq. (4.13), $\psi^\dagger\psi$ is the time component of a 4-vector, namely the probability current. Using $(\gamma^0)^2 = 1$ (prove this; the result is independent of the γ matrix representation) one can show that the current eq. (4.13) takes the compact form:

$$j^\mu = \overline{\psi}\gamma^\mu\psi \,. \tag{4.15}$$

Note that since the probability density $\psi^\dagger\psi$ is positive definite, Dirac managed to solve one problem. But what about the negative energy solutions? In the rest frame, with $(E,p) = (m,0)$, we find solutions to (4.6) of the form $A_\mp e^{\mp imt}$, provided that

$$(\pm\gamma^0 - 1)A_\mp = 0 \implies A_- \propto \begin{pmatrix} A_1 \\ A_2 \\ 0 \\ 0 \end{pmatrix}, A_+ \propto \begin{pmatrix} 0 \\ 0 \\ A_3 \\ A_4 \end{pmatrix}, \tag{4.16}$$

where we used the Pauli-Dirac basis. So there are four modes, two of which have positive energy and two of which have negative energy. The two positive energy modes are interpreted (as we shall soon see) as the two different spin states of a spin-half particle. Dirac's proposal to deal with the negative energy states was as follows. Since the Pauli exclusion principle for these spin-half fermions forbids multiple occupation of states, one can postulate that the vacuum corresponds to a state in which all of the negative energy states are filled. Then, Dirac argued, if one has enough energy, one might be able to promote one of these negative-energy particles to a positive-energy particle. One would be left with a 'hole' in the sea of negative energy states, which would behave just like a particle with opposite charge to the original particles. Thus Dirac came up with the concept of antiparticles. The antiparticle of the electron, the positron, was duly found, bringing great acclaim to Dirac. But this picture of the *Dirac sea* was soon rendered obsolete by the emergence of quantum field theory.

It is not much harder to find the plane-wave solutions of the Dirac equation in any frame, so we do it for completeness. For the positive-energy solutions of (4.6), write $\psi = ue^{-ip\cdot x}$, such that $(\not{p} - m)u = 0$. Writing $u = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ implies

$$u = N \begin{pmatrix} \phi \\ \frac{\sigma\cdot\mathbf{p}}{E+m}\phi \end{pmatrix}. \tag{4.17}$$

[4] Sadly, we cannot show this without first showing how a spinor transforms. You will have to look in a book.

Finally, taking the two states to be $\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we obtain

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \; u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}. \tag{4.18}$$

For the negative-energy solutions, write $\psi = ve^{+ip\cdot x}$, such that $(\not p + m)v = 0$. Thus,

$$v = N \begin{pmatrix} \frac{\sigma\cdot\mathbf{p}}{E+m}\chi \\ \chi \end{pmatrix}, \tag{4.19}$$

such that

$$v_1 = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \; v_2 = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}. \tag{4.20}$$

We find it most convenient to normalize in such a way that that there is a number density $\rho = \psi^\dagger\psi = u^\dagger u = v^\dagger v$ of $2E$ particles per unit volume. This fixes $N = \sqrt{E+m}$.

We end our treatment of the Dirac equation by showing that it does indeed describe a spin-half particle. To do so, we show that there exists an operator $\mathbf{S}$, such that $\mathbf{J} \equiv \mathbf{L} + \mathbf{S}$ is a constant of the motion with $\mathbf{S}^2 = s(s+1) = \frac{3}{4}$. First note that the orbital angular momentum $\mathbf{L}$ does not commute with the Hamiltonian, defined, *à la* Schrödinger, to be everything that appears on the right of the Dirac equation when $i\frac{\partial\psi}{\partial t}$ appears on the left. Thus,

$$H = \gamma^0(\gamma^i p_i + m). \tag{4.21}$$

Then, for example

$$[L_3, H] = [x_1p_2 - x_2p_1, H] = [x_1, H]p_2 - [x_2, H]p_1 = i\gamma^0(\gamma^1 p_2 - \gamma^2 p_1) \neq 0. \tag{4.22}$$

The operator $\mathbf{S}$ that ensures $[H, J^i] = 0$ is given by $\mathbf{S} \equiv \frac{\mathbf{\Sigma}}{2}$, where $\Sigma^i \equiv \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$. As a check (in the chiral basis),

$$[S_3, H] = [\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \begin{pmatrix} -\sigma^i p_i & m \\ m & \sigma^i p_i \end{pmatrix}] = -i\gamma^0(\gamma^1 p_2 - \gamma^2 p_1) = -[L_3, H]. \tag{4.23}$$

Moreover, $\mathbf{S}^2 = \frac{1}{4}\sigma_i\sigma_i = \frac{3}{4}$, as required.

4.4 Maxwell's equations

This is a convenient juncture at which to introduce Maxwell's equations of electromagnetism, even though we make no effort to make a quantum mechanical theory out of them (since the number of photons is not fixed, it is doomed to fail). We shall need them for our later study of QFT, however.

In some system of units, Maxwell's equations may be written as

$$\nabla \cdot \mathbf{E} = \rho, \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0 \tag{4.24}$$

$$\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{B} = \mathbf{j} + \dot{\mathbf{E}}. \tag{4.25}$$

In terms of the scalar and vector potentials V and $\mathbf{A}$ we may solve the two homogeneous equations by writing

$$\mathbf{E} = -\nabla V - \dot{\mathbf{A}}, \tag{4.26}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4.27}$$

All of this is more conveniently (and covariantly) written in terms of the 4-vector potential, $A^\mu \equiv (V, \mathbf{A})$, the 4-current, $j^\mu \equiv (\rho, \mathbf{j})$ and the antisymmetric *field strength tensor*, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$; indeed, Maxwell's equations then reduce to the rather more compact form

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{4.28}$$

This rendering makes it obvious that Maxwell's equations are invariant (as are $\mathbf{E}$ and $\mathbf{B}$ themselves) under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$, where χ is an arbitrary function on spacetime. This 'gauge' is the same 'gauge' that appears in the title of these lectures, so it behoves you to play close attention whenever you see the word from now on!

One way we can deal with the gauge freedom is to remove it (wholly or partially) by *gauge fixing*. One common choice is the Lorenz (not Lorentz!) gauge $\partial_\mu A^\mu = 0$. In this gauge, each of the four components of the vector A^μ satisfies the Klein-Gordon equation with $m = 0$, corresponding to a massless photon. We can find plane wave solutions of the form $A^\mu = \epsilon^\mu e^{-ip\cdot x}$, with $p^2 = 0$. Since we have fixed the gauge $\partial_\mu A^\mu = 0$, we must have that $\epsilon \cdot p = 0$. Moreover, the residual gauge invariance implies that shifting the polarization vector ϵ_μ by an amount proportional to p^μ gives an equivalent polarization vector. Thus, there are only two physical degrees of polarization. These could, for example, be taken to be purely transverse to the photon 3-momentum.[5]

Finally, we discuss how to couple the electromagnetic field to Klein-Gordon or Dirac particles. The usual argument given in classical mechanics and non-relativistic QM is that one should use the rules of minimal substitution, replacing [6]

$$\partial^\mu \to D^\mu \equiv \partial^\mu + ieA^\mu \,. \tag{4.29}$$

[5]The fact that there are two polarizations does not mean that the photon has spin one-half! In fact, spin – which could be described as the total angular momentum of a particle in its rest frame – is not a well-defined concept for massless particles, which do not have a rest frame. Massless particles can instead be described by their *helicity*, which is defined as the angular momentum parallel to the direction of motion. It can take just two values (± 1 for the photon), leading to the two polarizations just found.

[6]This is completely unmotivated. We shall , very shortly, have the means at hand to provide a satisfactory discussion of how things *should* be done, but for now we beg the reader's leniency.

Thus, the Klein-Gordon equation becomes

$$(\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu)\phi + m^2\phi = 0. \tag{4.30}$$

It is interesting to note that, if we take a negative energy solution $\phi \propto e^{+i(Et+\mathbf{p}\cdot\mathbf{x})}$ with charge $+e$, the complex conjugate field $\phi^* \propto e^{-i(Et+\mathbf{p}\cdot\mathbf{x})}$ (which satisfies the complex conjugate of the Klein-Gordon equation) can be interpreted as a positive energy solution with opposite momentum and opposite charge $-e$. This presages the interpretation of the negative energy solutions in terms of antiparticles in quantum field theory.

For the Dirac equation, the coupling to electromagnetism is even more interesting. Blithely making the minimal substitution, we get

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi = 0. \tag{4.31}$$

Now, if we act on the left with $(i\gamma^\mu(\partial_\mu + ieA_\mu) + m)$ we do not obtain the Klein-Gordon equation (4.30). Instead, we find the equation (exercise – hint: use $2\gamma^\mu\gamma^\nu \equiv \{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]$)

$$(D^2 + m^2 + \frac{ie}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu})\psi = 0, \tag{4.32}$$

with the extra term $\frac{ie}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}$. Now, in the Pauli-Dirac basis, $\frac{i}{2}[\gamma^i, \gamma^j]$ is given by $\epsilon^{ijk}\Sigma^k$ where, as we saw before, $\frac{\Sigma_k}{2}$ represents the spin S^k. Thus, in a magnetic field, with $F_{ij} = \epsilon_{ijk}B_k$, we get the extra term $2e\mathbf{S}\cdot\mathbf{B}$. This factor of 2 is crucial – if one works out the D^2 term (which is present even for a spinless particle), one will also find an interaction between the orbital angular momentum $\mathbf{L}$ and $\mathbf{B}$ given by $e\mathbf{L}\cdot\mathbf{B}$. Thus, Dirac's theory predicted that the electron spin would produce a magnetic moment a factor of two larger that the magnetic moment due to orbital magnetic moment, as was observed in experiment.

In fact, increasing experimental precision eventually showed that the gyromagnetic ratio of the electron is not quite two, but rather $2.0023193\ldots$. In yet another heroic triumph for theoretical physics, Schwinger showed in 1948 that this tiny discrepancy could be perfectly accounted for by quantum field theory, to which we shortly turn.

4.5 Transition rates and scattering

Before we go further, we need to modify one more aspect of your quantum mechanics education. QM has its hegemony in atomic physics, where one is interested in energy spectra and so on. In particle physics, we are less interested in energy spectra. One reason is that (as we shall see) we are not able to compute them. A more pragmatic reason is that many of the particles in particle physics are very short-lived; we learn things about them by doing scattering experiments, in which we collide stable particles (electrons or protons) to form new particles, and then observe those new particles decay. The quantities of interest (that we would like to compute using quantum field theory) are therefore things like *decay rates* and *cross sections*. What a decay rate is should be obvious to you. A cross-section is only a bit more complicated. Clearly, the probability for two beams of particles to scatter depends on things like the area of the beams and their densities. The cross-section is a derived quantity which depends only on the nature of the particles making up the beams (and their four-momenta).

To derive formulæ for these, we start with something you should know from QM. *Fermi's Golden rule* decrees that the transition rate from state i to state f via a Hamiltonian perturbation H' is given by

$$\Gamma = 2\pi |T_{fi}|^2 \delta(E_i - E_f), \tag{4.33}$$

where

$$T_{fi} = \langle f|H'|i\rangle + \Sigma_{n\neq i} \frac{\langle f|H'|n\rangle\langle n|H'|i\rangle}{E_n - E_i} + \dots \tag{4.34}$$

Let's now try to apply this formula to the decay of a particle into n lighter particles, $a \to 1 + 2 + \cdots + n$. There are $n-1$ independent 3-momenta in the final state (momentum must be conserved overall in the decay). Now, for states normalized such that there is one particle per unit volume in position space, then we have one particle per $h^3 = (2\pi)^3$ volume in momentum space (recall the de Broglie relation $p = \frac{h}{\lambda}$ and recall that $\hbar = 1$ in our system of units). Thus, the decay rate to produce particles in the final state with momenta between p and $p + dp$ is

$$\Gamma = 2\pi \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_{n-1}}{(2\pi)^3} |T_{fi}|^2 \delta(E_a - E_1 - E_2 \cdots - E_n), \tag{4.35}$$

$$= (2\pi)^4 \int \frac{d^3\mathbf{p}_1}{(2\pi)^3} \cdots \frac{d^3\mathbf{p}_n}{(2\pi)^3} |T_{fi}|^2 \delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2 \cdots - \mathbf{p}_n)\delta(E_a - E_1 - E_2 \cdots - E_n), \tag{4.36}$$

where in the last line we have written things more covariantly.

There is one complication, which is that we will *not* normalize states to one particle per unit volume. Instead (as we just did for solutions of the Dirac equation), we will normalize to $2E$ particles per unit volume. The E is convenient because the density transforms under a Lorentz transformation like an energy does (the volume is Lorentz contracted). The 2 just makes some formulæ more streamlined. To compensate for this, we divide by $2E$ everywhere in the above formula, defining $|T_{fi}|^2 = \frac{|\mathcal{M}|^2}{2E_a 2E_1 \ldots 2E_n}$. Finally, we get

$$\Gamma = \frac{(2\pi)^4}{2E_a} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \cdots \frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n} |\mathcal{M}|^2 \delta^4(p_a^\mu - p_1^\mu - p_2^\mu \cdots - p_n^\mu). \tag{4.37}$$

For two-particle scattering, $a + b \to 1 + 2 + \cdots + n$, the transition rate is, analogously,

$$\frac{(2\pi)^4}{2E_a 2E_b} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \cdots \frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n} |\mathcal{M}|^2 \delta^4(p_a^\mu + p_b^\mu - p_1^\mu - p_2^\mu \cdots - p_n^\mu). \tag{4.38}$$

To get the cross-section formula with these conventions, we just divide by the flux of a particles on b in a given frame, which is $|v_a - v_b|$. In all,

$$\sigma = \frac{(2\pi)^4}{2E_a 2E_b |v_a - v_b|} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \cdots \frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n} |\mathcal{M}|^2 \delta^4(p_a^\mu + p_b^\mu - p_1^\mu - p_2^\mu \cdots - p_n^\mu). \tag{4.39}$$

It is useful to derive expressions from these general formulæ for two-body final states. For the two-body decay in the rest frame of a, we find (exercise)

$$\Gamma(a \to 1 + 2) = \frac{|\mathbf{p}_1|}{32\pi^2 m_a^2} \int |\mathcal{M}|^2 \sin\theta d\theta d\phi, \tag{4.40}$$

where particle 1 has 3-momentum $(|\mathbf{p}_1| \sin\theta \cos\phi, |\mathbf{p}_1| \sin\theta \sin\phi, |\mathbf{p}_1| \cos\theta)$. For two-body scattering in the CM frame, we similarly find

$$\sigma(a + b \to 1 + 2) = \frac{|\mathbf{p}_1|}{64\pi^2 |\mathbf{p}_a| s} \int |\mathcal{M}|^2 \sin\theta d\theta d\phi. \tag{4.41}$$

Here we have introduced the first of three *Mandelstam variables*

$$s \equiv (p_a^\mu + p_b^\mu)^2, \tag{4.42}$$

$$t \equiv (p_1^\mu - p_a^\mu)^2, \tag{4.43}$$

$$u \equiv (p_a^\mu - p_2^\mu)^2. \tag{4.44}$$

Note that these three variables are dependent, satisfying (exercise)

$$s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2. \tag{4.45}$$

5 Relativistic quantum fields

5.1 Classical field theory

Before we consider quantum field theory, it is useful to begin with a primer on classical field theory. Happily (though you may not know it) you are already experts on classical field theory. Indeed, most undergraduate physics is based on the solution of wave equations, *etc.*, and that is all classical field theory is. However, you may not be so expert on the Hamiltonian and Lagrangian formulations of classical field theory; just like in QM, it is these formulations which are most useful in going from the classical to the quantum regime.

Let us begin with the Lagrangian formulation. Imagine we have a field on spacetime, which we denote generically by $\phi(x^\mu)$. Just like in classical mechanics, the action, S, is obtained by integrating the Lagrangian, L, over time. Now, we shall restrict ourselves to theories in which the Lagrangian can be obtained by integrating something called the Lagrangian density, $\mathcal{L}$ over *space*.[7] Thus

$$S = \int dt L = \int d^4x^\mu \mathcal{L}(\phi(x), \partial^\mu \phi(x)). \tag{5.1}$$

From now on, we will almost always deal with the Lagrangian density only and will often simply call it the Lagrangian.

Given the Lagrangian, the classical (Euler-Lagrange) equations of motion are obtained by extremizing the action. Thus, consider the variation δS that results from a field variation $\delta\phi$:

$$\delta S = \int d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi} \delta\phi + \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi} \delta \partial^\mu \phi \right) \tag{5.2}$$

$$= \int d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi} - \partial^\mu \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi} \right) \delta\phi, \tag{5.3}$$

where we have integrated by parts. The action is thus extremal when

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial^\mu \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi} = 0. \tag{5.4}$$

As an example, the Klein-Gordon Lagrangian is the most general Lorentz-invariant with two or fewer derivatives and is given by [8]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \,. \tag{5.5}$$

You may easily show that the Klein-Gordon equation (4.4) follows from extremization (exercise).

[7]This is an extremely important assumption, in that it restricts us to theories which are local in space-time, in the sense that the fields only couple to other fields which are at the same point in space or are at most infinitesimally far away. It is not obvious that this is a necessary requirement. The only motivations for it are (i) that all observations so far seem to be consistent with it, (ii) even slightly non-local physics looks local if viewed from far enough away and (iii) we have almost no idea of how to write down a consistent theory, bar string theory, which violates locality. Perhaps you can find one.

[8]The factor of one-half is conventional but notice its absence in the case of complex scalar field eq.(5.10).

This formalism is particularly useful for identifying symmetries of the dynamics and the consequent implications. This is encoded in *Noether's theorem.* Suppose that the action is invariant under some symmetry transformation of the fields, $\phi \to \phi + \delta\phi$. Let the transformation is parameterized by the infinitesimal parameter α, i.e. $\delta\phi = \alpha\Delta\phi$, where $\Delta\phi$ is some function of the fields, depending on the specifics of the transformation (there could be more than one independent parameters in which case we sum over them).

The fact that the action is invariant means that the Lagrangian can change at most by a total derivative, $\alpha\partial_\mu K^\mu$, which integrates to zero in the action. Here K^μ is some vector and its form depends on the transformation. Thus, on one hand we have that

$$\delta\mathcal{L} = \alpha\partial_\mu K^\mu \ , \tag{5.6}$$

while on the other

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial^\mu\phi}\delta\partial^\mu\phi \tag{5.7}$$

$$= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi - \left(\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi}\right)\delta\phi + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi}\delta\phi\right). \tag{5.8}$$

But when the equations of motion hold – on classical trajectories – the first two terms on the right hand side cancel. Thus, classically, we have the conserved current

$$\partial_\mu J^\mu = 0, \ \text{ where } \ J^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial^\mu\phi}\Delta\phi - K^\mu. \tag{5.9}$$

As a first example, consider the theory of a complex Klein-Gordon field. Its Lagrangian is given by

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \ . \tag{5.10}$$

The action (and indeed the Lagrangian) is invariant under $\phi \to e^{i\alpha}\phi$. For this transformation, therefore, $K^\mu = 0$. We can derive the conserved current by taking α to be small, such that $\delta\phi = i\alpha\phi$ and $\delta\phi^* = -i\alpha\phi^*$. Thus

$$J^\mu = i\left((\partial^\mu\phi^*)\phi - \phi^*\partial^\mu\phi\right) \ , \tag{5.11}$$

which is precisely the probability current that we encountered in our discussion of the Klein-Gordon equation in QM.

Similarly, the Dirac Lagrangian is given by

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi. \tag{5.12}$$

The invariance of the Dirac Lagrangian under a global rephasing of ψ results in the conservation of this probability current, a fact that we pulled out of a hat in our earlier discussion.

The theories that we concern ourselves with here are also Lorentz- (indeed, Poincaré-) invariant and this too has consequences for the dynamics. Consider, as a second example, the effect of the invariance under spacetime translations $x^\mu \to x^\mu + \alpha^\mu$. A field transforms correspondingly as $\phi(x^\mu) \to \phi(x^\mu + \alpha^\mu) \simeq \phi(x^\mu) + \alpha^\nu\partial_\nu\phi(x^\mu)$, for small α^ν. For this transformation the Lagrangian also changes by $\mathcal{L} \to \mathcal{L} + \alpha^\mu \partial_\mu \mathcal{L}$ (i.e. in this case $K^\mu \neq 0$) and there are four resulting conserved currents (one for each ν) denoted by

$$T^\mu_\nu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}. \tag{5.13}$$

This is called the *energy-momentum tensor.* $\partial_\mu T^\mu_0 = 0$ corresponds to the invariance under time translations and hence expresses conservation of energy (T^0_0 is just the energy density) and $\partial_\mu T^\mu_i = 0$ expresses conservation of momentum. Similarly, invariance under rotations (a subgroup of Lorentz transformations) implies conservation of angular momentum.

At this point, the Lagrangians that we have written down may seem completely arbitrary. In fact, it usually turns out in particle physics that the form of the Lagrangian is essentially fixed, up to a few free parameters, once one has specified the particle content, the symmetries that one desires [9] as well as that the equation of motion is at most of second order.

Let us illustrate this by 'deriving' the Lagrangian for electromagnetism. Here the key symmetry principles are Lorentz invariance and gauge invariance. The second of these dictates that the Lagrangian should be built out of gauge-invariant objects, for which the only candidate is the field strength tensor, $F_{\mu\nu}$. The first dictates that all indices should be contracted together. If we are primarily interested in the long-distance (hence low energy) behaviour of the theory, then the dominant term will be the one with the smallest number of derivatives. Thus we arrive at the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{5.14}$$

It is worthwhile to point out that while the coefficient in front is merely conventional, the sign is not. The minus ensures that the term involving the spatial components of the gauge field (which 'contain' the physical degrees of freedom), $\dot{A^i}^2$, has a positive contribution to the kinetic energy (recall that $L = T - V$). In fact, we can get a lot further by means of symmetry considerations. We can even, for example, determine exactly how the electromagnetic field should couple to complex Klein-Gordon or Dirac fields. We have already seen how both of these fields have an invariance under a global phase rotation, say $\phi \to e^{ie\chi}\phi$. Now suppose that we try to increase the symmetry even further, by promoting this to a local transformation, in which the phase χ, previously a constant, becomes a function of spacetime $\chi(x^\mu)$. The mass terms in the Klein-Gordon or Dirac Lagrangians remain invariant under this enlarged symmetry. But the derivative terms do not, because

[9] It is interesting to ponder, in the long winter evenings, why Nature exhibits such a high degree of symmetry. (It is true that glancing casually at an atlas does not suggest that Nature is terribly symmetric. But we shall see that at short distances, Nature shows a breathtakingly high degree of symmetry.) Some attribute it to the genius of some higher intelligence. Others are more prosaic, arguing that it could not really be any other way. Indeed, as you well know, it is extremely difficult to build a mathematical theory of physics which is fully consistent in all regimes. Every theory breaks down somewhere. The only chance that a theory has to be consistent is for its dynamics to be very strongly constrained, so that nothing can go wrong. But this is precisely what symmetry achieves. A good analogy is a mechanical system, where experience tells us that the fewer moving parts, the less likely it is to break!

$\partial_\mu \phi \to e^{ie\chi}\partial_\mu \phi + ie\partial_\mu \chi e^{ie\chi}\phi$. But now suppose that we introduce an electromagnetic field A^μ whose gauge transformation is given by

$$A^\mu \to A^\mu - \partial^\mu \chi. \tag{5.15}$$

Then, the quantity $(\partial^\mu + ieA^\mu)\phi \equiv D^\mu \phi \to e^{ie\chi} D^\mu \phi$ and the kinetic terms in the action will be invariant.

Let us now pause for breath. What have we done? We have shown that if we take a complex Klein-Gordon or Dirac field with a global re-phasing invariance, we can promote it to a local symmetry at the expense of introducing a new, gauge field A^μ via the covariant derivative D^μ.[10] We have thus 'derived' the arbitrary principle of minimal substitution. But is the principle of local symmetry any less arbitrary? Our general 'theological' argument is that nature is symmetric because symmetry is necessary for consistency of physical laws. But making such an argument for a local symmetry looks like a con. After all, the local part of a symmetry is really just a redundancy of description: we can completely remove it by fixing the gauge. Nevertheless, requiring local symmetry does restrict the possible dynamics (in the sense that various possible terms in the Lagrangian are forbidden) and indeed it is the only way in which we can build a consistent theory of force-carrying vector particles.[11]

The principle of gauge invariance (together with Lorentz invariance) fixes the form of the action involving electrons (which are described by a Dirac field) and electromagnetic radiation (or photons) - it is precisely the one which gives rise to Maxwell's equations in the classical limit. The quantum version of this theory, which is called *quantum electrodynamics* or QED, explains at a stroke all of chemistry and most of physics as well. It has successfully predicted the results of measurements (like the gyromagnetic ratio of the electron) that are the most precise ever carried out in Science. Gauge invariance even dictates how the photon can couple to particles, like the Higgs boson, that do not carry electric charge and in fact this coupling was crucial in the recent discovery of the Higgs boson. Not bad for a humble re-phasing invariance, I would say.

[10] Note that the field strength can be written in terms of the covariant derivative as $F_{\mu\nu} \sim [D_\mu, D_\nu]$.

[11] This can be proven, but I won't do it here. For what comes later, I add that this is also true for non-renormalizable, effective theories. There, all terms are allowed in the Lagrangian, but the sizes of their coefficients are fixed by the principle of gauge invariance and this guarantees consistency.

5.2 Scalar field quantization

There exist two popular formalisms for QFT. Each has its advantages and disadvantages. Here we follow the approach of *canonical quantization*. Its great advantage, for our purposes, is that it is rather close to what you have already done in QM. Its great disadvantage is that it is not well-suited to gauge field theories. We shall circumvent this hurdle by studying only simple examples of QFTs, which are suited to canonical quantization, to begin with, and by using these examples to motivate the form of the *Feynman rules* for more complex theories. Those of you who view this course as the beginning of your career in physics (rather than the end) would be well advised to consult the literature for how to do canonical quantization properly and for the other, *path integral*, approach.

We begin with a real, scalar field. We have already encountered the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2). \tag{5.16}$$

The point of departure from QM is that we shall try to quantize the field ϕ, rather than the position x.[12] Thus, we compute the momentum conjugate to the field ϕ, namely $\pi \equiv \frac{\delta \mathcal{L}}{\delta \dot{\phi}}$ and impose the *equal time* commutation relations

$$[\phi(x^i, t), \pi(x'^i, t)] = i\delta^3(x^i - x'^i), \tag{5.17}$$

$$[\phi(x^i, t), \phi(x'^i, t)] = [\pi(x^i, t), \pi(x'^i, t)] = 0. \tag{5.18}$$

The δ function simply accounts for the fact that the fields at different space points are considered to be independent. [13] Notice that, since the operators ϕ and π depend on time, we are working in the *Heisenberg picture* of QM, rather than the *Schrödinger picture* (in the latter, operators are constant in time and states have all the time dependence). We'll have more to say about this later on.

The basic goal in QM is to find the spectrum of energies and eigenstates of the Hamiltonian. This looks like a hard problem for our field theory, for which the Hamiltonian (density) is given by

$$\mathcal{H}(\phi, \pi) \equiv \pi \dot{\phi} - \mathcal{L} = \frac{1}{2}(\pi^2 + (\nabla \phi)^2 + m^2 \phi^2). \tag{5.19}$$

Thankfully, it is rendered almost trivial if we make the Fourier transform

$$\phi(x, t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \left(a_p e^{-iEt + i\mathbf{p}\cdot\mathbf{x}} + a_p^\dagger e^{+iEt - i\mathbf{p}\cdot\mathbf{x}} \right), \tag{5.20}$$

with $E \equiv +\sqrt{p^2 + m^2}$. Note that we have forced ϕ to be real (or rather Hermitian, since it is now to be interpreted as an operator). Note also that we have normalized using the Lorentz-invariant integration measure $\frac{d^3\mathbf{p}}{(2\pi)^3 2E}$.[14]

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[12] Such a dramatic change makes it hard to imagine how QM can be recovered as a limit of QFT; we shall have to go through some acrobatics later on to do so.

[13] This statement can be generalized to independence on any space-like separation of which the above is a particular case.

[14] This is Lorentz invariant, because it can also be written as $\frac{1}{(2\pi)^3} \int d^4 p \delta(p^2 - m^2)$.

With this transformation, one may show (recall that $\int d^3\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3\delta^3(\mathbf{x})$) that the commutation relations (5.17) can be reproduced by

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = (2\pi)^3 2E\delta^3(\mathbf{p}-\mathbf{p}'), \tag{5.21}$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = 0. \tag{5.22}$$

This is encouraging, since (apart from a normalization factor) these are the usual commutation relations for the ladder operators a and $a^\dagger$ of the simple harmonic oscillator, with one oscillator for each $\mathbf{p}$. The delta function expresses the fact that the different oscillators are independent. Even better, the various contributions to the Hamiltonian (not the Hamiltonian density, for once) may be written as (note that $E = E'$ when $\mathbf{p}' = -\mathbf{p}$, etc)

$$\frac{1}{2}\int d^3x\, m^2\phi^2 = \frac{1}{(2\pi)^3 8E^2}\int d^3\mathbf{p}\, m^2\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iEt} + a^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}e^{+2iEt} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right) \tag{5.23}$$

$$\frac{1}{2}\int d^3x\, (\nabla\phi)^2 = \frac{1}{(2\pi)^3 8E^2}\int d^3\mathbf{p}\, p^2\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iEt} + a^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}e^{+2iEt} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right) \tag{5.24}$$

$$\frac{1}{2}\int d^3x\, \pi^2 = \frac{1}{(2\pi)^3 8E^2}\int d^3\mathbf{p}\, E^2\left(-a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iEt} - a^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}e^{+2iEt} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right). \tag{5.25}$$

All in all, we end up with

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E}\,\frac{E}{2}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right). \tag{5.26}$$

Again, this is nothing other than the Hamiltonian of a set of independent simple harmonic oscillators[15] (one for each $\mathbf{p}$) of frequency $\omega = E$, summed over $\mathbf{p}$ with the density of states factor. It is then simple to figure out the spectrum. Define the vacuum (a.k.a. the ground state) to be the state $|0\rangle$ annihilated by all of the *annihilation operators*, $a_{\mathbf{p}}$, *viz.* $a_{\mathbf{p}}|0\rangle = 0\forall\mathbf{p}$. Then, acting on the vacuum with a single *creation operator*, $a^\dagger_{\mathbf{p}}$, one produces a state $|\mathbf{p}\rangle \equiv a^\dagger_{\mathbf{p}}|0\rangle$ of momentum $\mathbf{p}$ and energy E. (To show this explicitly, one should act on the state $a^\dagger_{\mathbf{p}}|0\rangle$ with the Hamiltonian H and with the momentum $\mathbf{P}$, where $\mathbf{P}$ here is not the field momentum π, but rather is the operator corresponding to the generator of spatial translations. We shall do this later on.) In QM we call this the first excited state, but in QFT we interpret it as a state with a single particle of momentum $\mathbf{p}$. A two-particle state would be given by $|\mathbf{p},\mathbf{p}'\rangle \equiv a^\dagger_{\mathbf{p}'}a^\dagger_{\mathbf{p}}|0\rangle$, where the particles have momenta $\mathbf{p}$ and $\mathbf{p}'$, and so on. Note how the commutation relation $[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = 0$ implies immediately that a multiparticle wavefunction is symmetric under the interchange of any two particles: $\dots a^\dagger_{\mathbf{p}}\dots a^\dagger_{\mathbf{p}'}\dots|0\rangle = \dots a^\dagger_{\mathbf{p}'}\dots a^\dagger_{\mathbf{p}}\dots|0\rangle$. Thus, quantum field theory predicts that spinless excitations of the Klein-Gordon field obey Bose-Einstein statistics. Amazing.

The simple harmonic oscillator number operator $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ is now interpreted as counting the number of particles that are present with momentum $\mathbf{p}$ (this is easy to check by acting

[15] Recall that the SHO Hamiltonian may be written as $\omega\left(a^\dagger a + \frac{1}{2}\right) \equiv \frac{\omega}{2}\left(a^\dagger a + aa^\dagger\right)$.

on any state of the above type). Note that the total number of particles is measured by the operator

$$N = \int \frac{d^3p}{(2\pi)^3 2E}\, a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \tag{5.27}$$

which is not a conserved quantity for the real Klein-Gordon field (it does not correspond to a symmetry of the action). So the total number of particles, unlike in QM, is not fixed.

Notice also that the problem of negative energy solutions has gone away. Indeed, the negative frequency modes in the superposition (5.20) now have a different interpretation: they accompany the annihilation operators $a_{\mathbf{p}}$ and reflect the fact that annihilating a particle of energy E causes the total energy stored in the field to *decrease* by E.

In its place, a different problem appears. Let us try to calculate the energy of the vacuum state $|0\rangle$. It is

$$\langle 0|H|0\rangle = \int d^3\mathbf{p}\, \delta^3(0)\frac{E}{2}. \tag{5.28}$$

The first disturbing thing about this expression is that it contains $\delta(0)$. This in fact just corresponds to the volume of space: since $\int d^3\mathbf{x}\; e^{i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3\delta^3(\mathbf{p})$, we may write $V \equiv \int d^3\mathbf{x} = (2\pi)^3\delta^3(0)$. But even the Hamiltonian density is divergent, because it is a sum over all momentum modes of the SHO zero point energy $\frac{E}{2}$. At least if we forget about gravity, we can sidestep this problem by observing that we are only able to measure energy differences in experiment. Thus we can simply re-define the Hamiltonian to be $H - \langle 0|H|0\rangle$. Effectively, this can be implemented by ensuring that we always put operators in *normal order*, by which we mean that annihilation operators always appear to the right of creation operators. This guarantees that a normally-ordered operator will vanish when acting on the vacuum state. A normally-ordered operator is denoted by enclosing it in a pair of colons. The normally-ordered Hamiltonian, for example, is given by

$$: H :\equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E}\, E a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \tag{5.29}$$

This problem of the vacuum energy is only the first of many peccadillos that appear in quantum field theory. In this case, it seems relatively benign. The other peccadillos (which confused the founding fathers for decades) are now well understood. But this first problem of the vacuum energy reappears when we consider coupling quantum field theory to gravity, giving rise to the *cosmological constant problem*. It is arguably among the greatest unsolved problems in the Universe today.

5.3 Multiple scalar fields

Quantization of more than one scalar field is trivial, but it is helpful to point out one or two conceptual issues. Consider n real, scalar fields, ϕ_i. If we allow a maximum of two derivatives and two fields in each term, we claim that the Lagrangian can be written, without loss of generality, as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_i\partial^\mu\phi_i - m_i^2\phi_i^2). \tag{5.30}$$

Why? The most general kinetic term (the one involving the derivatives) could be written as $Z_{ij}\partial_\mu\phi_i\partial^\mu\phi_j$, but the matrix Z_{ij} may be diagonalized by an orthogonal transformation of the fields ϕ_i. An independent rescaling of the fields ϕ_i can then make each of the eigenvalues equal to ± 1. An eigenvalue of -1 would result in an inconsistent theory, since the kinetic energy would be unbounded below. So the kinetic term can always be written in the *canonical* form $\delta_{ij}\partial_\mu\phi_i\partial^\mu\phi_j$. Now, this kinetic term (which must be present in order to have a consistent theory) has a global $O(n)$ symmetry,[16] corresponding to orthogonal rotations of the fields ϕ_i. This then is the largest possible symmetry that a theory based on n real scalar fields can have, since the kinetic term must always be present for a dynamical field. This observation will be important when we come to consider gauge theories, since the name of the game there will be to promote a subgroup of this to a local symmetry.

As for the mass term, this too could be an arbitrary symmetric matrix, in the basis in which the kinetic term is canonical. This too can be diagonalized by an orthogonal transformation, without changing the form of the kinetic term. Hence we arrive at the Lagrangian written above. Note that the mass terms break the $O(n)$ symmetry, unless we force all the m_i to be equal.

A particularly interesting example is $n = 2$, with $m_1 = m_2 \equiv m$. This theory has $SO(2)$ symmetry, which you may know is (locally) equivalent to a $U(1)$ symmetry.[17] One possibility is to simply quantize the two fields, ϕ_1 and ϕ_2 independently, as we did in the last section. Evidently there are two types of 'particle', related somehow by the $SO(2)$ symmetry. More illuminating is to define a complex scalar field, $\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, in terms of which the Lagrangian may be written as

$$\mathcal{L} = (\partial_\mu\phi^*\partial^\mu\phi - m^2|\phi|^2). \tag{5.31}$$

This can be quantized via the mode expansion

$$\phi(x,t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E}\left(a_\mathbf{p} e^{-iEt+i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_\mathbf{p} e^{+iEt-i\mathbf{p}\cdot\mathbf{x}}\right), \tag{5.32}$$

with

$$[a_\mathbf{p}, a^\dagger_{\mathbf{p}'}] = (2\pi)^3 2E\delta^3(\mathbf{p}-\mathbf{p}'), \tag{5.33}$$

$$[b_\mathbf{p}, b^\dagger_{\mathbf{p}'}] = (2\pi)^3 2E\delta^3(\mathbf{p}-\mathbf{p}'), \tag{5.34}$$

with all other commutators vanishing. It is not surprising that there are now two particle creation operators, since there were two real scalar fields to begin with. In the complex field formalism here, we need two mode operators in the Fourier expansion because ϕ is complex. The Hamiltonian is given by

$$: H := \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E}\, E\left(a^\dagger_\mathbf{p} a_\mathbf{p} + b^\dagger_\mathbf{p} b_\mathbf{p}\right). \tag{5.35}$$

[16] $O(n)$ just means the group of $n\times n$ orthogonal matrices. We'll say more about it later on.

[17] Again, if you don't know what $SO(2)$ and $U(1)$ mean yet, don't panic: I'll say more about them later on. For now, $SO(2)$ is the group of 2×2, orthogonal matrices with unit determinant and $U(1)$ is the group of 1×1, unitary matrices, a.k.a complex numbers of the form $e^{i\theta}$.

As expected, since the two types of particle have the same mass, they contribute in the same way to the total energy.

What about the $SO(2)$ invariance? In the complex field formalism, it maps to the simple $U(1)$ rephasing: $\phi \to e^{i\alpha}\phi$. Noether's theorem tells us that there is a conserved charge and in terms of creation and annihilation operators it is given by

$$Q = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b^\dagger_{\mathbf{p}} b_{\mathbf{p}} \right). \tag{5.36}$$

Note, crucially, that it is the number of particles of type a minus the number of particles of type b that is conserved. We call the particles of type b *antiparticles*. They have the same mass as the particles, but the opposite charge. Recall that when we couple such a field to electromagnetism, we do so precisely by gauging the phase invariance $\phi \to e^{i\alpha}\phi$, so the charge Q is to be interpreted as the electric charge.

What we achieved up to here is not too complicated technically but is rather deep conceptually. Perhaps this is a good place to pause for a while and reflect on what we just derived.

This leads us naturally on to study *charge conjugation*. Roughly speaking, this operation is defined as exchanging particles with their antiparticles and is related to complex conjugation; many treatments therefore define it in association with various flips of i to minus i and e to minus e, *etc.*

This, in my view, is deeply confusing, since i and e are supposed to be fixed constants of Nature (indeed, we have known since the old testament that we should only exchange an i for an i ...). Much better is to define charge conjugation as a symmetry in exactly the way that we defined other symmetries above: a transformation acting on *fields* that leaves the action invariant.

We'll begin with the Klein-Gordon field. The Lagrangian is

$$\mathcal{L} = (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - m^2|\phi|^2. \tag{5.37}$$

I hope it is obvious that this is invariant under the transformation $A_\mu \to -A_\mu$ and $\phi \to \phi^*$. More particularly, the transformation corresponds to the symmetry *group* $\mathbb{Z}_2$, because transforming twice takes $A_\mu \to -A_\mu \to A_\mu$ and $\phi \to \phi^* \to \phi$, which is the same as the identity transformation. Because it is a discrete transformation, Noether's theorem does not imply a conserved charge in this case. Note that the transformation $A_\mu \to -A_\mu$ is just what we expect for charge conjugation from Maxwell's equations, which will be unchanged if we also flip the sign of the charge and the current (which in QFT will be generated by field configurations like ϕ and ψ).

Now let's do it for the Dirac field. Here it is not so simple to guess what the symmetry transformation is by looking at the Lagrangian, so we'll find our way along with the help of Simplicio, Salviati, and Sagredo, the three fictional characters of the Galilean trialogue.

The Dirac Lagrangian is

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi, \tag{5.38}$$

with $D_\mu = \partial_\mu + ieA_\mu$. Simplicio knows, from his study of Maxwell's equations, that the transformation of A_μ must be $A_\mu \to -A_\mu$ and he guesses that he can just complex conjugate ψ, as he did for the Klein-Gordon field. This doesn't work well at all. Consider the mass term for example, this transforms as

$$\overline{\psi}\psi \to \psi^T \gamma^0 \psi^* = -\psi^\dagger (\gamma^0)^T \psi = -\psi^\dagger \gamma^0 \psi = -\overline{\psi}\psi. \tag{5.39}$$

This argument is a bit subtle: in the second step we have used the fact that the whole quantity is just a number (not a matrix) and therefore equals its transpose. But as we shall see in the next subsection, this theory can only make sense as a QFT if the field anticommutes with itself. Thus, the transpose of a product of two fields is equal to *minus* the reversed product of the transposed fields. Once we take this into account, we see that charge conjugation cannot just involve complex conjugation of the fields, because the mass term in the Lagrangian would not be invariant. If we wanted the electron to be charged, it would have to be massless, which it is not. Simplicio is stuck.

Now Salviati enters the fray. He realises that complex conjugation is somewhat ambiguously defined for a multi-component spinor, since one could also mix up the different components at the same time. So he says, "Maybe it should be $\psi \to C\gamma^0\psi^*$,[18] for some matrix C. Then we'd find

$$\begin{aligned} \overline{\psi}\psi &\to \overline{\psi}\psi, \\ \overline{\psi}\gamma^\mu\psi' &\to -\overline{\psi'}\gamma^\mu\psi, \end{aligned} \tag{5.40}$$

provided $CC^\dagger = 1$ and $C^\dagger\gamma^\mu C = -(\gamma^\mu)^T$." Note that Salviati carefully wrote the second relation for a bi-linear combination of two different fields ψ and ψ', to stress that they get flipped by C.

Only now does Sagredo realise the true genius of Salviati. Sagredo realises that if we set $\psi' = \psi$ in (5.40), we find $\overline{\psi}A_\mu\gamma^\mu\psi \to \overline{\psi}A_\mu\gamma^\mu\psi$, whereas if we set $\psi' = \partial_\mu\psi$, we find $\overline{\psi}\partial_\mu\gamma^\mu\psi \to -\left(\partial_\mu\overline{\psi}\right)\gamma^\mu\psi = +\overline{\psi}\partial_\mu\gamma^\mu\psi$ (where in the last step we integrated by parts). So all terms in the Lagrangian will be invariant.

Simplicio hasn't really followed any of this, but he does point out that a suitable C is $i\gamma^2\gamma^0$. Thus, we can now forget the trialogue and remember only that charge conjugation can be implemented on Dirac spinors as $\psi \to i\gamma^2\psi^*$.

Let me make one last point, which will be important when we study non-Abelian gauge theories. Imagine that ψ carries an extra index i and that A_μ is really a matrix with indices i and j. Then, by an obvious generalization of Salviati's result, $\overline{\psi_i}\gamma^\mu\psi_j \to -\overline{\psi_j}\gamma^\mu\psi_i$ and charge conjugation will only be a symmetry of the Lagrangian if we also define $A^\mu_{ij} \to -A^\mu_{ji}$. So a matrix-valued gauge field must go to minus its transpose under charge conjugation.

5.4 Spin-half quantization

We now wish to quantize the Dirac Lagrangian[19]

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi. \tag{5.41}$$

[18] The γ^0 is conventional.

[19] We'll worry about the coupling to photons later, so for now we put $D \to \partial$.

To do so, we first derive the Hamiltonian. The field momenta conjugate to the fields ψ and $\overline{\psi}$ are

$$\pi \equiv \frac{\delta \mathcal{L}}{\delta \dot{\psi}} = i\psi^\dagger, \tag{5.42}$$

$$\overline{\pi} \equiv \frac{\delta \mathcal{L}}{\delta \dot{\overline{\psi}}} = 0, \tag{5.43}$$

whence the Hamiltonian is

$$\mathcal{H} = -\overline{\psi} i\gamma \cdot \nabla \psi + m\overline{\psi}\psi. \tag{5.44}$$

We guess from our experience with the Klein-Gordon system that our best chance at solving this system is to do a Fourier transform. For this, we need a complete set of plane wave solutions to the Dirac equation. For the positive-energy solutions, we write these as $\psi = u^s_{\mathbf{p}} e^{-ip\cdot x}$; plugging into the Dirac equation, we find that they satisfy

$$(\not{p} - m)u^s_{\mathbf{p}} = 0. \tag{5.45}$$

There are two solutions (one for each of the two possible spin states), which we label by $s \in \{1, 2\}$. We found explicit expressions for these earlier in the Pauli-Dirac basis, but we do not need them here. Instead we simply note that since the u provide a complete set of states, the combination

$$\sum_s u^s_{\mathbf{p}} \overline{u}^s_{\mathbf{p}} \tag{5.46}$$

must satisfy a completeness relation. Moreover, this must be proportional to $\not{p} + m$, since acting on the left with $\not{p} - m$ then gives something proportional to $\not{p}^2 - m^2 = p^2 - m^2 = 0$. This is as it should be, since $(\not{p} - m)u^s_{\mathbf{p}} = 0$. We fix the normalization so that the proportionality constant is unity (this corresponds to $2E$ particles per unit volume, as for the Klein-Gordon field). Thus

$$\sum_s u^s_{\mathbf{p}} \overline{u}^s_{\mathbf{p}} = \not{p} + m. \tag{5.47}$$

Similarly, for the two negative energy solutions, we write $\psi = v^s_{\mathbf{p}} e^{+ip\cdot x}$; plugging into the Dirac equation, we find that they satisfy

$$(\not{p} + m)v^s_{\mathbf{p}} = 0 \tag{5.48}$$

with completeness relation

$$\sum_s v^s_{\mathbf{p}} \overline{v}^s_{\mathbf{p}} = \not{p} - m. \tag{5.49}$$

Our mode expansion is then

$$\psi = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \left(c^s_{\mathbf{p}} u^s_{\mathbf{p}} e^{-ip\cdot x} + d^{s\dagger}_{\mathbf{p}} v^s_{\mathbf{p}} e^{+ip\cdot x} \right), \tag{5.50}$$

where a sum on s is implicit. As for the complex Klein-Gordon case, since ψ is complex we need two operators c and d.

So far, we have made no mention of commutation relations, with good reason. To see why, let us compute the form of the conserved charge, $Q \equiv \int d^3\mathbf{x}\psi^\dagger\psi$ (corresponding to the re-phasing symmetry $\psi \to e^{i\alpha}\psi$). We find

$$Q = \int \frac{d^3\mathbf{p}}{(2\pi)^3(2E)^2} \left(u_{\mathbf{p}}^{s\dagger} u_{\mathbf{p}}^{s'} c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^{s'} + v_{\mathbf{p}}^{s\dagger} v_{\mathbf{p}}^{s'} d_{\mathbf{p}}^{s} d_{\mathbf{p}}^{s'\dagger} + u_{\mathbf{p}}^{s\dagger} v_{-\mathbf{p}}^{s'} c_{\mathbf{p}}^{s\dagger} d_{-\mathbf{p}}^{s'\dagger} e^{+2iEt} + v_{\mathbf{p}}^{s\dagger} u_{-\mathbf{p}}^{s'} d_{\mathbf{p}}^{s} c_{-\mathbf{p}}^{s'} e^{-2iEt} \right), \tag{5.51}$$

or something similar. We can simplify things using our completeness relations. Consider, for example

$$\sum_s u_{\mathbf{p}}^s \overline{u}_{\mathbf{p}}^s = \not{p} + m. \tag{5.52}$$

Multiplying this matrix equation on the right by γ^0 and then taking the trace, we get

$$\sum_s u_{\mathbf{p}}^{\dagger s} u_{\mathbf{p}}^s = \mathrm{tr}[(\not{p} + m)\gamma^0] = 4E. \tag{5.53}$$

But since this corresponds to a sum over two orthogonal spin states, we must have that

$$u_{\mathbf{p}}^{\dagger s} u_{\mathbf{p}}^{s'} = 2E\,\delta^{ss'}. \tag{5.54}$$

We similarly derive $v_{\mathbf{p}}^{\dagger s} v_{\mathbf{p}}^{s'} = 2E\,\delta^{ss'}$. To get an expression for $u_{\mathbf{p}}^{s\dagger} v_{-\mathbf{p}}^{s'}$, which appears in Q above, requires a little more ingenuity. Consider $\sum_s u_{\mathbf{p}}^s \overline{v}_{\mathbf{p}}^s$. This must vanish when we act on the left with $\not{p} - m$ (since $(\not{p} - m)u_{\mathbf{p}} = 0$), whence it is proportional to $\not{p} + m$. But it also must vanish when we act on the right with $\not{p} + m$, so it is proportional to $\not{p} - m$. Hence it vanishes identically. But the $\overline{v}_{\mathbf{p}}$ are proportional to $v_{-\mathbf{p}}^\dagger$ (one may easily check that they both satisfy the same equation). Hence $u_{\mathbf{p}}^{s\dagger} v_{-\mathbf{p}}^{s'} = 0$. In all, Q simplifies to

$$Q = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \left(c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^{s} + d_{\mathbf{p}}^{s} d_{\mathbf{p}}^{s\dagger} \right). \tag{5.55}$$

Similarly, one may show that

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E}\, E \left(c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^{s} - d_{\mathbf{p}}^{s} d_{\mathbf{p}}^{s\dagger} \right). \tag{5.56}$$

Now, if we impose *commutation* relations on c and d, we may simply permute the d with the $d^\dagger$ to get operators into normal order, but we end up with a disaster: not only will the charge count the numbers of both particles and antiparticles, but also the antiparticles will give a negative contribution to the total energy as measured by the Hamiltonian. Now, you may try as you like to insert factors of i to try to patch things up, but nothing will work. What *does* work is to make the simple but bold step of declaring that for fermions the commutation relations should be replaced by anticommutation relations. Thus,

$$\{c_{\mathbf{p}}^s, c_{\mathbf{p}'}^{s'\dagger}\} = (2\pi)^3 2E\delta^3(\mathbf{p} - \mathbf{p}')\delta^{ss'}, \tag{5.57}$$

$$\{d_{\mathbf{p}}^s, d_{\mathbf{p}'}^{s'\dagger}\} = (2\pi)^3 2E\delta^3(\mathbf{p} - \mathbf{p}')\delta^{ss'} \tag{5.58}$$

with other *anti*-commutators vanishing. Then the charge measures the number of particles minus the number of antiparticles and both particles and antiparticles contribute positively to the energy. Moreover, any n-particle state $\ldots c^\dagger \ldots c^\dagger \ldots |0\rangle$ is manifestly antisymmetric under the interchange of two particles. As Pauli realized, this means that if we try to put two particles into the *same* state, we find $(c^{\dagger s}_{\mathbf{p}})^2|0\rangle = 0$. So the Pauli exclusion principle of QM follows from the fact that in QFT, we can only quantize spin-half fields consistently by using anticommutation relations. Amazing.[20]

[20]Another philosophical discourse: Even if QFTs of both fermions and bosons are mathematically consistent, why did Nature choose to realize them both? One possibility is that consistency of the laws of Nature at a more fundamental level (e.g. including gravity) requires an even larger symmetry, called supersymmetry. If you want to know more, take courses on supersymmetry and string theory.

5.5 Gauge field quantization

To quantize the electromagnetic field presents a thorny problem, which has a variety of more or less elegant workarounds. The basic problem is that the field component A^0 does not appear in the Lagrangian with a time derivative. It is non-dynamical, and as a result, its conjugate momentum vanishes: $\pi^0 \equiv \frac{\delta \mathcal{L}}{\delta \dot{A}^0} = 0$. The Hamiltonian is given by

$$H = \frac{1}{2} \int d^3x \, \left(\mathbf{E}^2 + \, \mathbf{B}^2 - A_0 \nabla \cdot \mathbf{E} \right). \tag{5.59}$$

Here, A_0 appears as a Lagrange multiplier, enforcing Gauss' law, $\nabla \cdot \mathbf{E} = 0$ as a constraint. Thus, the problem we face (and the problem in quantizing gauge theories in general) is the problem of how to quantize a dynamical system with constraints. This is a most interesting problem, first studied by (who else?) Dirac, with a variety of elegant solutions. Here we shall follow what is perhaps the least elegant solution (but most direct) of all, which is to make sure that we first fix the gauge completely.[21] To do so, we set $\partial_\mu A^\mu = 0$ and $A^0 = 0$, removing the non-dynamical field A^0. This is called Coulomb gauge. A plane-wave solution then takes the form $A^i = \epsilon^i e^{-ip\cdot x}$, with $p^2 = 0$ and the condition $\nabla \cdot \mathbf{A} = 0 \implies \epsilon \cdot \mathbf{p} = 0$. Thus ϵ^i has two independent polarizations.

The components of the gauge field A^i can then be quantized like massless Klein-Gordon fields

$$A_i(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \sum_P \left(a^P_{\mathbf{p}} \epsilon^P_i e^{-ip\cdot x} + a^{\dagger P}_{\mathbf{p}} \epsilon^{*P}_i e^{+ip\cdot x} \right) \tag{5.60}$$

where ϵ^P_i are the polarization vectors for the two physical components. These satisfy the completeness relation

$$\sum_P \epsilon^P_i \epsilon^{*P}_j = \delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2}, \tag{5.61}$$

whose tensor structure is fixed by the requirement that $\epsilon \cdot \mathbf{p} = 0$. For example, if we choose the two states to be circularly polarized, for waves travelling in the z direction, we have

$$\epsilon^{L,R}_\mu = \frac{1}{\sqrt{2}}(0, -1, \pm i, 0). \tag{5.62}$$

The required commutation relations are

$$[a^P_{\mathbf{p}}, a^{\dagger P'}_{\mathbf{p}'}] = (2\pi)^3 2E \delta^{PP'} \delta^3(\mathbf{p} - \mathbf{p}') \tag{5.63}$$

and they result in the Hamiltonian

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \sum_P E a^{\dagger P}_{\mathbf{p}} a^P_{\mathbf{p}}, \tag{5.64}$$

after normal ordering, where now $E = \sqrt{\mathbf{p}^2}$.

[21] This approach will not work for the non-Abelian gauge theories that we study later. But there we shall bypass the details of the quantization procedure.

5.6 How to go back again

We have opened the Pandora's box that is quantum field theory. Having come this far, the poor reader might be forgiven for wondering how on Earth he or she might go back again to the mundane world of QM! That is to say, starting from quantum field theory, how can one re-derive quantum mechanics (relativistic or otherwise) as a limiting case?[22]

At first glance, passing from quantum field theory to quantum mechanics would seem to be child's play. Indeed, the Euler-Lagrange equation of motion for either the Klein-Gordon or Dirac field is precisely the respective quantum-mechanical Klein-Gordon or Dirac equation. We can even take the non-relativistic limit in either case to obtain the Schrödinger equation. For the complex Klein-Gordon field, for example,[23] satisfying

$$(\partial_\mu \partial^\mu - m^2)\phi = 0, \tag{5.65}$$

we make the substitution $\phi = e^{-imt}\chi$. This substitution accounts for the fact that, in the low energy limit, the energy E in the argument of the plane-wave exponential is dominated by the rest mass m. The remaining piece, χ should then have a small time dependence, such that $\dot{\chi} \ll m\chi$. Making the substitution in the Klein-Gordon equation, we directly obtain the Schrödinger equation $i\frac{\partial \chi}{\partial t} = -\frac{1}{2m}\nabla^2 \chi$.

Unfortunately, this argument is unsatisfactory for a number of reasons. For one thing, the Euler-Lagrange equation of motion corresponds to the classical limit, $\hbar \to 0$,[24] rather than the limit of quantum mechanics. Moreover, in this framework, the position x is just a label, *not* an operator, as it should be in QM. Finally, the interpretation of $\chi^*\chi$ as the probability density in QM is missing.

How, then, does QM really arise as the limit of QFT? Well, let us first recall that QM is a theory with a fixed number of particles, which forces us to consider (i) the non-relativistic limit and (ii) a theory in which the number of particles can be conserved by a symmetry. Otherwise the limit cannot be consistent. This immediately rules out there being such a limit for the real Klein-Gordon field, for which there is no candidate conserved charge that could correspond to particle number in the low energy limit. For the complex Klein-Gordon field, there is a candidate charge, but in the full theory it conserves the number of particles minus the number of antiparticles, rather than the number of particles (which is what we want in order for QM to be consistent). Nevertheless, we shall now show that it is possible to have a consistent theory of QM in the low-energy limit.

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[22] This is a topic that does not seem to be adequately addressed in the majority of field theory textbooks and so I beg the reader's indulgence here in allowing me to treat it in some detail. Those who wish to remain in blissful ignorance may skip it.

[23] The case of the Dirac field is set as an exercise.

[24] One way to see that $\hbar \to 0$ is the limit of classical mechanics is to note that all commutation relations vanish in this limit, meaning that operators can be replaced by numbers. A much more elegant way is to note that in the path integral formulation of QM or QFT, amplitudes are obtained by integrating over all paths in spacetime weighted by a factor of $e^{iS/\hbar}$, where S is the action. In the limit $\hbar \to 0$, the path integral is dominated by paths for which $\delta S = 0$, *viz.* those that satisfy the classical equations of motion. The units $\hbar = 1$ are obviously not ideal for the present discussion!

To do so, we make the same substitution $\phi = e^{-imt}\chi$ as before, but in the Lagrangian. We get

$$\mathcal{L}' = i\chi^\dagger \dot{\chi} - \frac{1}{2m}\nabla\chi^\dagger\nabla\chi \tag{5.66}$$

where we have integrated by parts, taken the non-relativistic limit $\dot{\chi} \ll m\chi$, and divided by $2m$. The canonical momentum conjugate to the field χ is then $\pi \equiv \frac{\delta\mathcal{L}}{\delta\dot{\chi}} = i\chi^\dagger$ and the Hamiltonian is

$$\mathcal{H}' = +\frac{1}{2m}\nabla\chi^\dagger\nabla\chi. \tag{5.67}$$

The canonical commutation relations are then

$$[\chi, \chi^\dagger] = \delta(\mathbf{x} - \mathbf{y}) \tag{5.68}$$

(with all others vanishing). Now, the important point is that we can consistently realize these commutation relations with a single particle annihilation operator defined by

$$\chi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} a_{\mathbf{p}} e^{ip\cdot x}, \tag{5.69}$$

with

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3\delta(\mathbf{p} - \mathbf{q}) \tag{5.70}$$

This can be traced back to the fact that the Lagrangian is first-order in the time derivative. As a result, it is possible to quantize, in the low energy limit, in a way in which there are only particles in the theory, with no antiparticles. Intuitively, the reason this is possible is because in the non-relativistic limit, starting from a configuration of particles only, there is insufficient energy to produce particle-antiparticle pairs from the vacuum.

It is important to note that this cannot be the only possible way to quantize the theory at low energy, since it is also perfectly possible to have configurations consisting of antiparticles only, or indeed of both particles and antiparticles.

The fact that it is possible to quantize the theory in terms of particles only is not enough to guarantee the consistency of QM. (Indeed, we already know that this can be done for the real Klein-Gordon field and we shall soon show that this does not have a consistent QM limit.) We must also show that the number of particles is a conserved quantity. This is easily done: the low-energy Lagrangian has a symmetry $\chi \to e^{i\alpha}\chi$ whose conserved charge is $Q = \int \frac{d^3\mathbf{p}}{(2\pi)^3} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$. This charge simply counts the number of particles in a state (as one may easily show for, *e.g.* the one-particle states $a^\dagger_{\mathbf{p}}|0\rangle$.

So, we have shown that there is a consistent limit of the theory in which there is a fixed number of particles. It remains to show that this limit really corresponds to QM, with its commutation relations, the Schrödinger equation, and so on.

To do so, one may first easily show that the Hamiltonian and the conserved momentum [25] arising from the Noether current corresponding to the symmetry of the Lagrangian under

[25] Note, this is not the momentum π conjugate to the field χ.

time and space translations are given by[26]

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{p^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}, \tag{5.71}$$

$$\mathbf{P} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \tag{5.72}$$

Note that the momentum $\mathbf{P}$ is indeed an operator and it is this momentum that should obey the usual QM commutation relation $[X_i, P_j] = i\delta_{ij}$. To show this explicitly, we must first identify the position operator $\mathbf{X}$. We claim that it is

$$\mathbf{X} \equiv \int d^3\mathbf{x}\mathbf{x}\chi^\dagger(\mathbf{x})\chi(\mathbf{x}). \tag{5.73}$$

To verify this, note that $\mathbf{X}$ acting on a one-particle state at $\mathbf{x}$, *viz.* $|\mathbf{x}\rangle \equiv \chi^\dagger(\mathbf{x})|0\rangle$, returns eigenvalue $\mathbf{x}$: $\mathbf{X}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$. An arbitrary state, with wavefunction $\psi(\mathbf{x})$, may then be written as

$$|\psi\rangle \equiv \int d^3\mathbf{x}\psi(\mathbf{x})|\mathbf{x}\rangle, \tag{5.74}$$

and one may then show (exercise) that

$$\mathbf{X}|\psi\rangle = \int d^3\mathbf{x}\mathbf{x}\psi(\mathbf{x})|\mathbf{x}\rangle, \tag{5.75}$$

$$\mathbf{P}|\psi\rangle = \int d^3\mathbf{x}(-i\nabla\psi(\mathbf{x}))|\mathbf{x}\rangle. \tag{5.76}$$

Thus we have the usual correspondence $P \to -i\frac{\partial}{\partial x}$ of QM and the usual commutation relation $[X_i, P_j] = i\delta_{ij}$. Similarly, one may show that

$$H|\psi\rangle = \int d^3\mathbf{x} \left(-\frac{1}{2m}\nabla^2\psi(\mathbf{x})\right)|\mathbf{x}\rangle, \tag{5.77}$$

so that $\psi(\mathbf{x})$ satisfies the usual time-dependent Schrödinger equation $i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi(\mathbf{x})$. Finally, the probability for the particle to be found at $\mathbf{X}$ is given by $|\langle\mathbf{x}|\psi\rangle|^2$, which one may show (exercise) is given by $|\psi(\mathbf{x})|^2$.

To check that you understand things, you should now worry how we can obtain the usual QM commutation relations $[X, P] = i$ for the non-relativistic limit of the Dirac theory, in which all operators obey *anti*commutation relations. (Hint: X and P both involve *two* creation or annihilation operators.)

[26]These expressions are not unexpected: they sum the kinetic energies and momenta for each state labelled by p, multiplied by the occupation number of each state.

5.7 Interactions

If you have understood this far, you may rightly feel pleased with yourself. We have successfully quantized relativistic field theories containing particles with spin (or helicity) zero, one-half, and one. This covers everything we have seen thus far in Nature, with the exception of the spin-two graviton.

You may, however, have noticed the elephant in the room: thus far we have only dealt with Lagrangians that are quadratic in the fields. These correspond to linear equations of motion, which everybody knows are far easier to solve than non-linear equations of motion, in that solutions may be superposed. We call the quantum versions of such theories *free* or *non-interacting* theories. They are decidedly dull, in that particles that are present remain present for ever. Interacting theories, which contain terms with more than two powers of fields in the Lagrangian, are far more interesting: they provide the catalyst by which particles can appear or disappear, being transformed into other sources of energy and momentum. So rich, in fact, are such theories, that no one has been able to solve them, except in a few very special cases (if you manage it, let me know – we can write a paper together). We are forced to resort to perturbation theory. Let us now develop the necessary formalism to do this. Unfortunately, this is one of the things that is perhaps more easily done in the path integral approach to field theory. Since our ultimate goal is to get to the Feynman rules, which provide a straightforward mnemonic for doing real calculations, I will merely sketch how things go in canonical quantization.

Thus far, we have been working in the Heisenberg picture of QM, in which operators (like $\phi(x,t)$) depend on time, but states do not. You have probably spent much of your previous career working in the Schrödinger picture, in which the opposite happens. It is simple to go between the two. In the Schrödinger picture, everyone knows that the time-dependence of the states is given by $i\frac{\partial}{\partial t}|\psi\rangle_S = H|\psi\rangle_S$, where the subscripts are to remind us that this is the Schrödinger picture. In the Heisenberg picture, we define

$$O_H(t) = e^{iHt} O_S e^{-iHt} \tag{5.78}$$

$$|\psi\rangle_H = e^{iHt}|\psi\rangle_S. \tag{5.79}$$

The pictures are equivalent, because we always sandwich operators between states to compute amplitudes, which are the things we use to make physical predictions.

For doing perturbation theory, a third picture, the *interaction picture*, is useful. In this picture, we split the Hamiltonian into a free part H_0 (that we can solve) and a perturbation H_1 and we instead define

$$O_I(t) = e^{iH_0 t} O_S e^{-iH_0 t} \tag{5.80}$$

$$|\psi\rangle_I = e^{iH_0 t}|\psi\rangle_S. \tag{5.81}$$

As a result, the operators evolve according to H_0 (meaning that operator expressions like eq. 5.20, which was written in the Heisenberg picture of the free theory, are equally valid in the interaction picture), while the states evolve according to (exercise) $H_I \equiv e^{iH_0 t}(H_1)_S e^{-iH_0 t}$:

$$i\frac{\partial}{\partial t}|\psi\rangle_I = H_I|\psi\rangle_I. \tag{5.82}$$

The above is easy to see (note that H_0 commutes with e^{iH_0t} but not H_1 (or H_I) and H_I is explicitly time dependent):

$$i\frac{\partial}{\partial t}|\psi\rangle_I = i\frac{\partial}{\partial t}(e^{iH_0t}|\psi\rangle_S) = -H_0|\psi\rangle_I + e^{iH_0t}(H_0+H_1)|\psi\rangle_S = \left(-H_0 + e^{iH_0t}(H_0+H_1)e^{-iH_0t}\right)|\psi\rangle_I\,.$$

Given an initial state $|\psi(t_0)\rangle_I$, Dyson showed that a formal solution to this last equation is given by $|\psi(t)\rangle_I = U(t,t_0)|\psi(t_0)\rangle_I$, where

$$U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right). \tag{5.83}$$

Here, the time-ordering operator acting on a product of fields is defined by

$$TO_1(t_1)O_2(t_2) = \begin{cases} O_1(t_1)O_2(t_2), \text{ if } t_1 > t_2 \\ O_2(t_2)O_1(t_1), \text{ if } t_2 > t_1 \end{cases} \tag{5.84}$$

Acting on an exponential, the time ordering is obtained by Taylor expanding the exponential and then acting on the individual terms in the expansion (which are simple products of fields). You may wonder why time ordering is needed. The point is that H_I, being time dependent, does not commute with itself at different times. So $H_I(t)e^{-i\int^t dt' H_I(t')}$ is not the same thing as $e^{-i\int^t dt' H_I(t')}H_I(t)$. But with time ordering, $\frac{\partial}{\partial t}$ acting on $U(t,t_0)$ unambiguously gives $-iH_I(t)U(t,t_0)$, since t is a later time than any time appearing in $U(t,t_0)$. Hence (5.83) solves (5.82). Intuitively, the role of time ordering is to enforce *causality* in the theory: colloquially, it prevents particles from being destroyed before they are created.

Formally, we have now solved quantum field theory. Unfortunately, nobody knows how to compute $U(t,t_0)$ for non-trivial H_I. The best we can do is to attempt a perturbative expansion. Provided H_I is small enough,[27] we may expand

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')dt' + \frac{(-i)^2}{2}T\left(\int_{t_0}^{t}dt'\int_{t_0}^{t}dt''\ H_I(t')H_I(t'')\right) + \dots \tag{5.85}$$

In the H_I^2 term, we integrate over a square region in (t',t'') we may simplify the time-ordering operation by splitting the integration region into two triangles: one with $t'' > t'$ and one with $t'' < t'$. Thus,

$$T\left(\int_{t_0}^{t}dt'\int_{t_0}^{t}dt''\ H_I(t')H_I(t'')\right) = \\ \int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\ H_I(t')H_I(t'') + \int_{t_0}^{t}dt''\int_{t_0}^{t''}dt'\ H_I(t'')H_I(t'). \tag{5.86}$$

Perversely, we chose to do the first integral with respect to t'' and then t', but we did the second integral the other way round. Actually this is not so perverse, since it shows that

[27] I make no attempt to define 'small enough'; it turns out that the perturbative expansion of QFT almost *never* converges, being at best an asymptotic expansion. This is in some sense a good thing, since there are devils to be found in the details: many of the rich phenomena that have been discovered in QFT in recent decades are non-perturbative.

the two contributions are identical, once we interchange the dummy variables $t' \leftrightarrow t''$. Thus we have

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')dt' - \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')H_I(t'') + \ldots \tag{5.87}$$

In particle physics experiments, we typically prepare some particles (a pair of protons at the LHC, for example), arrange for them to collide, and try to detect the products. Now, the relevant time and distance scales for particle physics tend to be so small that, to a very good approximation, we may consider the initial and final states to be in the *infinite* far past and future, respectively, and we also may safely integrate over all of space in computing the Hamiltonian from the Hamiltonian density. We thus claim that the quantities of interest for particle physics are the amplitudes

$$\langle f|U(+\infty,-\infty)|i\rangle. \tag{5.88}$$

We now have an idea how to compute U as a perturbation series in H_I (and shall do so explicitly for some examples presently). But how do we compute $|i\rangle$ and $|f\rangle$? They are eigenstates of the full interacting theory (albeit in the interaction picture). One might hope that since the particles are well separated in space, they might be considered to be the n-particle eigenstates of H_0, *e.g.* $a^\dagger|0\rangle$, that we computed before. Unfortunately, this is not quite correct, because even though the particles are well-separated from each other, they are not well-separated from the vacuum, which, in QFT, is a complicated place, with particles being created and annihilated on quantum timescales.[28] Fortunately, the theorists have declared that it is safe to consider $|i\rangle$ and $|f\rangle$ as free eigenstates, provided we make one or two modifications to the Feynman rules later on. We will take their word for it for now.

Once we accept this, doing calculations in QFT is easy, if tedious. All we do is to take initial and final states (of the form $a^\dagger|0\rangle$), sandwich them between products of time-ordered Hamiltonians (expressed in terms of creation and annihilation operators as $a^\dagger a$), and (anti-)commute the as and $a^\dagger$s until we are left with a c-number. This is the desired amplitude, which we should square to find the decay rate, cross-section or whatever (taking into account phase space, of course). In fact, it is even easier than that. Feynman showed that the whole tedious business can be reproduced by the mnemonic of drawing *Feynman diagrams*, from which the amplitudes are reconstructed via the *Feynman rules*. Our strategy in later lectures will be to take the Feynman rules as a starting point and compute from there, but here we shall compute two processes the tedious way, so that you can fully appreciate the difference.

5.8 e^+e^- pair production

Our first process is conversion of a photon γ into an electron-positron pair. This cannot happen in free space, because of energy-momentum conservation (exercise), but it can

[28] In fact, the vacuum is so complicated that we can compute everything in QFT from it: as we have seen, every amplitude is just given by $\langle 0|\ldots|0\rangle$, where $\ldots$ represent some operator.

occur in a material (which recoils). We have already seen that the electromagnetic interaction is given by $\mathcal{H}_I = +eA_\mu \overline{\psi}\gamma^\mu \psi$ and that the scattering amplitude, at leading order in perturbation theory, is given by $-i\langle f|\int d^4x \mathcal{H}_I|i\rangle$. Let's examine the different pieces of this in turn. Firstly, the initial state is to be a photon, of momentum k, say, and polarization P. So $|i\rangle = a_{\mathbf{k}}^{\dagger P}|0\rangle$. Similarly, we want the final state to consist of an electron of momentum p_1 and spin s_1 and a positron of momentum p_2 and spin s_2.[29] So $|f\rangle = c_{\mathbf{p}_1}^{\dagger s_1} d_{\mathbf{p}_2}^{\dagger s_2}|0\rangle \implies \langle f| = \langle 0|d_{\mathbf{p}_2}^{s_2} c_{\mathbf{p}_1}^{s_1}$. The bit in the middle is $e\int d^4x A_\mu \overline{\psi}\gamma^\mu\psi$. When we plug in the Fourier mode expansions, we have that $A_\mu \sim a + a^\dagger$, but only the a piece will give a non-vanishing contribution to the matrix element (the $a^\dagger$ piece can be commuted to the left, where it will annihilate $\langle 0|$. Similarly, only the $d^\dagger$ and $c^\dagger$ pieces of ψ and $\overline{\psi}$, respectively, contribute. Moreover, all of these contributions can be reduced to c-numbers by commutation. For example, we can commute the a piece through the $a^\dagger$ in $|i\rangle$ to get a delta-function (as in (5.63)) together with a term that annihilates $|0\rangle$. Doing this, our amplitude reduces to[30][31]

$$
\begin{aligned}
-i\langle f|\int d^4x\, \mathcal{H}_I|i\rangle = -ie\int d^4x\, \epsilon_\mu^P \overline{u}^{s_1}\gamma^\mu v^{s_2} e^{-i(k-p_1-p_2)\cdot x} = \\
-ie(2\pi)^4\delta^4(k-p_1-p_2)\epsilon_\mu^P \overline{u}^{s_1}\gamma^\mu v^{s_2}. \quad (5.89)
\end{aligned}
$$

It is pleasing to see that conservation of 4-momentum is manifest. This happens because we took the Fourier transform. To check conservation of angular momentum, you'd need to work out the spin and polarization states explicitly.

For what comes later, it is useful to extract the overall $(2\pi)^4\delta(p_f - p_i)$ (which always appears, *cf.* our discussion of Fermi's Golden rule), defining the *matrix element* by $\langle f|U(+\infty,-\infty)|i\rangle \equiv i(2\pi)^4\delta(p_f - p_i)\mathcal{M}$. Hence, we have

$$
i\mathcal{M} = -ie\epsilon_\mu^P \overline{u}^{s_1}\gamma^\mu v^{s_2}. \quad (5.90)
$$

We can think of this as arising from the following factors: a factor ϵ_μ^P representing an incoming photon; $\overline{u}^{s_1}$ and v^{s_2} representing an outgoing electron and positron, respectively; and $-ie\gamma^\mu$ representing the interaction vertex. When we get to the Feynman rules, our process will be represented by the diagram in Fig. 1 with the external lines telling us to include the various ingoing and outgoing factors and with the dot representing the vertex factor. You should now convince yourself (exercise) that the matrix element for $e^-(s_1) + \gamma(P) \to e^-(s_2)$ is $i\mathcal{M} = -ie\epsilon_\mu^P \overline{u}^{s_2}\gamma^\mu u^{s_1}$, so that the vertex factor for an incoming electron is u^{s_1}.

5.9 Compton scattering

For our second process, we wish to compute the amplitude for a photon to scatter off an electron. Again, this cannot happen for free particles, but it can happen for an electron

[29] If this doesn't make sense to you, go back and read §5.4.

[30] Previously, we worked in Coulomb gauge, $A_0 = 0$ and wrote the polarization vector of a photon as a 3-vector ϵ_i^P; more generally, we may write it as a 4-vector, ϵ_μ^P.

[31] This sort of argument is straightforward, but is liable to make one's eyes glaze over. Suffice to say that you will only really get to grips with it if you sit down and work out all the intermediate steps for yourself (exercise).

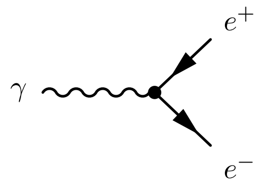


Figure 1. Feynman diagram representing the process $\gamma \to e^+e^-$.

that is bound in an atom. It is called *Compton scattering* and you will doubtless have heard it touted in your QM courses as evidence for the corpuscular nature of light. Touted as it was, you probably did not go beyond computing the kinematics. That is because to compute the scattering amplitude requires at least relativistic QM, and better still QFT. Let's do it at last.

Compton scattering is more complicated than pair production, because it cannot happen in leading order perturbation theory (exercise). So we need the second order perturbation

$$\langle f|T\int_{t,t'} H_I(t)H_I(t')|i\rangle \tag{5.91}$$

and the issue of time-ordering rears its ugly head. You have by now realised that the game in computing QFT matrix elements is to move all the annihilation operators to the right and all the creation operators to the left, where they vanish when acting on $|0\rangle$. But this is precisely what we previously called normal ordering. So it would be very useful to have a theorem that tells us how to convert from time-ordering to normal ordering. That theorem is called *Wick's theorem.* It decrees that

$$T\phi(x_1)\phi(x_2)\cdots =: \phi(x_1)\phi(x_2)\cdots : +\text{contractions}, \tag{5.92}$$

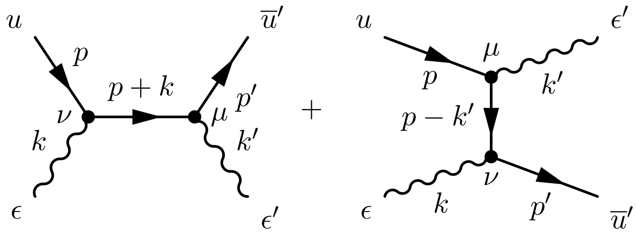
where 'contractions' instructs us to take all possible pairs of operators from the list and replace them with something called the *propagator.* We shall not prove Wick's theorem in general, nor shall we derive the propagator for all fields. Rather, we shall content ourselves with showing how things work for a product of two Klein-Gordon fields.

For these, there is only one possible contraction, so we write

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +\Delta_F(x-y), \tag{5.93}$$

where $\Delta_F(x-y)$ is known as the Feynman propagator and our goal is to determine it, or at least to find an expression for it in momentum space. Let us first consider the case $x^0 > y^0$, such that $T\phi(x)\phi(y) = \phi(x)\phi(y)$. Then, when we write out the mode expansion for $\phi(x)\phi(y)$, the piece which is not in normal order is the piece containing $a_{\mathbf{p}}e^{-ip\cdot x}a^\dagger_{\mathbf{p}'}e^{+ip'\cdot y}$. When we normally order it, we generate the additional contribution $[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}]e^{-ip\cdot x}e^{+ip'\cdot y} = (2\pi)^3 2E\delta^3(\mathbf{p}-\mathbf{p}')e^{-ip\cdot(x-y)}$. If instead $x^0 < y^0$, we shall find a piece $(2\pi)^3 2E\delta^3(\mathbf{p}-\mathbf{p}')e^{-ip\cdot(y-x)}$. Thus, we may write

$$\Delta_F(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E}\,\left(\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{-ip\cdot(y-x)}\right). \tag{5.94}$$



$$(5.97)$$

Figure 2. Feynman diagram representing Compton scattering, $e^- + \gamma \to e^- + \gamma$.

This involves a Lorentz-invariant measure and indeed it may be written as (exercise)

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}, \tag{5.95}$$

where $\epsilon > 0$ is a small quantity telling us how to avoid the poles at $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$ in the complex p^0 plane.[32]

We can now see how to compute the matrix element for Compton scattering. We must first apply Wick's theorem to the expression

$$(-ie)^2 \langle f|T \int_{x,x'} A_\mu(x)\overline{\psi}(x)\gamma^\mu\psi(x)A_\nu(x')\overline{\psi}(x')\gamma^\nu\psi(x')|i\rangle. \tag{5.96}$$

Given that the initial and final states both contain an electron and a photon, the only contractions in (5.92) that give a non-vanishing contribution involve one ψ and one $\overline{\psi}$. There are two such contractions and these are represented by the Feynman diagrams in Fig. 2, where the propagator is represented by the line joining the two blobs, which are called vertices. This propagator is the *Dirac propagator* given by

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p} - m + i\epsilon} e^{-ip\cdot(x-y)}, \tag{5.98}$$

Its form is easy to understand: it too is a Green function, but this time for the Dirac equation. The uncontracted fields act on the states $|i\rangle$ and $|f\rangle$; for them we derive the same in/outgoing electron/photon factors that we derived above. In all the amplitude is given by (ignoring the $i\epsilon$s)

$$i\mathcal{M} = (-ie)^2 \epsilon^{*\prime}_\mu \overline{u}' \left(\gamma^\mu \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2}\gamma^\nu + \gamma^\nu \frac{i(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2}\gamma^\mu \right) u\epsilon_\nu. \tag{5.99}$$

Since there are two contributions to the amplitude, the cross-section (which goes as $|\mathcal{M}|^2$) contains interference terms.

In the examples, we'll turn this into a cross-section.

[32] These poles are present because $\Delta_F(x-y)$ is a Green function of the Klein-Gordon equation and is defined only up to a solution of the homogeneous equation until boundary conditions are specified. In this case the $i\epsilon$ prescription amounts to specifying the boundary conditions to be Lorentz-invariant and causal (meaning that $\Delta_F(x-y)$ should vanish outside the light cone). Note that the latter condition is forced upon us by the time ordering. So insisting on causality in time (together with Lorentz invariance) guarantees causality in spacetime.

6 Gauge field theories

Our construction of the edifice of QFT thus far has been painful to say the least. We went down many blind alleys, broke Lorentz invariance (by giving t a special rôle in the equations) and recovered it again, violated gauge symmetry, swept infinities under the rug, and more. All this without ever calculating a cross-section. But I hope at least that you learnt something. We started with quantum mechanics and we ended up with quantum field theory, more or less. With the foundations in place, we can now relax a bit. For the rest of the course, we shall not worry too much about the unpleasantries of quantization. We shall start from the Lagrangian and from that write down the Feynman rules. As we have hinted, even the Lagrangian itself is fixed to a large extent, once we have specified the field content and the symmetries that we desire the theory to have.

6.1 Quantum electrodynamics

Consider, for example, quantum electrodynamics (QED). This is a theory containing a spin-half Dirac field ψ (the electron) and a vector (helicity-one) field A_μ (the photon). We insist that the theory possess the local (gauge) symmetry

$$\psi \to e^{ie\alpha(x)}\psi,\ A^\mu \to A^\mu - \partial^\mu \alpha. \tag{6.1}$$

This together with Lorentz invariance, fixes the form of the Lagrangian to be

$$\mathcal{L}_{\text{QED}} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{6.2}$$

provided we allow terms which are at most cubic in the fields (the reasons for this will be discussed in the next Section). Recall that the *covariant* derivative is given by $D_\mu = \partial_\mu + ieA_\mu$ and that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The theory has just two free parameters, the mass m of the electron and the electron charge e (*n.b.* $e < 0$). Note how a mass term for the photon, $\sim A_\mu A^\mu$, which is allowed by Lorentz invariance, is forbidden by gauge invariance.

We now claim that a valid set of Feynman rules (in momentum space) for computing the matrix element, $i\mathcal{M}$, in QED are as follows.

1. The basic building blocks of Feynman diagrams are: a photon propagator, an electron propagator, and an electron-photon-electron interaction vertex, as shown in Fig 3. (The arrow on the electron propagator denotes the direction of particle number flow. It is conserved at a vertex, meaning arrows never clash.)

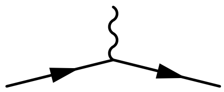
2. Draw all possible diagrams containing these elements with the required initial and final states, with the number of vertices fixed by the desired order of perturbation theory.

3. Assign momenta to the various internal lines so that the 4-momentum is conserved at each vertex.

4. For each internal photon line with 4-momentum q, associate the propagator $\frac{-ig_{\mu\nu}}{q^2+i\epsilon}$. For an external in(out)-going photon of polarization P, assign the factor $\epsilon^P_\mu (\epsilon^{*P}_\mu)$.

$$= \frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \tag{6.3}$$

$$= \frac{i}{\not{q} - m + i\epsilon} \tag{6.4}$$



$$= -ie\gamma^{\mu} \tag{6.5}$$

Figure 3. Feynman rules for QED.

5. For each (in)outgoing electron, assign a factor $(u^s)\overline{u}^s$. For each (in)outgoing positron, assign a factor $(\overline{v}^s)v^s$. For each internal propagator with momentum q in the direction of the arrow, write $\frac{i}{\not{q}-m+i\epsilon}$. For each vertex, write $-ie\gamma^{\mu}$.

6. Any loop in a diagram will have an unfixed 4-momentum, k. Integrate over it with measure $\int \frac{d^4k}{(2\pi)^4}$.

7. Fret about the overall sign.

The last rule perhaps requires some further clarification. Since fermions anticommute, it happens that different diagrams contributing to the same amplitude have a relative minus sign (the overall sign is not important, because we always take the modulus squared of the amplitude). The sign can be easily figured out by going back to canonical quantization and studying the positions of the fermion operators. In particular, it turns out that any closed loop of fermions will always contribute a minus sign.

These rules should make sense to you after what we have done so far and we shall not make an exhaustive derivation of them. In particular, we have written the propagator for the photon as $\frac{-ig_{\mu\nu}}{q^2+i\epsilon}$, when in fact the propagator is undefined until we deal with the gauge fixing. For a proper treatment, see the textbooks.

As an exercise, you should try to compute the amplitude for electron-electron scattering, at order e^2. Hint: there are two diagrams and you need to worry about the relative sign. You can figure it out by going back to canonical quantization and moving the creation and annihilation operators around.

Figure 4. Probably you are far too young to find this amusing. Never mind.

6.2 Janet and John do group theory

We have been going on and on about the central rôle played by symmetry in QFT. You surely know by now that the correct mathematical language in which to study symmetry is called group theory,[33] and so it is proper that we discuss how group theory enters in QFT.

The reason I have held off mentioning group theory until now is that, unfortunately, the group theory that you learnt in Part IB is not the sort of group theory that will pass muster here. The key difference is that whilst you learnt all about discrete groups, of finite order, we shall only be interested in continuous groups, of infinite order. The ones we are interested in are called *Lie groups*.[34]

Let's start slowly, by seeing how group theory appears in QED. The symmetry is $\psi \to e^{ie\alpha(x)}\psi$, or in the global case, $\psi \to e^{ie\alpha}\psi$. This is a continuous symmetry, because every value of $\alpha \in [0, 2\pi]$ corresponds to a different symmetry transformation. In contrast, if we allowed only, say, $\alpha \in \{0, \pi\}$, we would have the discrete symmetry $\mathbb{Z}_2$.

There is, by the way, a good reason why we are only interested in continuous symmetries for gauge theory. The reason is that to promote a global symmetry to a gauge symmetry, $\alpha \to \alpha(x)$, the derivative $\partial\alpha(x)$ needs to be well-defined, since it appears in the rule for the transformation of the gauge field.

Getting back to QED, we note that $U \equiv e^{ie\alpha}$ can be thought of as 1 x 1 matrix. Moreover, it is a unitary matrix, in that $U^\dagger U = e^{-ie\alpha}e^{ie\alpha} = 1$. We are thus entitled to say, somewhat pompously, that QED is a $U(1)$ *gauge theory*.

Back in the good old days,[35] the only particles knocking around were electrons, positrons

[33] Funnily enough, most of the group theory used by physicists is actually called representation theory by mathematicians.

[34] As always our level of rigour will be embarassingly low. For a more thorough treatment, you could start by reading [8].

[35] This was a time when the Cavendish could be said to have had something of a monopoly on particle physics, having discovered both the neutron and the electron and Dirac having predicted the positron. It is probably stretching it a bit far to claim that Newton's corpuscular theory of light pre-empted the photon,

and photons (well, and nuclei), and QED described all these quite nicely. But then someone had the misfortune to discover (in cosmic rays) a new particle called the muon. It is rather heavier that the electron (about 200 times), but it was straightforwardly incorporated into QED. Indeed, consider two fields ψ_1 and ψ_2, transforming as

$$\psi_1 \to e^{ie_1\alpha(x)}\psi_1,\ \psi_2 \to e^{ie_2\alpha(x)}\psi_2. \tag{6.6}$$

Then we can write down the locally $U(1)$ invariant Lagrangian

$$\mathcal{L}_{\text{QED}} = \overline{\psi}_1(i\partial\!\!\!/ - e_1 A\!\!\!/ - m_1)\psi_1 + \overline{\psi}_2(i\partial\!\!\!/ - e_2 A\!\!\!/ - m_2)\psi_2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{6.7}$$

which describes two particles, each of arbitrary mass and charge, coupled to the photon. In a sense, this Lagrangian asks more questions than it answers, since it allows both particles to have arbitrary mass and charge, whereas experiment showed that the charge of the muon is *exactly* the same as that of the electron. In the intervening decades, we have managed to discover many new particles and *all* of them have charges with are integer multiples of $\frac{e}{3}$. Neither QED nor indeed the Standard Model explains this basic feature of Nature, but we shall see later on how it might be explained in the context of a *grand unified theory*.[36]

This way of thinking about QED as a theory based on the group $U(1)$ begs the question of whether it might be possible to build a gauge theory based on a larger symmetry group, for example the $N \times N$ unitary matrices, $U(N)$. This question was answered in the affirmative by Yang and Mills in the '50s, who showed that the resulting theory is far richer than QED, but it took a long time for us to realise that Nature actually chooses to do things this way. By now, the pendulum has come full circle, in that our current 'theory of everything' (the Standard Model of particle physics) is nothing but a gauge theory.[37]

The basic reason why gauge theories can be much richer (read: harder to answer exam questions on) than QED is that QED is an *Abelian* theory.[38] That is, two successive $U(1)$ transformations commute (it is, after all, just the product of two complex numbers). But two $N \times N$ matrices do not commute, in general, and so we have the possibility of a *non-Abelian* theory. Let's consider unitary matrices in more detail.[39] A generic unitary matrix U can be re-written as e^{iH}, where H is an Hermitian matrix, $H^\dagger = H$, and the exponential is defined by the power series. Since this is a continuous group, and since every group contains the identity element $1 = e^0$, we may consider elements that are close to the identity, writing them in terms of a basis for Hermitian $N \times N$ matrices, $\{T^a\}$ and some real parameters ϵ^a. For elements close to the identity, the ϵ^a are small, and we may expand $e^{i\epsilon^a T^a} = 1 + i\epsilon^a T^a + \dots$. Now consider two elements (parameterised by ϵ^a and η^a) and

however. It certainly pre-dated the Cavendish, in any case.

[36]Even if we could explain the muon charge in this way, nobody yet has a good explanation for why the muon, a heavy cousin of the electron, exists at all. Do you?

[37]The moral of this story is that if you have a theory that is too elegant not to be true, but doesn't seem to be realised in Nature, you just need to be patient.

[38]The 'A' in Abel is pronounced as in 'gargle'.

[39]It will turn out that all of the groups that we consider can be written in terms of unitary matrices, so there is no loss of generality.

compute[40]

$$e^{i\epsilon^a T^a} e^{i\eta^b T^b} e^{-i\epsilon^a T^a} e^{-i\eta^b T^b} = 1 - \epsilon^a \eta^b [T^a, T^b] + O(\epsilon^2, \eta^2, \eta\epsilon). \tag{6.8}$$

This is a product of group elements and so must itself be a group element (by the axiom of closure). Since $\{T^a\}$ form a basis, it must be possible to write

$$[T^a, T^b] = i f^{abc} T^c, \tag{6.9}$$

for some real constants f^{abc}, which are manifestly antisymmetric in the first two indices and in fact may be taken to be antisymmetric in all three. This type of structure is called a *Lie algebra*. The arguments we just made apply equally for a subgroup of the unitary matrices, for which the T^a form a basis for the relevant subalgebra. We call the number of basis elements the *dimension* of the Lie algebra. For $N \times N$ unitary matrices, for example, a basis for the $N \times N$ Hermitian matrices contains N^2 elements.

The algebra is a much simpler object to work with than the group itself. (Locally, in the vicinity of the identity element, the two are equivalent, but we shall see that groups with the same algebra can have a distinct global structure. Everything we will say applies at the level of the algebra.) Remarkably, just from the form of the relation (6.9), it is possible to classify *all* of the possible Lie algebras. They are built from building blocks consisting of three infinite series, corresponding to: $N \times N$ unitary matrices (which can be thought of as matrices such that $U^\dagger \delta U = \delta$) with unit determinant, called $SU(N)$; $N \times N$ orthogonal matrices (which can be thought of as matrices such that $U^T \delta U = \delta$) with unit determinant (called $SO(N)$); and $2N \times 2N$ matrices satisfying $U^T \Omega U = \Omega$, with $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ (called $Sp(2N)$).[41] On top of these three infinite series, there are five *exceptional algebras* called G_2, F_4, E_6, E_7, and E_8. The subscript denotes the *rank* of the Lie algebra, which is the maximal number of commuting generators that one can find. If you are lucky, you may never need to worry about the exceptional algebras, though they do crop up in grand unified theories and in string theory.

The algebra (6.9) is also sufficiently strongly constraining to determine the possible *representations* that each Lie algebra has. Recall that a representation is any set of matrices that respects the multiplicative structure of the group (or, equivalently, the algebra (6.9)). Recall too that representations can be divided up into those that are *reducible* and those that are *irreducible* (henceforth, 'irreps'), meaning that they cannot be further decomposed. Representations are important for gauge theories, because it turns out (as we shall see) that matter (such as the electrons of QED) must transform in some representation of the gauge group.

Some representations are easy to find. For example, for $SU(N)$ we have the *defining* representation carried by vectors in $\mathbb{C}^N$, on which the $N \times N$ matrices act by multiplication.

[40] This corresponds to the 'difference' between the product and its reverse, so will vanish for an Abelian group.

[41] In this picture, the Lorentz group consist of matrices such that $U^T \eta U = \eta$, with $\eta = \mathrm{diag}(1, -1, -1, -1)$. This group is called $SO(3, 1)$. Though clearly related, it does not appear in our classification because it cannot be represented by (finite-dimensional) unitary matrices.

It turns out that one can build all of the other representations by taking tensor products of this (together with its complex conjugate representation) and decomposing into irreps and we shall do things in that way. $SO(N)$ similarly has a defining representation on vectors in $\mathbb{R}^N$, but it is not possible to obtain all irreps from tensor products of this: one misses the *spinor* representations. You have already met these before in QM, in the form of the spin-$\frac{1}{2}$ (or doublet) representation of angular momentum operators, which are nothing but the Lie algebra corresponding to the group $SO(3)$ of spatial rotations. We also met spinors in the context of the Lorentz group $SO(3,1)$, for which the Dirac field comes in a 4-dimensional spinor representation, whereas a gauge field comes in the vector representation (which is also 4-dimensional, but inequivalent to the spinor).

One representation, called the *adjoint*, is especially important, and is present for every Lie algebra. To find it, we note that the Lie algebra (6.9) implies the *Jacobi identity*

$$[T^a,[T^b,T^c]] + \text{cyclic permutations} = 0, \tag{6.10}$$

which you can confirm by simply expanding. But $[T^a,[T^b,T^c]] = if^{bcd}[T^a,T^d] = -f^{bcd}f^{ade}T^e$ and so

$$f^{bcd}f^{ade} + f^{abd}f^{cde} + f^{cad}f^{bde} = 0. \tag{6.11}$$

So far this is just mindless algebra, but if we define $(T^a_{\text{adj}})^{bc} \equiv -if^{abc}$, we see that we can recast this as

$$[T^a_{\text{adj}}, T^b_{\text{adj}}] = if^{abc}T^c_{\text{adj}}. \tag{6.12}$$

That is, the matrices T^a_{adj} form a representation of the algebra! This representation exists for any Lie group and is called the *adjoint* representation. The dimension of the adjoint representation is the same as the dimension of the Lie algebra itself. As examples, $SU(N)$ is generated by traceless, Hermitian matrices, and so has dimension N^2-1; $SO(N)$ is generated by antisymmetric, Hermitian matrices, and so has dimension $\frac{N}{2}(N-1)$.

One last point: the algebra (6.9) implies that the overall normalization of the generators in any representation is fixed, once we have decided on the normalization for the f^{abc}, or equivalently the generators T^a_{adj}. This is the underlying reason why charges are quantized in non-Abelian gauge theories.

6.3 Non-Abelian gauge theory

Suppose we wish to build a non-Abelian gauge theory with gauge group G with matter transforming in rep r of G. Under a global G transformation, the matter fields (fermions, say) transform as

$$\psi \to U\psi \equiv e^{ig\alpha^a T^a_r}\psi. \tag{6.13}$$

Remember that each T^a_r is really an $n_r \times n_r$ matrix, where n_r is the dimension of the representation r. Thus ψ is really a vector of dimension n_r, but we write everything in matrix notation to avoid drowning in a sea of indices.[42] For now g is just a constant, but

[42]Don't forget that ψ is also a spinor of the Lorentz group. Agh!

it will become the gauge coupling (lie e in QED). To have a chance of promoting G to a local symmetry (such that $\alpha^a \to \alpha^a(x)$, we need a derivative which tranforms covariantly. Following our noses, we assume that this takes the same form $D_\mu = \partial_\mu + igA_\mu$, as in QED and deduce how A must transform ($A_\mu \to A'_\mu$), in order that $D_\mu\psi \to UD_\mu\psi$. We find that

$$\partial_\mu + igA'_\mu = U(\partial_\mu + igA_\mu)U^{-1}. \tag{6.14}$$

But since $\partial_\mu U^{-1} = U^{-1}\partial_\mu + (\partial_\mu U^{-1})$ (remember that this is an operator relation), we find that

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}U\partial_\mu U^{-1} = UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}. \tag{6.15}$$

Note that for QED, where everything commutes, we recover $A^\mu \to A_\mu - \partial_\mu\alpha$.

It is clear that A_μ is an $n_r \times n_r$ matrix, but the transformation law for the gauge field may be defined in a way that makes no reference to the representation r. Writing $A_\mu \equiv A^a_\mu T^a_r$ and considering an infinitesimal transformation, we find that (exercise)

$$A'^a_\mu = A^a_\mu - \partial\alpha^a - gf^{bca}\alpha^b A^c_\mu. \tag{6.16}$$

So, the transformation of the A^a is fixed solely by the structure constants f^{abc} and indeed, apart from the derivative term, A^a obeys the transformation law for a field in the adjoint representation. This is hardly surprising, given that the number of fields A^a is equal to the dimension of the Lie algebra.

We have not yet completed our formulation of the gauge theory, because we have no dynamical terms for the gauge field in the action. In QED, we found the gauge-invariant field strength tensor $F_{\mu\nu}$ by inspection, but here we shall have to be more clever. To find an analogue of the field strength tensor, we use the covariance property $D_\mu \to UD_\mu U^{-1}$ of the covariant derivative. This means that $[D_\mu, D_\nu]$ also transforms covariantly. Call this $igF^a_{\mu\nu}T^a_r$ (which amounts to an implicit definition of $F^a_{\mu\nu}$). Now,

$$[D_\mu, D_\nu] = ig([A_\mu, \partial_\nu] + [\partial_\mu, A_\nu]) - g^2[A_\mu, A_\nu] = ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu]. \tag{6.17}$$

We now expand $A_\mu = A^a_\mu T^a_r$ (recall that r is any representation) and use the Lie algebra to get (exercise)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu. \tag{6.18}$$

This is a bit like the QED field strength tensor, except that it is not gauge invariant (it transforms covariantly) and it is not linear in the fields. But $\frac{1}{2g^2}\mathrm{tr}[D_\mu, D_\nu][D^\mu, D^\nu] = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ is gauge invariant and is the appropriate generalization of the Maxwell Lagrangian. But note that it necessarily contains terms that are cubic and quartic in the gauge fields. Thus, a non-Abelian gauge theory (unlike QED) automatically contains self-interactions of the gauge field! Physically, the difference with QED is easy to understand: in QED, the gauge field does not transform under a global $U(1)$ transformation, so we think of it as uncharged; in a non-Abelian gauge theory, the gauge field itself transforms as an adjoint under a global G transformation, so carries charge, so couples to itself.

6.4 The strong nuclear force: quantum chromodynamics

It is the self interactions of the gauge field that give rise to much of the aforementioned richness of non-Abelian gauge theory and indeed much of the richness of the world around us. As our first example, it was convincingly demonstrated in the 1970s and 1980s that the strong nuclear force is actually an $SU(3)$ gauge theory, called *quantum chromodynamics* or QCD. There are $N^2 - 1 = 8$ gauge bosons, which we call *gluons*, which couple to fermions, which we call *quarks*, which transform in the defining 3-dimensional representation of $SU(3)$. The three different values for the index are sometimes labelled by different colours (red, green, and blue), hence the name *chromo*dynamics. It turns out that there is more than one quark (they are called different *flavours*), all transforming as colour triplets. The different flavours are called *up, down, strange, charm, bottom*, and *top*, in order of increasing mass. The QCD Lagrangian is thus given by

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{f \in \{u,d,s,c,b,t\}} \overline{\psi} \left(i\partial\!\!\!/ - g_s A\!\!\!/^a \frac{\lambda^a}{2} - m_f \right) \psi. \tag{6.19}$$

Here, the *Gell-Mann matrices*

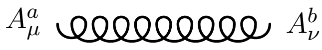
$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

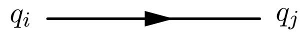
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

provide an explicit basis for the defining triplet representation. Note that it is conventional to denote the gluon field strength by $G^a_{\mu\nu}$ and the strong coupling constant by g_s. The Feynman rules are given in Fig. 5. Actually, they are not really the Feynman rules. The subtleties of gauge-fixing in non-Abelian theories (which we have completely circumvented) mean that one needs to modify the rules in general. But the rules we give suffice for tree-level computations (that is, diagrams without loops of propagators).

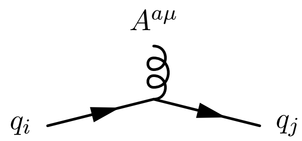
Now, it turns out (for reasons that will become clearer later on) that the force between two quarks – the analogue of the Coulomb interaction in QED – is strong at low energies. So strong, in fact, that it is physically impossible to isolate a single quark. Rather quarks are confined in nuclei. This 'explains' at a stroke both why we have never seen a single quark in the laboratory and why it took so long to establish QCD as the correct theory of the strong nuclear force: the force is so strong at the relatively low energy scales of nuclear physics that we are well beyond the realm of perturbation theory. In fact, nobody has yet managed to start from the Lagrangian of QCD and show analytically that it predicts the



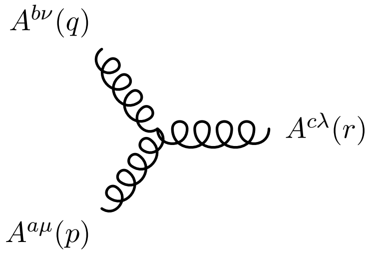
$$= \frac{-ig_{\mu\nu}\delta^{ab}}{q^2 + i\epsilon} \tag{6.20}$$



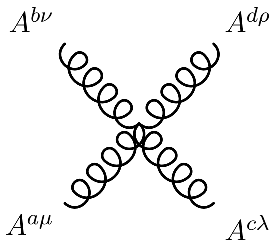
$$= \frac{i\delta_{ij}}{\not{q} - m + i\epsilon} \tag{6.21}$$



$$= -ig_s\gamma^\mu \frac{\lambda^a_{ij}}{2} \tag{6.22}$$



$$= -g_s f^{abc}(\eta^{\mu\nu}(p-q)^\lambda + \eta^{\nu\lambda}(q-r)^\mu + \eta^{\lambda\mu}(r-p)^\nu) \tag{6.23}$$



$$= -ig_s^2[f^{eac}f^{ebd}(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\rho}\eta^{\nu\lambda}) \tag{6.24}$$

$$+ f^{ead}f^{ebc}(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho}) \tag{6.25}$$

$$+ f^{eab}f^{ecd}(\eta^{\mu\lambda}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\lambda})] \tag{6.26}$$

Figure 5. Feynman rules for QCD. All momenta are defined to be ingoing.

confinement of quarks in nuclei. We have strong indications from numerical simulations that it is so, but we do not have a proof.[43]

The flipside of this (and the reason we know that QCD must be the correct theory of the strong nuclear force) is that QCD is perturbative at high energies (like at the LHC), so we can use the formalism we have already developed there. For example, the relevant Feynman diagram for computing the amplitude for scattering two quarks of distinct flavours (e.g. an up quark and a down quark) is shown in Fig. 6. Compared to the analogous QED

[43]If you think you have found a proof, scribble it down and send it off to these people: http://www.claymath.org/millennium/Yang-Mills_Theory/. If they think you are right, they will send you back a cheque for a million dollars.

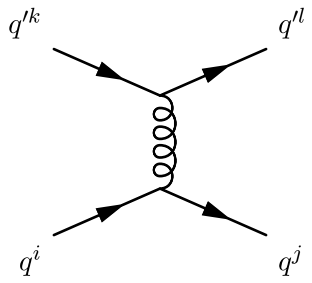
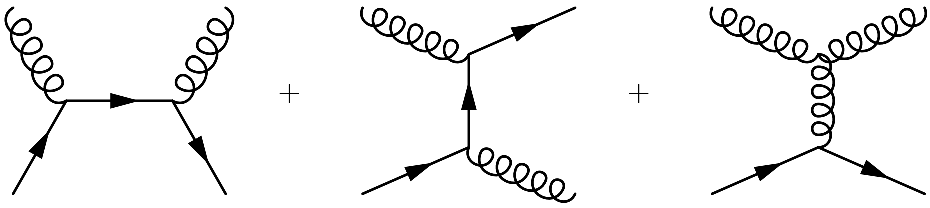


Figure 6. Feynman diagram for scattering of quarks of different flavours.



(6.29)

Figure 7. Feynman diagrams for quark-gluon scattering.

scattering the only different factor in the matrix element is the representation matrix, so that

$$\mathcal{M}_{\text{QCD}} = T^a_{ij} T^a_{kl} \mathcal{M}_{\text{QED}}, \tag{6.27}$$

where i, j, k, and l are colour indices. To get the cross-section for unpolarized scattering, we need to average over the initial colours and sum over the final state colours. In all, we get

$$\frac{\sigma_{\text{QCD}}}{\sigma_{\text{QED}}} = \frac{1}{3 \cdot 3} \sum_{i,j,k,l} T^a_{ij} T^a_{kl} (T^b_{ij} T^b_{kl})^* = \frac{1}{9} (\text{tr} T^a T^b)^2 = \frac{2}{9}. \tag{6.28}$$

The analogue of Compton scattering in QED, quark-gluon scattering, is more complicated, because the three-gluon vertex comes into play. Fig. 7 shows the contributing diagrams at leading order.

6.5 The weak nuclear force and $SU(2) \times U(1)$

Having built a gauge theory for the strong nuclear force, we now try to build a gauge theory for the weak nuclear force. We'll try to do this in the same way as our ancestors did, piecing together the experimental facts one by one. This makes for a longer and more arduous journey, but I think it is far more instructive than presenting the final theory as a *fait accompli*.

So, what do you know about the weak force? The one thing you should know, is that it is responsible for things like β decay, in which $n \to p + e + \overline{\nu}$. Our theory of the strong force tells us that a proton is basically made up of two up quarks and a down quark and that the neutron is made up of two downs and an up, so at a more fundamental level, β decay involves $d \to u + e^- + \overline{\nu}$. How could we describe this using a non-Abelian gauge theory? Suppose we regard this process as occurring via exchange of a gauge boson. In a non-Abelian theory, the effect of a gauge boson vertex is to take one component of a field carrying some representation and to change it to another (as an example, in QCD, the quark colour is changed when it interacts with a gluon). Since baryon and lepton number are conserved to a very good degree in Nature, we expect that the gauge boson should turn an up quark into a down quark at one vertex (conserving quark or baryon number) and turn an electron into a neutrino at the other (conserving lepton number). Our representations must contain at least two elements (since one particle gets turned into a different one at a vertex). Are there any reps which contain *only* two elements? There is one, which is the fundamental (defining) representation of the simplest non-Abelian Lie group, $SU(2)$. Let's try to build a theory of the weak interactions using $SU(2)$. Fortunately (though you may not know it), you are already quite good at doing $SU(2)$ group theory. The reason (already mentioned above) is that symmetry under spatial rotations corresponds to the group $SO(3)$ (orthogonal rotations in 3 dimensions), but the Lie algebra of $SO(3)$ is exactly the same as the Lie algebra of $SU(2)$. (Remember we said before that two Lie groups can have the same Lie algebra? Well, here's an example.) This means that the theory of angular momentum in QM (recall that angular momentum operators are really the Lie algebra of spatial rotations) is really just the representation theory of $SU(2)$. So, for example, the smallest rep is of dimension two (you call it spin-half) and the generators in that rep are just given by the Pauli matrices (divided by two, in the usual normalization convention $\mathrm{tr} T^a T^b = \frac{\delta^{ab}}{2}$). Another way of seeing why the Pauli matrices appear is to note that the Lie algebra of $SU(2)$ should be represented by a basis for 2×2, traceless (because of the 'S' in $SU(2)$), Hermitian (because of the 'U' in $SU(2)$) matrices. The Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.30}$$

are just that. With the Pauli matrices in hand, we can easily work out the Lie algebra of $SU(2)$. It is (exercise) $[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = i\epsilon_{ijk}\frac{\sigma^k}{2}$.

Denoting the $SU(2)$ gauge field by W^i_μ, the covariant derivative for the 2-dimensional rep is then given by

$$D_\mu = \partial_\mu + i\frac{g}{2}W^i_\mu \sigma^i = \partial_\mu + i\frac{g}{2}\begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix} = \partial_\mu + i\frac{g}{2}\begin{pmatrix} W^3_\mu & \sqrt{2}W^+_\mu \\ \sqrt{2}W^-_\mu & -W^3_\mu \end{pmatrix}, \tag{6.31}$$

where we have defined a complex gauge field $W^\pm_\mu \equiv \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu)$ (the $\sqrt{2}$ is included so that we get the usual normalization for the kinetic term of a complex field). The reason for introducing $W^\pm_\mu$ becomes clear when we package the quarks and leptons up into $SU(2)$ doublets $l \equiv \begin{pmatrix} \nu \\ e \end{pmatrix}$ and $q \equiv \begin{pmatrix} u \\ d \end{pmatrix}$: the part of the Lagrangian involving the covariant derivative

$$\mathcal{L} \supset \bar{l} i \not{D} l + \bar{q} i \not{D} q \tag{6.32}$$

contains interactions like $ig\sqrt{2}\overline{\nu}\not{W}^{+}e^{-}$ and the $\pm$ superscript on $W_\mu^\pm$ is just the electric charge (which is conserved) carried by the gauge boson. Even more satisfyingly, recall from our discussion of charge conjugation that a matrix gauge field should transform into minus its transpose. This sends $W_\mu^\pm \to W_\mu^\mp$, meaning that the particle is sent into its antiparticle, as we expect.

This is starting to look like a good model for weak interactions, but now we encounter its first big flaw. The flaw is that it was observed in the 1950s by Madam Wu and collaborators (at the suggestion of Lee and Yang) that the weak interactions do not conserve parity. That is to say, the Lagrangian is not invariant under the spatial inversion $\mathbf{x} \to -\mathbf{x}$. This result shocked the physics community. Hitherto, no one had really bothered to question the status of such symmetries; with the discovery that they were in fact broken, the race was on to find out how and why.[44]

6.6 Intermezzo: Parity violation and all that

To understand how parity can be violated in a gauge theory, we need to go back and work out how to implement parity in a theory containing fermions. This is not too difficult. Start with the Dirac equation

$$(i\gamma^0\partial_t + i\gamma^i\partial_i - m)\psi = 0 \tag{6.33}$$

and premultiply by γ^0. Now, γ^0 commutes with itself, but anticommutes with γ^i. Thus

$$(i\gamma^0\partial_t - i\gamma^i\partial_i - m)\gamma^0\psi = 0 \tag{6.34}$$

and $\psi'(t, -x^i) \equiv \gamma^0\psi(t, x^i)$ satisfies the Dirac equation in a space-reflected Universe (where $\partial_i \to -\partial_i$).

We want to know how to write down a Lagrangian that violates parity, but is still Lorentz invariant. It is easy to show that the Lorentz invariant terms we have been writing down, like $\overline{\psi}\psi$ and $\overline{\psi}\not{\partial}\psi$, are also parity invariant. For example,

$$\overline{\psi}\psi \to \overline{\psi}'\psi' = \psi^\dagger(\gamma^0)^3\psi = \overline{\psi}\psi. \tag{6.35}$$

As an exercise, you can now show parity invariance of $\overline{\psi}\not{\partial}\psi$. But if we introduce the matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{6.36}$$

[44] At the same time, a race began to determine the status of similar symmetries like time reversal invariance and charge conjugation. In turns out that none of these symmetries is sacrosanct in QFT (and surprise, surprise, none is sacrosanct in nature), though the combined operation of CPT is. CP violation is particularly interesting in that the Standard Model gives a very good description of all CP violation observed in experiments up until now, but it is also known that amount of CP violation in the SM is too small to explain the predominance of matter over antimatter that we see in the Universe. This predominance should be pretty important to you, because you would not be here without it — your proto-self would long ago have annihilated with your anti-self.

(equal to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in the chiral basis), we find that it anticommutes with γ^μ. Hence, objects like $\overline{\psi}\gamma^5\psi$ and $\overline{\psi}\gamma^5\partial\!\!\!/\psi$ are odd under parity.[45] For example,

$$\overline{\psi}\gamma^5\psi \to \overline{\psi}'\gamma^5\psi' = \psi^\dagger(\gamma^0)^2\gamma^5\gamma^0\psi = -\overline{\psi}\gamma^5\psi. \tag{6.37}$$

Exercise: show parity oddness of $\overline{\psi}\gamma^5\partial\!\!\!/\psi$.

These considerations have even more far reaching consequences than mere parity violation. The combinations $P_{L,R} \equiv \frac{1}{2}(1 \mp \gamma^5)$ have the properties of a set of projection operators when acting on a Dirac fermion ψ.[46] We define $\psi_{L,R} \equiv P_{L,R}\psi$ and call them left- and right-handed fermions.[47] Let's now write the Dirac Lagrangian in terms of $\psi_{L,R}$. We get[48]

$$\mathcal{L} = i(\overline{\psi_L}\partial\!\!\!/\psi_L + \overline{\psi_R}\partial\!\!\!/\psi_R) - m(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L). \tag{6.38}$$

This rendering makes two points clear. The first point is that, for massless fermions, we could write a Lagrangian using just ψ_L (or ψ_R) alone. Such a theory describes a massless *Weyl fermion*. Note that there are only two degrees of freedom (corresponding to a particle spinning one way and an antiparticle spinning the other way). It violates parity, but not Lorentz invariance. There is, furthermore, nothing to stop us promoting the derivative to a covariant derivative and making a gauge theory involving Weyl fermions. The second (related) point is that even in a theory which contains *both* left- and right-handed components, we can assign the different components to different representations of the gauge group. But if we do so, the mass term (which couples left to right) will no longer be gauge invariant.

There is a third point, which is not relevant to our present discussion, but which will be relevant when we discuss neutrino masses. The point is that we can write a different mass term for a Weyl fermion, ψ_L say, called a *Majorana* mass term. It takes the form

$$\mathcal{L} \supset -\frac{1}{2}m\psi_L^T C\psi_L + \text{h. c.}, \tag{6.39}$$

[45] Smart alecs will sniff that we have not shown Lorentz invariance of $\overline{\psi}\gamma^5\psi$, to which my churlish retort is that we never showed Lorentz invariance of $\overline{\psi}\psi$ either. And so the house of cards collapses ...

[46] A set of projection operators should add up to the unit operator ($P_L + P_R = 1$), should be orthogonal ($P_L P_R = 0$), and should be idempotent ($P_{L,R}^2 = P_{L,R}$), so that repeated projections have no further effect.

[47] Why left- and right-handed? Well, consider the limit in which a fermion is massless and moving in the $+z$ direction. The Dirac equation in the chiral basis is just $p\!\!\!/\psi = 0 \implies \begin{pmatrix} 0 & E(1+\sigma^3) \\ E(1-\sigma^3) & 0 \end{pmatrix}\psi = 0$. Now γ^5 is diagonal in this basis (which is why we chose the basis in the first place), and so ψ_L has only the top two components non-vanishing, whilst ψ_R has only the bottom two components non-vanishing. We find that the Dirac equation implies that $\psi_L \propto \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T$ and $\psi_R \propto \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T$. But these are eigenstates of the spin operator $\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$, spinning opposite to, and along, the direction of motion, respectively.

[48] One has to be a bit careful with the notation here, because $\overline{(\psi_L)} = \psi_L^\dagger\gamma^0 = \psi^\dagger\frac{1-\gamma^5}{2}\gamma^0 = \psi^\dagger\gamma^0\frac{1+\gamma^5}{2} \equiv (\overline{\psi})_R$.

where $C = i\gamma^2\gamma^0$ is called the charge conjugation matrix (since $\psi \to C\gamma^0\psi^*$ is nothing but charge conjugation) and the '+h. c.' instructs us to add the Hermitian conjugate term to make the action real. Note that only ψ_L is required. The flipside is that ψ_L is coupled to itself, rather than to its complex conjugate. Thus this term is not invariant under a $U(1)$ phase rotation $\psi_L \to e^{i\alpha}\psi_L$ and cannot describe a particle carrying electromagnetic charge. It could describe a neutrino, however.

When we come to study grand unification, it will be useful to know that charge conjugation switches a left handed field to a right-handed field.[49] Thus we can replace any right-handed field by its charge conjugate and consider all fields as being left-handed.

6.7 Back to the weak interactions

Now we know how to violate parity, we can incoporate it into the weak interactions. We do it by declaring that only the left-handed parts of the quarks and leptons couple to the W^μ via $SU(2)$. (This introduces a further problem of how the quarks and leptons can have a mass, which we shall only be able to solve after another intermezzo.) This can be straightforwardly implemented in the Feynman rules by including a projection factor P_L in the vertex.

So far, we checked that $W^\pm_\mu$ could be the culprit behind β decay. But what about W^3? Could it be the Z boson? From (6.32), we find the couplings $\frac{ig}{2}\not{W}^3(\overline{\nu_L}\nu_L - \overline{e_L}e_L)$. This is a bit like the Z boson, but unfortunately it turns out that the Z also couples to right-handed quarks and leptons.[50] *Quel chagrin*!

Salvation comes by noticing that there are two neutral bosons in Nature: the Z boson and the photon. Both couple to left- and right-handed fermions. But could it be that they are mixtures of W^3_μ (which couples to only left-handed fermions) and a second $U(1)$ boson (call it B_μ) which couples to both left and right-handed fermions?

Before we go further, it is useful to pause and appreciate what this means. The suggestion is that the weak force and electromagnetism are not distinct phenomena, but are somehow mixed up in a unified *electroweak theory*. The claim is that these two forces, which manifest themselves completely differently to our eyes (quite literally), are really different aspects of the same thing. Gadzooks!

Let's see how it works. We put the left handed fermions in doublets q_L and l_L of $SU(2)$ as before (and call the coupling constant g) and also give them each a charge, called *weak hypercharge* $Y_{q,l}$, under a $U(1)$ phase transformation gauged by B_μ (for which the coupling constant is denoted g'). We make the right-handed fermions u_R, d_R, e_R[51] singlets of $SU(2)$ (meaning they don't transform) and give them weak hypercharges $Y_{u,d,e}$. We then demand that the physical gauge boson eigenstates A_μ and Z_μ be some mixture of W^3_μ and B_μ, such

[49] Proof: $\gamma^2 P_L = P_R\gamma^2 \ldots$

[50] You might wonder how we know this. A direct way is to produce polarised electrons and positrons and scatter them off each other.

[51] We discuss the possibility of a ν_R later on.

that

$$W^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu, \tag{6.40}$$
$$B_\mu = -\sin\theta_W Z_\mu + \cos\theta_W A_\mu. \tag{6.41}$$

Here θ_W is the *Weinberg angle*. Roughly, $\sin^2\theta_W = 0.231$.

Now we try to work out what the charges must be. On the one hand, the covariant derivative for the right handed fermions contains a piece

$$\mathcal{L} \supset -\overline{\psi_R} g' Y_\psi \not{B} \psi_R \supset -\overline{\psi_R} g' \cos\theta_W Y_\psi \not{A} \psi_R. \tag{6.42}$$

Thus we have no choice but to identify $g'\cos\theta_W$ with the electric charge $|e|$ and Y_ψ with the electric charge of that particle. Thus[52]

$$Y_e = -1,\ Y_u = +\frac{2}{3},\ Y_d = -\frac{1}{3}. \tag{6.43}$$

On the other hand, the covariant derivative for the left-handed fermions contains a piece

$$\mathcal{L} \supset -\overline{\psi_L}(g\frac{\sigma^3}{2}\not{W}^3 + g' Y_\psi \not{B})\psi_L \supset -\overline{\psi_L}(g\sin\theta_W \frac{\sigma^3}{2} + g'\cos\theta_W Y_\psi)\not{A}\psi_L. \tag{6.44}$$

Now, both l and q doublets contain two states whose electric charges differ by one (in units of e). This can only happen here if we set $g\sin\theta_W = |e|$. Furthermore, we can only get the absolute values of the charges right if we set $Y_q = +\frac{1}{6}$ and $Y_l = -\frac{1}{2}$.

Thus we are able to fix everything up so that the photon couples in the same way to left- and right-handed fields (and with the correct charge for each particle). This brings us back to our original, parity-invariant theory of QED. But the couplings of the Z^μ are not the same for left and right. Specifically the charges are (exercise)

$$g\cos\theta_W I_3 - g'\sin\theta_W Y = \frac{|e|}{\sin 2\theta_W}(I_3 - Q\sin^2\theta_W), \tag{6.45}$$

where $I_3 = 0, \pm\frac{1}{2}$ is the weak *isospin* (the eigenvalue of the third $SU(2)$ generator) and Q is the electric charge in units of $|e|$.

Yet again, you may or may not have noticed an elephant in the room and the time has come to chase it out. The elephant is manifest in two ways. The first way is that we have put left and right fermions in different representations of $SU(2) \times U(1)$. This forbids us from writing a mass term for fermions, contrary to what we observe in Nature.[53]

The second way is that we claimed to have made a conceptual breakthrough in mixing neutral gauge fields to obtain the physical photon and the Z boson. This is nonsense, because we never specified what we meant by physical.

The resolution to both of these problems lies in what is apparently a third problem - our theories of the weak force and electromagnetism are basically the same. Ok, the

[52] Oops! I didn't tell you what the electric charges of the quarks are. But you can work it out for yourself from the fact that $p \sim uud$ and $n \sim udd$.

[53] In fact, the top quark is the heaviest particle yet discovered!

charges and the symmetry groups are different, but that turns out not to be a big deal. This flies totally in the face of what we observe in Nature. Specifically, the photon as far as we are able to tell, is strictly massless, which translates to electromagnetism being a long-range force. The weak interaction, on the other hand, is mediated over a very short range, meaning that the corresponding gauge boson must have a mass (via the uncertainty principle). We can even work out roughly what the mass should be. The Fermi constant that describes beta decay has mass dimension minus two and value

$$10^{-5}\text{GeV}^{-2}, \tag{6.46}$$

from which we infer a mass scale of about 10^2GeV.

Uh oh! We said at the very beginning that gauge invariance forbids a gauge boson mass. The particular kind of gauge invariance we have here (different symmetry for left and right fermions) also forbids fermion masses. How do we get all our masses back?

Enter the Higgs boson. The Higgs mechanism[54] solves both of these problems via the mechanism of *spontaneous symmetry breaking*. That is a big deal. It also predicts the existence of the Higgs boson and we have spent several decades and several billion dollars looking for it. And now, serendipitously, it would seem that the LHC has found it. Hurrah.

So, what is spontaneous symmetry breaking and what is the Higgs mechanism? Time for another intermezzo.

6.8 Intermezzo: Spontaneous symmetry breaking

Let's start simply. Consider a complex scalar field, with the Klein-Gordon Lagrangian

$$\mathcal{L} = \partial\phi^*\partial\phi - m^2|\phi|^2. \tag{6.47}$$

This has a global symmetry $\phi \to e^{i\alpha}\phi$. We could also add an interaction, whilst maintaining the symmetry, of the form $-\lambda|\phi|^4$. This is candidly called *phi-to-the-fourth* theory and you now know how to go and compute the effect of λ in perturbation theory. Let's not bother. Instead, let's go back and think about the structure of the vacuum. The terms in the Lagrangian which do not involve derivatives may be thought of as a potential for the field, of the form

$$V(\phi) = m^2|\phi|^2 + \lambda|\phi|^4. \tag{6.48}$$

This potential has its minimum (which gives the classical vacuum) at the origin. That's why, back in the dark ages of canonical quantization, we started with $\phi = 0$ and considered fluctuations about that point. Indeed, you can go back and verify that $\langle 0|\phi|0\rangle$, which we call the *vacuum expectation value* (VEV), vanishes.

What would happen if m^2 was actually negative? The global minima of the potential would now be at points such that

$$|\phi| = \sqrt{\frac{-m^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}} \tag{6.49}$$

[54]Conceived in the 1960s by a number of people, only one of whom is named Higgs, and only two of whom were rewarded with the Nobel prize.

and we should quantize about one of those points instead.[55] For our purposes though, it is enough to think about what happens classically. Firstly, notice that (6.49) describes not a single point in field space, but rather a circle of points in the complex ϕ plane. Any one of these points (which are degenerate in energy) could be the minimum. But whichever point the theory picks, the symmetry $\phi \to e^{i\alpha}\phi$ will be broken by the vacuum configuration. This is the phenomenon of *spontaneous symmetry breaking*.[56] It has an immediate consequence, which is that fluctuations of the field about the minimum in the degenerate direction have no associated potential energy. So provided the wavelength of the fluctuations is large enough, the kinetic (and hence total) energy cost of the fluctuation will be small. This is formalized as *Goldstone's theorem* and in Lorentz-invariant theories, it means that spontaneous symmetry breaking always implies the existence of a massless particle.

You can check that it works for ϕ^4 theory right now. Choose the vacuum direction to be along the real ϕ axis and expand

$$\phi = \frac{1}{\sqrt{2}}(v + \phi_1 + i\phi_2), \tag{6.50}$$

where $\phi_{1,2}$ are real scalar fields. You should find (by substituting in the Lagrangian and picking out the quadratic terms — exercise) that ϕ_1 has mass $\sqrt{-2m^2}$ and that ϕ_2 is massless.

Now let's ask what would happen if we had promoted the symmetry $\phi \to e^{i\alpha}\phi$ to a $U(1)$ gauge symmetry, *viz.* $\alpha \to \alpha(x)$. Then the Lagrangian would be

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi - m^2|\phi|^2 - \lambda|\phi|^4, \tag{6.51}$$

with $D_\mu = \partial_\mu + ieA_\mu$ as always. This is called the *Abelian Higgs model*. When we allow ϕ to have a VEV, $\langle 0|\phi|0\rangle = \frac{v}{\sqrt{2}}$, we find the gauge boson mass term[57]

$$\mathcal{L} \supset +\frac{e^2v^2}{2}A^\mu A_\mu. \tag{6.52}$$

So spontaneous breaking of a gauge field gives rise to a gauge boson mass! There is something a bit fishy here, which is that a massive vector boson has three polarizations (corresponding to the three directions the spin can point it in its rest frame), whilst a massless vector boson has only two (corresponding to whether its helicity is plus or minus). We seem to have got a degree of freedom 'for free', just by flipping the sign of a parameter in the Lagrangian. This is not so. Indeed, we musn't forget about the freedom to do gauge transformations. In particular, there exists a transformation, given by $\alpha = -\tan\frac{\phi_2}{v+\phi_1}$, in which the degree of freedom ϕ_2' (that was previously the Goldstone boson) of the gauge-transformed scalar field vanishes. This is nothing other than a choice of gauge fixing, called

[55]Note that in quantum mechanics (or in QFT in $d = 1 + 1$), we would instead find that the vacuum is some linear superposition of states localized about each of the points. But QFT in $d > 1 + 1$ is different.

[56]Note that if you tried this trick for a fermion or a vector, rather than a scalar, you would end up breaking Lorentz invariance as well.

[57]Note that this is a positive mass squared term in the potential for the spatial components of the gauge field.

the *unitary gauge*. Colloquially, we say that the massless Goldstone boson gets 'eaten' by the gauge field to become the third polarization of a massive vector field.

All of this discussion generalizes directly to theories with non-Abelian symmetry group G. Depending on what rep of G the scalar field comes in and depending on how the VEV is aligned, the group G will get broken to some subgroup $H \subset G$. In the global version, there will be as many massless Goldstone bosons as there are generators of G (more precisely, its Lie algebra) which are not in H. In the local (gauged) version, the gauge boson mass term is given by

$$\frac{g^2}{2} v^\dagger T_r^a T_r^b v A^{\mu a} A_\mu^b = \frac{(m^2)^{ab}}{2} A^{\mu a} A_\mu^b; \tag{6.53}$$

gauge bosons which correspond to broken generators ($T^a v \neq 0$) become massive, whilst those corresponding to unbroken generators remain massless.

We are now in a position to go back and work out the final details of the weak interactions. Before we do, you might be worrying that I am trying to pull the wool over your eyes. I gave you gauge symmetry with one hand and I took it away with the other, by breaking it. Aren't we back where we started?

The answer is a resounding no. Actually, as we hinted earlier on, gauge symmetry is not really a symmetry at all, or at least it is no *more* of a symmetry than the underlying global symmetry. One way to see this is to note there are no extra conservation laws that appear once one gauges a symmetry. Rather, gauge symmetry is a convenient *redundancy* of description, which can be got rid of by gauge fixing.

Moreover, spontaneous symmetry breaking is not really a symmetry breaking. The symmetry is still present, but acts on the physical degrees of freedom in a different way. In particular, for a globally symmetric theory, in the unbroken version, the scalar fields transform linearly, like a representation: $\phi \to e^{i\alpha}\phi$. But in the 'broken' version, the Goldstone boson transforms non-linearly: $\phi_2 \to \phi_2 + v\alpha + \dots$. So pedants say that the symmetry is not broken, but rather is non-linearly realized. And they are right, as they usually are. The symmetry still restricts the form of the Lagrangian and indeed allows us to have a consistent theoretical description of a massive vector boson force-carrier.

6.9 Back to the electroweak interaction

Let's now show what happens for the electroweak theory, a.k.a. the Standard Model. You are probably getting tired of repeating the mistakes of your predecessors by now, so I will just lay down the facts.

We have a gauge theory of $SU(2) \times U(1)$, containing gauge bosons $W_\mu^\pm, W_\mu^3$ and B_μ. We want to break things in such a way that the $W_\mu^\pm$, together with the combination of W_μ^3 and B_μ that we called Z_μ, become massive, while the combination A_μ stays massless. Clearly we need to break $SU(2) \times U(1)$ down to $U(1)$, where the unbroken $U(1)$ is the 'right' combination of the original $U(1)$ and a $U(1)$ subgroup of $SU(2)$. It can be done as follows. Introduce a scalar field (the *Higgs field*), H, transforming as a doublet of $SU(2)$, with hypercharge $Y = \frac{1}{2}$. The Higgs potential takes the form

$$-\mu^2 H^\dagger H + \lambda (H^\dagger H)^2. \tag{6.54}$$

This is minimized when

$$\sqrt{H^\dagger H} \equiv \frac{v}{\sqrt{2}} = \sqrt{\frac{\mu^2}{2\lambda}} \tag{6.55}$$

and we may choose, without loss of generality,

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}, \tag{6.56}$$

with v real. The covariant derivative

$$D_\mu H = (\partial_\mu + ig\frac{\sigma^i}{2}W^i_\mu + i\frac{g'}{2}B_\mu)H \tag{6.57}$$

then results in a gauge boson mass matrix

$$\frac{1}{8}\begin{pmatrix} 0 & v \end{pmatrix}\begin{pmatrix} gW^3_\mu + g'B_\mu & \sqrt{2}gW^+_\mu \\ \sqrt{2}gW^-_\mu & -gW^3_\mu + g'B_\mu \end{pmatrix}\begin{pmatrix} gW^3_\mu + g'B_\mu & \sqrt{2}gW^+_\mu \\ \sqrt{2}gW^-_\mu & -gW^3_\mu + g'B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{6.58}$$

or, using (6.40) together with $\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$

$$\frac{(gv)^2}{4}W^+_\mu W^{-\mu} + \frac{(g^2+g'^2)v^2}{8}Z_\mu Z^\mu \tag{6.59}$$

Taking into account the different normalizations (the mass term is $m^2\phi^*\phi$ for a complex field but $\frac{m^2}{2}\phi^2$ for a real field), we find

$$m_W = \frac{gv}{2},\ m_Z = \frac{\sqrt{g^2+g'^2}v}{2} = \frac{m_W}{\cos\theta_W},\ m_A = 0. \tag{6.60}$$

Miraculously, we find massive W and Z bosons, together with a massless photon. Moreover, the theory predicts the ratio of W and Z masses to be given by $\cos\theta_W$, in agreement with experiment ($m_W = 80.2$ and $m_Z = 91.2$ GeV).[58] Was it really a miracle? In many ways, no. Once we fixed the charges of the Higgs and of the fermions, we had no choice but to break $SU(2) \times U(1)$ to electromagnetism (or not to break it at all). The m_W/m_Z mass ratio prediction is non-trivial, in that choosing a different representation for the Higgs would spoil it. Then again, choosing an arbitrary representation for the Higgs would not give the right pattern of symmetry breaking. In the end, everything which appears miraculous can be traced back to the choices of charges for the fermions and the Higgs. They are what they are observed to be, but still the question remains of why Nature chose them that way. Why for example, are all the hypercharges quantized in units of one-sixth (recall that it need not be so; indeed, we could have chosen a charge of π for one of the fermions, *a priori*)? Could it be that Nature *had* to choose them that way, in the sense that the theory could not be consistent otherwise? Questions like these drive us to look for theories of physics that go beyond the Standard Model, in the hope that we may gain a deeper level of understanding of why things are the way they are.

[58]Strictly speaking, the ratio disagrees with experiment, because it receives corrections from higher orders in perturbation theory. But once these are taken into account everything fits nicely.

6.10 Fermion Masses

We have explained how the gauge bosons get their masses by the Higgs mechanism, but what about the quarks and leptons? Again, the answer is straightforward. Given a Higgs field transforming as a doublet of $SU(2)$ with hypercharge one-half, we can write down the *Yukawa couplings*

$$\mathcal{L} \supset -\lambda^u \overline{q_L} H^c u_R - \lambda^d \overline{q_L} H d_R - \lambda^e \overline{l_L} H e_R + h.c. \tag{6.61}$$

where $H^c \equiv i\sigma^2 H^*$ is an $SU(2)$ doublet field with hypercharge minus one-half.[59] These terms represent interactions, but when we plug in the Higgs VEV, lo and behold, we get fermion masses

$$m_u = \frac{\lambda^u v}{\sqrt{2}},\ m_d = \frac{\lambda^d v}{\sqrt{2}},\ m_e = \frac{\lambda^e v}{\sqrt{2}}. \tag{6.62}$$

It just works.™

6.11 Three Generations

We have described what happens for the first generation of quarks and leptons. In fact there are three generations (we already know about the muon and the various flavours of quarks) and it turns out that the extension of the theory just described gives an elegant (and more to the point, correct) description of flavour physics (namely transitions between the generations). In particular, the Yukawa couplings in (6.61) can be complex, and this is what gives rise to CP violation, once we have three generations. We don't have time to describe it here, but I encourage you to look it up.

6.12 The Standard Model and the Higgs boson

We have almost finished our description of the Standard Model. To recap, we show in Table 1 the different fields and their representations under the SM gauge group $SU(3) \times SU(2) \times U(1)$ (recall that $SU(3)$ corresponds to QCD, or the strong nuclear force).

We have worked out the properties of all of the particles, but one: the Higgs boson. What Higgs boson? Remember in the Abelian Higgs model that the Goldstone boson got eaten by the gauge field, but we were left with one massive scalar mode, corresponding to fluctuations in the radial direction in the complex plane of the field ϕ. For the Higgs field H in the Standard Model, we have four real scalar degrees of freedom (since H is a complex doublet); three of these get 'eaten' to form the longitudinal polarizations of the $W^{\pm}_{\mu}$ and Z_{μ}. One scalar remains: the Higgs boson. We can work out its properties by going to the unitary gauge, in which the three Goldstone bosons are manifestly eaten. In the SM, this amounts to choosing

$$H(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}. \tag{6.63}$$

[59] It is easy to see that H^c transforms with $Y = -\frac{1}{2}$, since it involves the complex conjugate of H. It is a doublet of $SU(2)$ because the complex conjugate of $SU(2)$ transforms as an anti-doublet of $SU(2)$, which is equivalent to the doublet representation. The $i\sigma^2$ is just the similarity transform that takes us from one rep to the other. Go and look in the group theory book if you're worried about it.

Field	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
g	8	1	0
W	1	3	0
B	1	1	0
$q_L = (u_L d_L)^T$	3	2	$+\frac{1}{6}$
u_R	3	1	$+\frac{2}{3}$
d_R	3	1	$-\frac{1}{3}$
$l_L = (\nu_L e_L)^T$	1	2	$-\frac{1}{2}$
e_R	1	1	-1
H	1	2	$+\frac{1}{2}$

Table 1. Fields of the Standard Model and their $SU(3) \times SU(2) \times U(1)$ representations

The Higgs boson, $h(x)$, is a real scalar field. It is not charged under electromagnetism (it can't be, since it is real). Its couplings to other fields can be worked out by replacing v with $v + h$ in our previous expressions. Thus, from (6.62), we find a Yukawa coupling to fermion i given by

$$\mathcal{L} \supset -\frac{m_i}{v} h \overline{\psi}_i \psi_i. \tag{6.64}$$

Similarly, from (6.59), we find couplings to gauge bosons given by

$$\mathcal{L} \supset m_W^2 \left(\frac{2h}{v} + \frac{h^2}{v^2} \right) W_\mu^+ W^{-\mu} + \frac{m_Z^2}{2} \left(\frac{2h}{v} + \frac{h^2}{v^2} \right) Z_\mu Z^\mu. \tag{6.65}$$

Finally, the Higgs boson has self interactions, coming from the potential

$$\mathcal{L} \supset +\frac{\mu^2}{2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4 \supset -\lambda v^2 h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4 = -\frac{m_h^2}{2} h^2 - \frac{m_h^2}{2v} h^3 - \frac{m_h^2}{8v^2} h^4. \tag{6.66}$$

Thus $m_h^2 = 2\lambda v^2$, such that we know the value of the coupling λ once we know the mass of the Higgs. The recent LHC measurement of $m_h \simeq 125$ GeV thus fixes $\lambda \simeq 0.13$.

With these couplings worked out, we can roughly work out the phenomenology of Higgs boson decays. The self interactions are not relevant here, because energy-momentum conservation obviously prevents the Higgs boson decaying to two or three Higgs bosons! For the same reason, if the Higgs is light, it will lie below the required mass threshold for decay to pairs of heavier particles, such as W^+W^- or ZZ or top quarks ($m_t \sim 173$ GeV, in case you didn't know). This consideration must be balanced against the fact that the Higgs boson couplings to particles all grow with the mass of the particle. Thus, for a lightish Higgs (above about 10 GeV), decays to bottom quark pairs will dominate ($m_b \simeq 4.1$ GeV). But by the time the Higgs has become very heavy ($m_h \gtrsim 2m_W$), decays to W^+W^- and ZZ must dominate. Interestingly enough, the crossover does not occur near the mass threshold $m_h = 2m_W \sim 160$ GeV, but somewhat below, nearer $m_h \simeq 140$ GeV. The reason is that QFT allows the Higgs boson to decay to a W^+W^- or ZZ pair in which one of the gauge

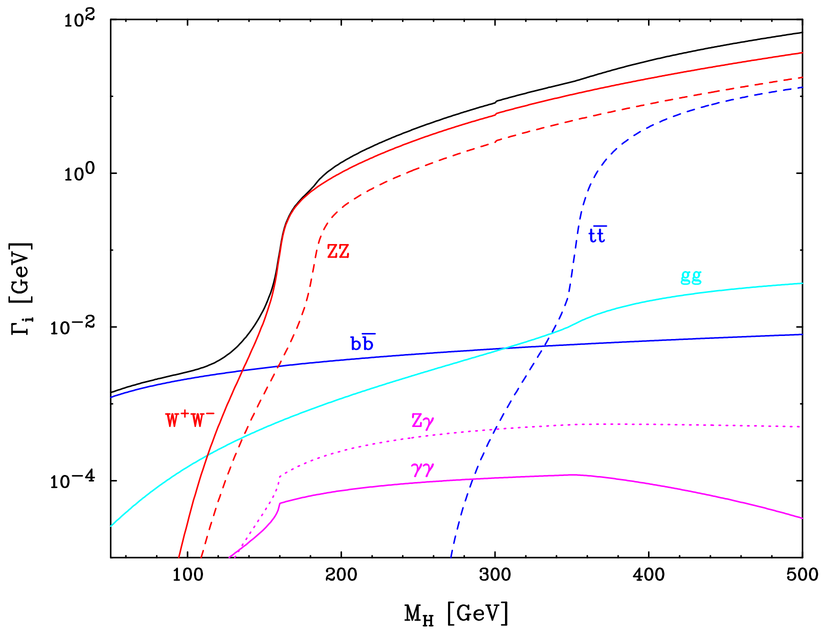


Figure 8. Higgs boson partial decay widths, from [9].

bosons is *virtual*, in that the mass-shell condition $E^2 = p^2 + m^2$ is not satisfied.[60] The virtual gauge boson then decays to real (on-shell) quarks or leptons by the usual gauge interaction. The partial decays widths and branching ratios, as a function of m_h, are shown in Figs. 8 and 9. Remarkably, at the point $m_h = 125$ GeV where the Higgs was found, we see comparable branching ratios to a variety of final states. This has the disadvantage of making it very difficult to discover the Higgs in the first place, since the number of Higgs decays in a single final state is suppressed compared to the fixed background of things that look like the Higgs decaying that way, but are not. But it has the great advantage that it makes it easy for us to make a variety of experimental tests that the Higgs boson that we claim to have discovered really does have the properties predicted in the SM. So far, the LHC data seems to confirm that.

There is one thing that may be bothering you in the Figures. They suggest that the Higgs has a small coupling to both a pair of photons $\gamma\gamma$ and to a pair of gluons gg. How can this be, when the Higgs carries neither colour nor electric charge? The answer is that loop Feynman diagrams, like those in Fig. 10 generate such couplings. Though small, they are very important for Higgs boson phenomenology at the LHC. Indeed, the LHC is a proton-proton collider. Protons are mostly made of up and down quarks, but the coupling of the Higgs boson to these is very small (it doesn't even appear in the Figures we just showed). But the proton also contains gluons, that bind the quarks together and these provide a way for us to produce the Higgs boson in pp collisions at the LHC. Similarly, the coupling to photons is small, but a pair of photons has a much lower background (from non-Higgs

[60] If you want to prove this for yourself, draw the Feynman diagram and show that the resulting amplitude is non-vanishing.

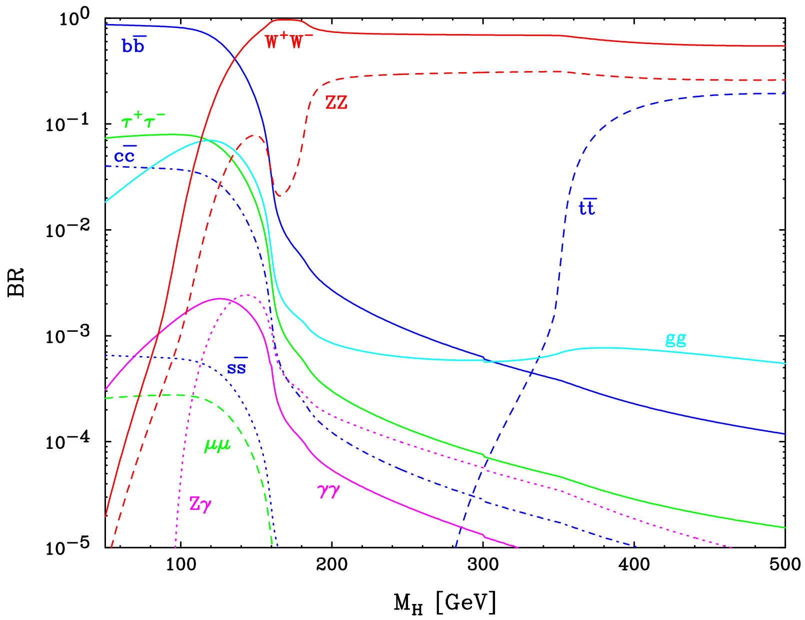


Figure 9. Higgs boson branching ratios, from [9].

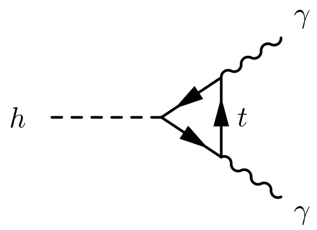


Figure 10. Feynman diagram with a loop of top quarks, contributing to the process $h \to \gamma\gamma$.

events) in LHC collisions than, say, a pair of b-quarks. So, even if you are experimentally-minded and think that theoretical physics is pointless, I hope you can appreciate that the nitty-gritty of theoretical QFT calculations was absolutely essential to the success of the LHC experiment. On a related note, I encourage you now to go back and work out the various Feynman rules for interactions involving the Higgs boson.

7 Renormalization

Congratulations! You now know (nearly) as much as anyone else about Nature, or at least the underlying particle physics. The state of the art is finding out all about the properties of the Higgs and you are *au fait* with it. Cock-a-hoop as we are, let's take our hubris to the next level and see if we can follow some of the theoretical speculation about what lies beyond the Standard Model. To do so, we need to delve a bit deeper into the seedy underbelly of QFT.

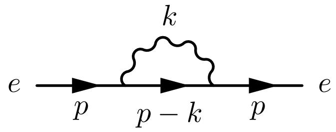
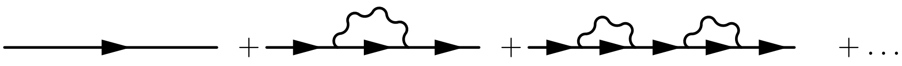


Figure 11. Loop contribution to the self-energy of the electron.



(7.3)

Figure 12. Contributions to the electron self-energy.

7.1 Ultraviolet divergences in quantum field theory

You are now in a position to write down the Feynman rules and compute the Feynman diagram for any process you like. Should you do so, you will, most likely, quickly encounter a problem. Most loop amplitudes that you calculate will be infinite. As an example, consider the one-loop correction to the electron propagator shown in Fig. 11. Referring back to the Feynman rules, we find

$$i\mathcal{M} = \int \frac{d^4k}{(2\pi)^4}\, \overline{u}(-ie\gamma^\mu)\frac{-ig_{\mu\nu}}{k^2}\frac{i(\not{p}-\not{k}-m)}{(p-k)^2-m^2}(-ie\gamma^\nu)u. \tag{7.1}$$

At large k, this goes like $\int d^4k\, \frac{\not{k}}{k^4}$, which is linearly divergent. In fact, the integral is only logarithmically divergent, because the integrand is odd under $k^\mu \to -k^\mu$, but it is divergent nevertheless.

These divergences crop up all over the place and they were a great source of insomnia for our predecessors. Eventually, they came up with a ruse for getting rid of them. Here's how it works in the example above. Call the divergent amplitude $i\Sigma$ and consider the sequence of diagrams shown in Fig. 12. We can sum them up to get

(7.2)

$$\frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}i\Sigma\frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}i\Sigma\frac{i}{\not{p}-m}i\Sigma\frac{i}{\not{p}-m} + \dots \tag{7.4}$$

$$= \frac{i}{\not{p}-m}\left(1 + i\Sigma\frac{i}{\not{p}-m} + \dots\right) \tag{7.5}$$

$$= \frac{i}{\not{p}-m}\left(1 - i\Sigma\frac{i}{\not{p}-m}\right)^{-1} \tag{7.6}$$

$$= \frac{i}{\not{p}-m+\Sigma}. \tag{7.7}$$

Thus Σ may be considered as an (infinite) shift of the mass parameter m in the Lagrangian. This would not pose a problem if m itself were chosen to be infinite, in just such a way that $m - \Sigma$ yields the measured electron mass of 511 keV.

This procedure of absorbing the divergences into the original parameters of the Lagrangian can only work if we are able to absorb all of the divergences in this way. Let's see if it has a chance of working. To do so, we need to do a bit of dimensional analysis. In units where $\hbar = c = 1$, this is easy, because we only have a dimension of energy or mass. So first let's figure out the dimensions of all the fields.

The action has the same dimensions as $\hbar$, so is dimensionless in our units. Since the 4-momentum corresponds to ∂_μ in these units, space and time both have (mass) dimension -1. The Lagrangian (density) must therefore have dimension 4, since $\int d^4x\mathcal{L}$ yields the dimensionless action. The field dimensions can then be figured out from the kinetic terms. Bosonic fields must have dimension one, since the kinetic term involves two derivatives. Fermions on the other hand must have dimension three-halves. You can then check that the mass parameters in the respective Lagrangians really do have dimensions of mass and that the gauge couplings are dimensionless.

This dimensional analysis enables us to quickly work out the degree of divergence of any Feynman diagram. We call it the *superficial degree of divergence*, D, because it may be that the real degree of divergence is smaller (*cf.* the log rather than linear divergence of the one-loop electron self-energy diagram in QED that we wrote down above). It is defined in the following way:

$$D = (\text{power of } k \text{ in numerator}) - (\text{power of } k \text{ in denominator}) \,, \tag{7.8}$$

where k stands for all loop momenta in the diagram. Note that the numerator accounts for the integration measures for all loops.

It is easy to calculate D. Consider a diagram with L loops, $F_{I,E}$ internal or external fermion propagators, $B_{I,E}$ internal or external boson propagators, and V vertices. We should also allow for interactions that involve derivatives of fields. Such interactions lead to vertices with explicit dependence on loop momenta. An example for such a (derivative) coupling is the 3-gluon coupling of QCD eq(6.23). Let's denote by P_j the power of loop momenta entering the vertex j. Then:

$$D = 4L - F_I - 2B_I + \sum_j P_j \,. \tag{7.9}$$

The coefficients of F_I and B_I in the equation above are the dimensions of the fermion and boson propagators, respectively (they are the same in any space-time dimension).

The above result can be cast in a much more useful form. As it turns out we can exclude all quantities depend on the internal structure of the diagram. First one can show that

$$L = F_I + B_I - V + 1. \tag{7.10}$$

The above equation can be easily understood: It states that the number of loop integrations (L) in a diagram equals the total number of propagators ($F_I + B_I$) minus the number of integrations undone with the help of delta functions (one delta function for each vertex, except for the overall delta function which does not depend on loop momenta).

Now let's think about the vertices. Each one comes from a dimension four term in the Lagrangian. If vertex j involves F_j and B_j fermionic and bosonic fields, together with a power P_j of the loop momenta, and its coupling constant has dimension $\dim(g_j)$ then the dimension of the vertex is:

$$4 = \frac{3}{2}F_j + B_j + P_j + \dim(g_j). \tag{7.11}$$

In the above equation we used that the dimension of the fermionic (bosonic) fields is 3/2 (1), respectively. Furthermore, since every internal propagator ends on two vertices and every external propagator lands on one vertex, it must be that

$$\sum_j F_j = 2F_I + F_E,\ \sum_j B_j = 2B_I + B_E, \tag{7.12}$$

where we sum over all vertices in the diagram. Combining the above results we get:

$$\begin{aligned} D &= 4L - F_I - 2B_I + \sum_j P_j \\ &= 4(F_I + B_I - V + 1) - F_I - 2B_I + \sum_j P_j \\ &= 4 - \frac{3}{2}F_E - B_E + \sum_j \left\{ \frac{3}{2}F_j + B_j + P_j - 4 \right\} \\ &= 4 - \frac{3}{2}F_E - B_E - \sum_j \dim(g_j)\,. \end{aligned} \tag{7.13}$$

This relation is most instructive: it tells us the superficial degree of divergence for fixed initial and final states depends only on the dimensions of couplings that appear. Moreover, if any coupling has negative mass dimension, we have no chance of carrying out the renormalization programme, since more and more divergences appear as we include more and more vertices in diagrams. Conversely, renormalization might work for theories like QED or the SM (where we only have couplings of positive or vanishing mass dimension), because diagrams get less and less divergent as they get more complicated.

This is not the same as saying that it does work, however. To prove renormalizability of the electroweak theory took a heroic effort by 't Hooft and Veltman. Heroic enough to win them the Nobel prize, the real breakthrough being a clever choice of gauge by the young 't Hooft.

Our arguments also tell us immediately why gravity cannot be included straightforwardly within the quantum gauge field theory framework. The classical action for gravity is the Einstein-Hilbert action

$$S = \frac{1}{M_P^2}\int d^4x\,\sqrt{-\mathrm{det} g_{\mu\nu}}R^\sigma_\sigma, \tag{7.14}$$

where g and R are the metric and Riemann tensors, respectively. This *is* a gauge theory (the symmetry being diffeomorphism invariance), but the coupling constant $\frac{1}{M_P^2}$ has negative mass dimension. The theory cannot be perturbatively renormalizable.

To summarize, analysis based solely on the superficial degree of divergence shows that a diagram diverges if $D > 0$; if $D = 0$ then it is only logarithmically divergent (which is typical for renormalizable theories) and finite if $D < 0$.

7.2 Non-renormalizable interactions and effective theories: the modern view

Even though the SM is renormalizable and the infinities can be swept away, this procedure hardly seems aesthetically attractive. Nowadays we have a rather different view of renormalizability. The problems appear because we tried to define the theory up to arbitrarily high energy (and this short distance) scales, way beyond those which we are able to probe in our current experiments. We would not have to worry about infinities at all if we imposed some large momentum cut-off, Λ, on the theory, beyond the reach of our experiments. But since there are then no infinities, even non-renormalizable theories make perfect sense, provided we understand that they come with a cut-off, Λ. This is called an *effective field theory.*

In fact, this should have been obvious all along and indeed it is the way we have always done physics: we build a theory which works on the scales probed by our current experiments, accepting that we may need to revise it once we are able to probe new scales. QFT (which, via loop diagrams, prevents us from simply ignoring the effect of physics at other scales) merely brought this issue into focus. Moreover, even in quantum physics we have long had concrete examples of this. Perhaps the best is Fermi's theory of the weak interaction, containing a four-fermion interaction to describe β decay. A four-fermion interaction has mass dimension six and so the coupling, G_F has mass dimension minus two. The theory, considered as a QFT, is non-renormalizable, but this presents no problems provided that we do not ask questions about what happens at mass scales higher than the cut-off, *c.* 100 GeV, which is set by the mass scale associated with G_F. Moreover, the cut-off that is present in Fermi's description can be seen as a strong hint that something interesting happens in weak interactions at scales around 100 GeV. As we have seen, that is indeed what happens – we discover that the four-fermion effective interaction arises from the exchange of W and Z gauge bosons having that mass. Given the complete electroweak theory, we can go back to Fermi's theory, by considering only energies below 100 GeV, for which we can 'integrate out' the W and Z.[61]

If there is nothing wrong with non-renormalizable theories, then why is the Standard Model renormalizable? A better way to phrase this is as follows. We could extend the Standard Model by adding non-renormalizable operators to it, whilst still maintaining gauge invariance (we will do exactly that when we consider neutrino masses in the next Section). The fact that the SM gives a good description of all physics seen so far translates into the statement that the mass scale (a.k.a. the cut-off) associated with these higher-dimensional

[61]This procedure is called integrating out because in the path integral formalism of QFT it corresponds to doing the path integral with respect to the fields W and Z.

operators must be very large, meaning that the new physics (beyond the SM) that they provide an effective description of must be a long way out of our reach. No one knows why this must be the case and indeed there are strong (but indirect) arguments for why it should not be the case. Unfortunately, so far, experiments like the LHC indicate that the SM provides a very good description of physics at energy scales within reach.

8 Beyond the Standard Model

We now move on to consider some aspects of physics beyond the SM. With one exception, this is speculative, in that we have no concrete experimental evidence for it. We start with the exception.

8.1 Neutrino masses

The story of neutrino masses goes back several decades, beginning with the discovery in the 1960s that the flux of electron neutrinos from the sun was less than half of what was predicted by models of the nuclear reactions that fuel the sun. One way to resolve the deficit is to postulate that neutrinos can undergo oscillations between the different flavours, in much the same way as neutral mesons. In order for neutrino oscillations to be physical, there must be some distinguishing feature between the different neutrino generations. Since they have identical gauge couplings, the most obvious distinguishing feature is a neutrino mass, which may differ between the generations.

Despite many corroborating experimental hints, the hypothesis of solar neutrino oscillations into other flavours was not confirmed beyond doubt until 2001, by the Sudbury Neutrino Observatory. Whilst we do not have a direct measurement of the masses (though a bound on the sum of around an eV may be inferred from cosmological data), we do know that the two mass-squared differences are around 10^{-3} and 10^{-5} eV^2.

The challenge then, is to give a theoretical description of neutrino masses and, hopefully, to explain their smallness (in comparison, the lightest charged particle, the electron, has mass 511 keV). The renormalizable Standard Model cannot account for massive neutrinos. However, it turns out that the Standard Model does provide an elegant description of neutrino masses, when we consider it as a non-renormalizable, effective field theory.

Indeed, consider the Lorentz-invariant operators of dimension greater than four that respect the $SU(3) \times SU(2) \times U(1)$ gauge symmetry and hence could be added to the SM Lagrangian. The low-energy effects of the operators will be largest for the operators of lowest dimension. The lowest dimension greater than four is five and we find exactly one dimension five operator that can be added to the Lagrangian. It takes the form

$$\mathcal{L} \supset -\frac{1}{\Lambda}(l_L^T H^{c*} C (H^{c*})^T l_L) + h.\ c. \tag{8.1}$$

where $\frac{1}{\Lambda}$ is the coupling (written so that Λ has dimensions of mass) and where $+h.\ c.$ instructs us to add the Hermitian conjugate (so that the Lagrangian comes out to be real). This is an interaction involving two Higgs fields and two lepton doublets, but when the

Higgs field gets a VEV, we find a Majorana mass term for the neutrino of the form (6.39):

$$\mathcal{L} \supset -\frac{v^2}{2\Lambda}\nu_L^T C \nu_L + h.\ c. \tag{8.2}$$

The neutrino mass comes out to be $m = \frac{v^2}{\Lambda}$, which is in itself very interesting: we can explain the small mass of neutrinos $\sim 10^{-1}$ eV if Λ is very large, $\sim 10^{14}$ GeV. Why is this interesting? Recall from our discussion of effective field theories above that Λ corresponds to the scale at which the effective theory breaks down and must be replaced by a more complete description of the physics. The smallness of neutrino masses is indirectly telling us that the SM could provide a good description of physics all the way up to a very high scale of $\sim 10^{14}$ GeV. In comparison, the LHC probes energies around 10^3 GeV. Moreover, our effective field theory approach tells that neutrino masses are expected to be the first sign of deviation from the SM that we observe, in the sense that they are generated by the operator of lowest dimension: if all the higher-dimension operators are suppressed by the same mass scale (which, by the way, they need not be), then the neutrino mass operator above will have the largest effect at the relatively low energies at which we perform our experiments.

It is interesting to speculate what the new physics might be. One simple possibility is to add a new particle to the SM called a right-handed neutrino. This is simply a right-handed fermion which is completely neutral with respect to the SM gauge group. The most general, renormalizable Lagrangian then includes the extra terms

$$\mathcal{L} \supset \lambda^\nu \overline{l_L} H^c \nu_R - M \nu_R^T C \nu_R + h.\ c. \tag{8.3}$$

The first term is simply a generalization of the Yukawa couplings (6.61) and the second is a Majorana mass term (6.39). We can now identify two qualitatively different scenarios reproducing the observed small neutrino masses. The first way would be to allow the Yukawa coupling to be of order unity; then a small neutrino mass could only be accomplished by choosing the Majorana mass M around $\sim 10^{14}$GeV. Then, diagonalizing the mass matrix for ν_L and ν_R one finds one light eigenstate with mass around 0.1 eV and one heavy state around 10^{14} GeV. This is often called the *see-saw mechanism*. We could then integrate out the heavy state (which is mostly ν_R) to obtain the effective theory description containing only ν_L given above. The second scenario is to imagine that the Majorana mass term is forbidden. One could do this example by declaring that the theory should be invariant under a global phase rotation of all leptons, including ν_R. This corresponds to insisting on conservation of lepton number and is enough to forbid the Majorana mass term.[62] Then neutrino masses come from the Yukawa term alone, and both left- and right- handed neutrinos are light. In fact, they are degenerate, since they together make up a Dirac fermion. Notice that in this second picture we cannot integrate out a heavy neutrino to obtain an effective theory as in (8.1). This is an important caveat: the scale $\Lambda \sim 10^{14}$ GeV indicated by (8.1) is only an *upper bound* for the scale at which new physics should appear.

[62]It is important to note that this is very different from what happens in the SM. There we find that once we insist on the gauge symmetry, lepton (and baryon) numbers are automatically conserved by all operators of dimension four or less. They are called *accidental symmetries* of the theory.

8.2 The gauge hierarchy problem

In our modern view of quantum field theory as an effective field theory, non-renormalizable operators are not a problem. We recognize that they represent the effects of new physics at high energy scales. They are suppressed by the scale Λ of new physics. Provided that Λ is rather large, they give small contributions that we can take into account using the tools of perturbation theory.

But this interpretation shows that there is now a problem with the *renormalizable* operators. Indeed, in our enlightened understanding, we take the view that the physics at our low scale is determined by the physics at higher scales, which corresponds to some more fundamental theory. But then all mass scales in our current theory should be set by the higher scale theory. This includes not only the operators of negative mass dimension, but also the operators of positive mass dimension. Concretely, in the SM there is exactly one coupling of positive mass dimension: the mass parameter, μ of the Higgs field. Why on Earth does this have a value of around 100 GeV when we believe that it is ultimately determined by a more fundamental theory at a much higher scale? We certainly have evidence for the existence of physics at higher scales: neutrino masses indicate new physics at 10^{14} GeV and the mass scale associated with gravity is the Planck mass, 10^{19} GeV.

This problem of how to explain the hierarchy between the scale of weak interactions and other scales believed to exist in physics is called the *gauge hierarchy problem.* It is compounded by the fact that QFT has loops which are sensitive to arbitrarily high scales. This may all sound rather abstract to you, but I assure you that the problem can be viewed concretely. Take a theory with two scalar fields. One like the Higgs, should be set to be light. Make the other one heavy. Then compute the corrections to the mass of the light scalar from loop diagrams containing the heavy scalar. You will find that the mass of the light scalar gets lifted up to the mass of the heavy one.

Several beautiful solutions to this hierarchy problem have been put forward, involving concepts like *supersymmetry, strong dynamics, and extra dimensions.* They all involve rich dynamics (usually in the form of many new particles) at the TeV scale. We are looking for them at the LHC, but so far our searches have come up empty-handed.

8.3 Grand unification

There is yet another compelling hint for physics beyond the SM. It turns out that one consequence of renormalization is that the parameters of the theory must be interpreted as being dependent on the scale at which the theory is probed. I'm afraid you will have to read a QFT textbook to see why. It turns out that the QCD coupling gets smaller as the energy scale goes up (this is why we are able to do QCD perturbation theory for understanding LHC physics as the TeV scale, whilst needing non-perturbative insight in order to able to prove confinement of quarks into hadrons at the GeV scale), while the electroweak couplings g and g' get bigger. Remarkably, if one extrapolates far enough, one finds that all three couplings are nearly[63] equal[64] at a very high scale, *c.* 10^{15} GeV. Could it be that, just as electromagnetism and the weak force become the unified electroweak force at the 100 GeV scale, all three forces become unified at 10^{15} GeV?

The fact that the couplings seem to become equal is a hint that we could try to make all three groups in $SU(3) \times SU(2) \times U(1)$ subgroups of one big group, with a single coupling constant. The group $SU(5)$ is an obvious contender and in fact it is the smallest one. How does $SU(3) \times SU(2) \times U(1)$ fit into $SU(5)$? Consider $SU(5)$ in terms of its defining representation: 5×5 unitary matrices with unit determinant acting on 5-dimensional vectors. We can get an $SU(3)$ subgroup by considering the upper-left 3×3 block and we can get an independent $SU(2)$ subgroup from the lower right 2×2 block. There is one more Hermitian, traceless generator that is orthogonal to the generators of these two subgroups: it is $T = \sqrt{\frac{3}{5}}\mathrm{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$, with the usual normalization. Our goal will be to try to identify this with the hypercharge $U(1)$ in the SM. To do so, we first have to work out how the SM fermions fit into reps of $SU(5)$. To do so, it is most convenient to write the right-handed fermions of the SM as charge conjugates of left-handed fermions. Then the multiplets are $q_L, u^c_L, d^c_L, l, e^c_L$, with the charges as given in Table 1, except that we must take the conjugate reps for the multiplets with a 'c'.

Before going further, let's do a bit of basic $SU(N)$ representation theory. The defining, or *fundamental*, representation is an N-dimensional vector, acted on by $N \times N$ matrices. We can write the action as $\alpha^i \to U^i_j \alpha^j$, with the indices i, j enumerating the N components. Given this rep, we can immediately find another by taking the complex conjugate. This is called the antifundamental rep. It is convenient to denote an object which transforms according to the antifundamental with a downstairs index, β_i. Why? The conjugate of $\alpha^i \to U^i_j \alpha^j$ is $\alpha^{*i} \to U^{*i}_j \alpha^{*j} = U^{\dagger j}_i \alpha^{*j}$. So if we define things that transform according to the conjugate with a downstairs index, we can write $\beta_i \to U^{\dagger j}_i \beta_j$. The beauty of this is that $\alpha^i \beta_i \to \alpha^j U^i_j U^{\dagger k}_i \beta_k = \alpha^j \delta^k_j \beta_k = \alpha^k \beta_k$, where we used $UU^\dagger = 1$. Thus when we contract an upstairs index with a downstairs index, we get a singlet. This is, of course, much like what happens with μ indices for Lorentz transformations. Note that the Kronecker delta, δ^k_j, naturally has one up index and one down and it transforms as $\delta^l_i \to U^i_k \delta^k_j U^{\dagger j}_l$. But $UU^\dagger = 1 \implies \delta^l_i \to \delta^l_i$ and so we call δ^l_i an *invariant tensor* of $SU(N)$. Note, furthermore, that there is a second invariant tensor, namely $\epsilon_{ijk\ldots}$ (or $\epsilon^{ijk\ldots}$) , the totally antisymmetric tensor with N indices. Its invariance follows from the relation $\det U = 1$.

These two invariant tensors allow us to find all the irreps $SU(N)$ from (tensor) products of fundamental and antifundamental representations. The key observation is that tensors which are symmetric or antisymmetric in their indices remain symmetric or antisymmetric under the group action (exercise), so cannot transform into one another. So to reduce a generic product rep into irreps, one can start by symmetrizing or antisymmetrizing the indices. This doesn't complete the process, because one can also contract indices using

[63] Nearly enough to be impressive, but not quite. The discrepancy might be resolved by extra, supersymmetric particles, however.

[64] At the moment, this is an trivial statement: the normalization of g' is arbitrary and can always be chosen to make all three couplings meet at the same point. But we will soon be able to give real meaning to it.

either of the invariant tensors, which also produces objects which only transform among themselves (exercise).

Let's see how it works for some simple examples, reproducing some results which were probably previously introduced to you as dogma. Start with $SU(2)$, which is locally equivalent to $SO(3)$ and whose representation theory is known to you as 'The theory of angular momentum in quantum mechanics'. The fundamental rep is a 2-vector (a.k.a. spin-half); call it α^j. Via the invariant tensor ϵ_{ij} this can also be thought of as an object with a downstairs index, *viz.* $\epsilon_{ij}\alpha^j$, meaning that the doublet and anti-doublet are *equivalent* representations (the ϵ_{ij} also gives rise to the peculiar minus signs that appear, usually without explanation, in introductory QM courses). So all tensors can be thought of as having indices upstairs, and it remains only to symmetrize (or antisymmetrize). Take the product of two doublets for example. We decompose $\alpha^i\beta^j = \frac{1}{2}(\alpha^{(i}\beta^{j)} + \alpha^{[i}\beta^{j]})$, where we have explicitly (anti)symmetrized the indices. The symmetric object is a triplet irrep (it has (11), (22), and (12) components), while the antisymmetric object is a singlet (having only a [12] component). We write this decomposition as $2 \times 2 = 3 + 1$ and you will recognize it from your studies of the Helium (two-electron) atom.

The representation theory of $SU(3)$ is not much harder. The fundamental is a triplet and the anti-triplet is inequivalent.[65] The product of two triplets contains a symmetric sextuplet and an antisymmetric part containing three states. We can use the invariant tensor ϵ_{ijk} to write the latter as $\epsilon_{ijk}\alpha^{[i}\beta^{j]}$, meaning that it is equivalent to an object with one index downstairs, *viz.* an anti-triplet. Thus the decomposition is $3 \times 3 = 6 + \overline{3}$. On the other hand, we cannot symmetrize the product of a 3 and a $\overline{3}$, because the indices are of different type. The only thing we can do is to separate out a singlet obtained by contracting the two indices with the invariant tensor δ^i_j. Thus the decomposition is $\alpha^i\beta_j = \left(\alpha^i\beta_j - \frac{1}{3}\alpha^k\beta_k\delta^i_j\right) + \frac{1}{3}\alpha^k\beta_k\delta^i_j$, or $3 \times \overline{3} = 8 + 1$. The 8 is the adjoint rep. Again, you have probably seen this all before under the guise of 'the eightfold way'.

For $SU(5)$, things are much the same. The only reps we shall need are the smallest ones, namely the (anti)fundamental $5(\overline{5})$ and the 10 which is obtained from the antisymmetric product of two 5s.

Now let's get back to grand unified theories. We'll try to do the dumbest thing imaginable which is to try to fit some of the SM particles into the fundamental five-dimensional representation of $SU(5)$. I hope you can see that this breaks up into a piece (the first three entries of the vector) that transform like the fundamental (triplet) rep of $SU(3)$ and the singlet of $SU(2)$ and a piece (the last two entries of the vector) which does the opposite. For this to work the last two entries would have to correspond to l_L (since this is the only SM multiplet which is a singlet of $SU(3)$ and a doublet of $SU(2)$), in which case the hypercharge must be fixed to be $Y = -\sqrt{\frac{5}{3}}T$. Then the hypercharge of the first three entries is $+\frac{1}{3}$. This is just what we need for d^c_L, except that d^c_L is a colour anti-triplet rather than a triplet. But we can fix it up by instead identifying $Y = +\sqrt{\frac{5}{3}}T$ and then identifying

[65] It is inequivalent, because we cannot convert one to the other using ϵ_{ij}, which has been replaced by ϵ_{ijk}.

(d_L^c, l_L) with the *anti-fundamental* rep of $SU(5)$.[66]

What about the other SM fermions? The next smallest rep of $SU(5)$ is ten dimensional. It can be formed by taking the product of two fundamentals and then keeping only the antisymmetric part of the product. But since we now know that under $SU(5) \to SU(3) \times SU(2) \times U(1)$, $5 \to (3, 1, -\frac{1}{3}) + (1, 2, +\frac{1}{2})$, you can immediately deduce[67] that $10 \to (3, 2, +\frac{1}{6}) + (\overline{3}, 1, -\frac{2}{3}) + (1, 1, +1)$. These are precisely q_L, u_L^c, and e_L^c.

That things fit in this way is nothing short of miraculous. Let's now justify our statement about the couplings meeting at the high scale. The $SU(5)$ covariant derivative is

$$D_\mu = \partial_\mu + ig_{\mathrm{GUT}} A_\mu \supset ig_{\mathrm{GUT}} \left(W_\mu^3 T^3 + i\sqrt{\frac{3}{5}} Y B_\mu \right), \tag{8.4}$$

so unification predicts that $\tan\theta_W = \frac{g}{g'} = \sqrt{\frac{3}{5}} \implies \sin^2\theta_W = \frac{3}{8}$. *This* is the relation which is observed to hold good (very nearly) at the unification scale.

There is another GUT which is based on the group $SO(10)$. This is perhaps even more remarkable, in that the fifteen states of a single SM generation fit into a 16 dimensional rep (it is in fact a spinor) of $SO(10)$. You might be thinking that this doesn't look so good, but — wait for it — the sixteenth state is a SM gauge singlet and plays the rôle of a right handed neutrino. It almost looks too good to be true.

9 Afterword

Particle physics has had a tremendous winning streak. In a century or so, we have come an enormously long way. These lecture notes are, in a sense, a condensation of that.

Despite the glorious successes of the past, it is fair to say that the golden age of particle physics is happening right now. Not only have we just discovered the Higgs boson (and are busily checking that it conforms to the predictions of the SM), but we have strong indications that there should be physics beyond the SM and the LHC and other experiments are comprehensively searching for it. So far, nothing has been found, but now the LHC is being upgraded to run at even higher energies.

Who knows what lies around the corner? If your interest is piqued by what I have discussed, then I wholeheartedly encourage you to begin a proper study of particle physics in general, and gauge field theory, in particular. Maybe it will be you who makes the next big breakthrough ...

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[66] This discussion hinges on the group theoretical fact that a representation and its complex conjugate are inequivalent, in general.

[67] At least you can if you know a bit of group theory, for example that the antisymmetric product of two 2s of $SU(2)$ is a singlet and similarly that the antisymmetric product of two 3s of $SU(3)$ is a $\overline{3}$.

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