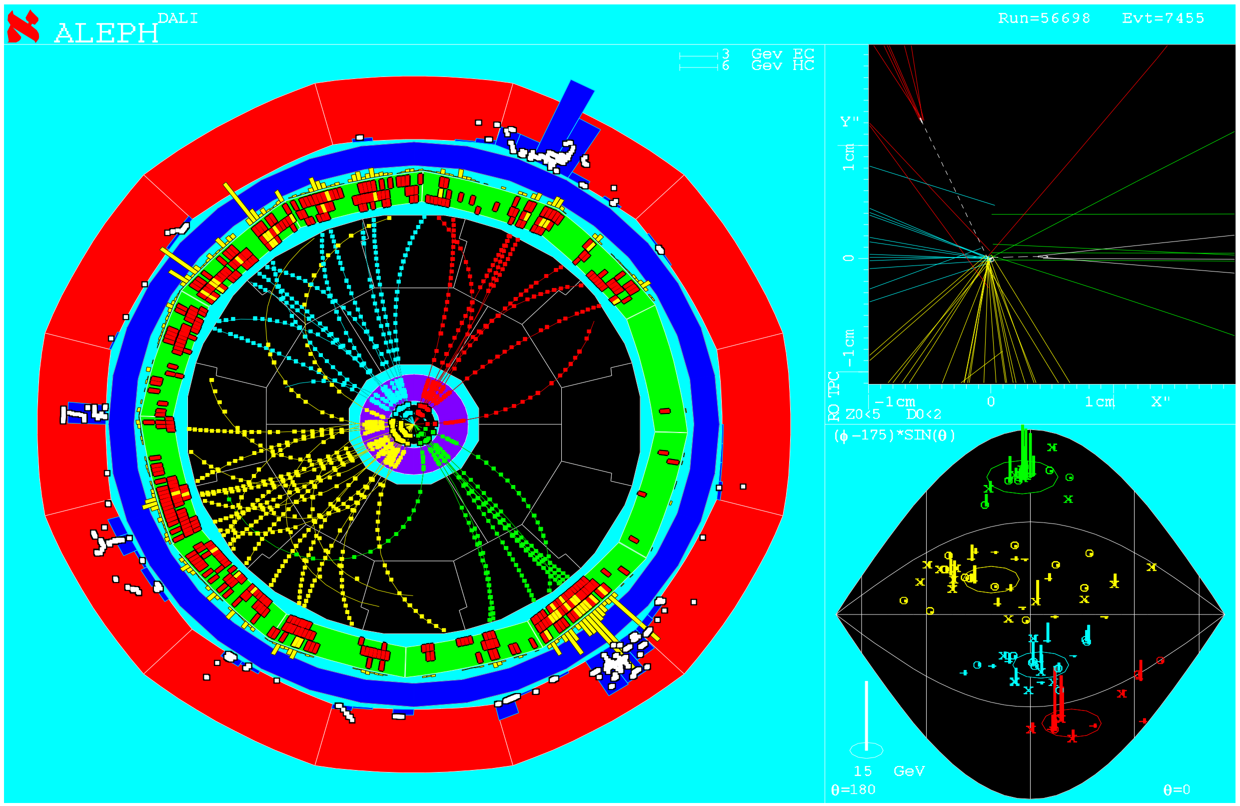
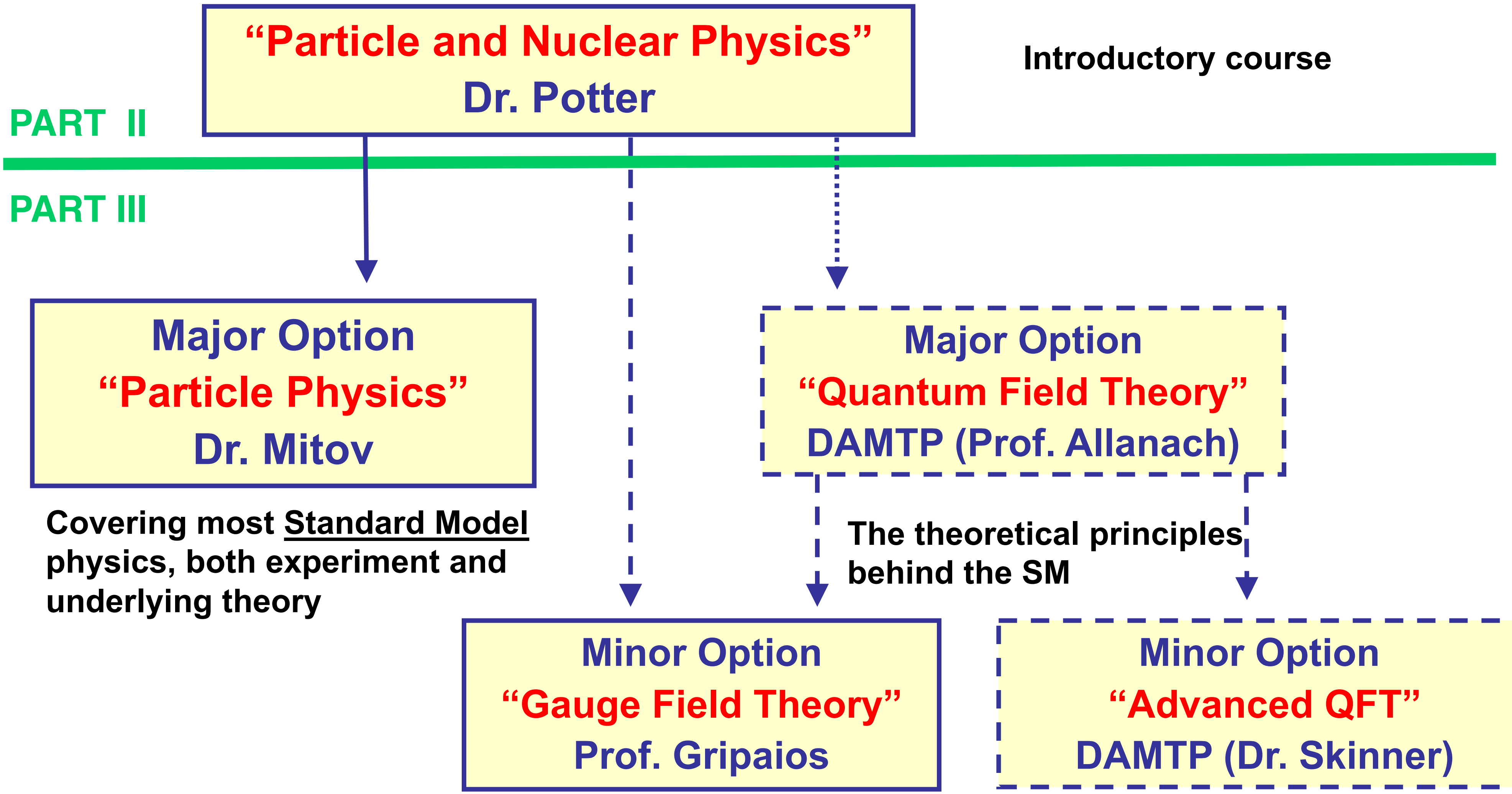
Particle Physics

Dr. Alexander Mitov



Handout 1 : Introduction

Cambridge Particle Physics Courses



Course Synopsis

★ Will concentrate on the modern view of particle physics with the emphasis on how theoretical concepts relate to recent experimental measurements

★ Aim: by the end of the course you should have a good understanding of both aspects of particle physics

Preliminaries

Web-page: http://www.precision.hep.phy.cam.ac.uk/people/mitov/teaching/

- All course material, old exam questions, corrections, interesting links etc.
- Detailed answers will posted after the supervisions (password protected)

Format of Lectures/Handouts:

- First part of each handout contains the "the course material".
- Some handouts contain additional theoretical background in non-examinable appendices at the end.
- Please let me know of any corrections: adm74@cam.ac.uk

Books:

★ The handouts are fairly complete, however there a number of decent books:

- "Modern Particle Physics" Mark Thomson BASED ON THIS COURSE!
- "Particle Physics", Martin and Shaw (Wiley): fairly basic but good.
- "Introductory High Energy Physics", Perkins (Cambridge): slightly below level of the course but well written.
- "Introduction to Elementary Physics", Griffiths (Wiley): about right level but doesn't cover the more recent material.
- "Quarks and Leptons", Halzen & Martin (Wiley): good graduate level textbook (slightly above level of this course).

★ Before we start in earnest, a few words on units/notation and a very brief "Part II refresher"…

Preliminaries: Natural Units

- **S.I. UNITS:** **kg m s** are a natural choice for "everyday" objects

  e.g. $M_{\text{(Dr. Mitov)}}$ ~ 90 kg ~ O(1) kg

- not very natural in particle physics
- instead use **Natural Units** based on the language of particle physics
  - **From Quantum Mechanics** - the unit of action : $\hbar$
  - **From relativity** - the speed of light: c
  - **From Particle Physics** - unit of energy: GeV (1 GeV ~ proton rest mass energy)

★ **Units become (i.e. with the correct dimensions):**

Energy	GeV	Time	$(\mathrm{GeV}/\hbar)^{-1}$
Momentum	GeV/c	Length	$(\mathrm{GeV}/\hbar c)^{-1}$
Mass	GeV/c^2	Area	$(\mathrm{GeV}/\hbar c)^{-2}$

★ **Use units in which :** '$\boxed{\hbar = c = 1}$' (though note both = signs are "wrong" !)

- Now all quantities expressed in powers of GeV

Energy	GeV	Time	GeV^{-1}
Momentum	GeV	Length	GeV^{-1}
Mass	GeV	Area	GeV^{-2}

To convert back to S.I. units, need to restore missing factors of $\hbar$ and c . Tip:

$\hbar c$ = 197 MeV fm

Preliminaries: Heaviside-Lorentz Units

- **Electron charge <u>defined</u> by Force equation:** $F = \dfrac{e^2}{4\pi\varepsilon_0 r^2}$, so:

$$[e] = [F4\pi\varepsilon_0 r^2]^{\frac{1}{2}}$$
$$= [\,\varepsilon_0 (FL) L\,]^{\frac{1}{2}}$$
$$= [\varepsilon_0 EL]^{\frac{1}{2}} = [\varepsilon_0 ETL/T]^{\frac{1}{2}} = [\varepsilon_0 \hbar\, c]^{\frac{1}{2}}$$

(E, L and T standing for an energy, length and time, respectively)

(recall [ET]=[$\hbar$])

This motivates, for problems involving electric charge, units in which: $\boxed{\varepsilon_0 = 1}$

e.g. now: $F \longrightarrow \dfrac{e^2}{4\pi r^2}$.

- Since $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}} = 1$, $\boxed{\mu_0 = 1}$ too!

- Therefore, in Heaviside-Lorentz units:

$$\boxed{\hbar = c = \varepsilon_0 = \mu_0 = 1}$$

(i.e. measure ang mom in `hbar's, speed in `c's, permittivities in `epsilon0's, and permeabilities in `mu0's)

★ **Unless otherwise stated, Natural Units are used throughout these handouts,** $E^2 = p^2 + m^2$, $\vec{p} = \vec{k}$, **etc.**

Review of The Standard Model

Particle Physics is the study of:

★ **MATTER: the fundamental constituents of the universe**
- the elementary particles

★ **FORCE: the fundamental forces of nature, i.e. the interactions between the elementary particles**

Try to categorise the PARTICLES and FORCES in as simple and fundamental manner possible

★ **Current understanding embodied in the STANDARD MODEL:**

- **Forces between particles due to exchange of particles**
- **Consistent with most experimental data !**
- **Does not account for Dark Matter**
- **But it is just a "model" with many unpredicted parameters, e.g. particle masses.**
- **As such it is not the ultimate theory (if such a thing exists), there are many mysteries.**

Matter in the Standard Model

★ In the Standard Model the fundamental "matter" is described by point-like spin-1/2 fermions

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	–1	0.0005	d	–1/3	0.3
	ν_1	0	≈0	u	+2/3	0.3
Second Generation	μ^-	–1	0.106	s	–1/3	0.5
	ν_2	0	≈0	c	+2/3	1.5
Third Generation	τ^-	–1	1.77	b	–1/3	4.5
	ν_3	0	≈0	t	+2/3	175

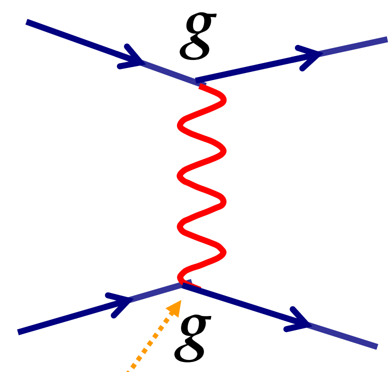
The masses quoted for the quarks are the "constituent masses", i.e. the effective masses for quarks confined in a bound state

- In the SM there are <u>three generations</u> – the particles in each generation are copies of each other differing <u>only</u> in mass. (not understood why three).
- The neutrinos are much lighter than all other particles (e.g. ν_1 has m<3 eV) – we now know that neutrinos have non-zero mass (don't understand why so small)

Forces in the Standard Model

★ Forces mediated by the exchange of spin-1 Gauge Bosons

Force	Boson(s)	J^P	m/GeV
EM (QED)	Photon γ	1^-	0
Weak	$W^\pm$ / Z	1^-	80 / 91
Strong (QCD)	8 Gluons g	1^-	0
Gravity (?)	Graviton?	2^+	0



- Fundamental interaction strength is given by charge g.
- Related to the **dimensionless** coupling "constant" α

e.g. QED $$g_{em} = e = \sqrt{4\pi\alpha\varepsilon_0\hbar c}$$

★ In Natural Units $$g = \sqrt{4\pi\alpha}$$

(both g and α are dimensionless, but g contains a "hidden" $\hbar c$)

★ Convenient to express couplings in terms of α which, being genuinely dimensionless does not depend on the system of units (this is not true for the numerical value for e)

Standard Model Vertices

★ Interaction of **gauge bosons** with **fermions** described by SM <u>vertices</u>

★ Properties of the **gauge bosons** and **nature of the interaction** between the bosons and fermions determine the properties of the interaction

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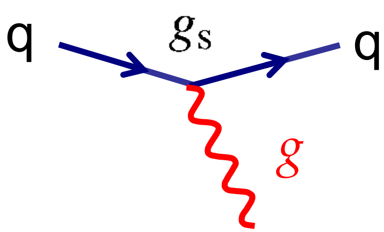
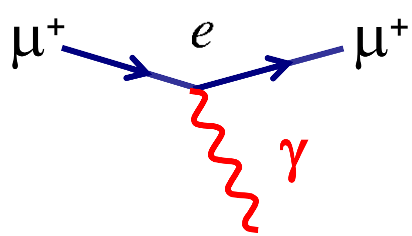


Only quarks

Never changes flavour

$$\alpha_S \sim 1$$

EM



All charged fermions

Never changes flavour

$$\alpha \simeq 1/137$$

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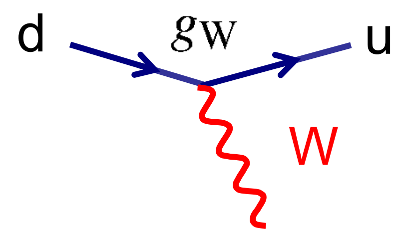

All fermions

<u>Always</u> changes flavour

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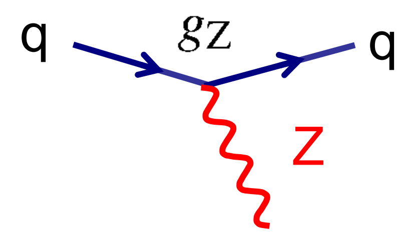


All fermions

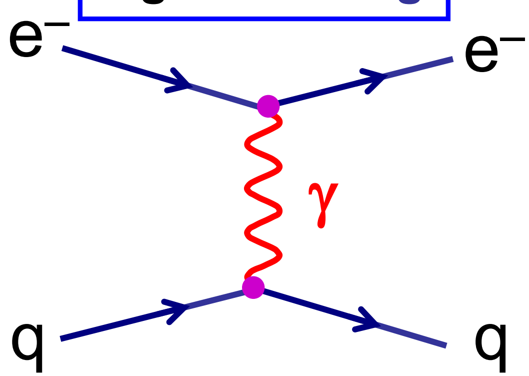
Never changes flavour

$$\alpha_{W/Z} \sim 1/40$$

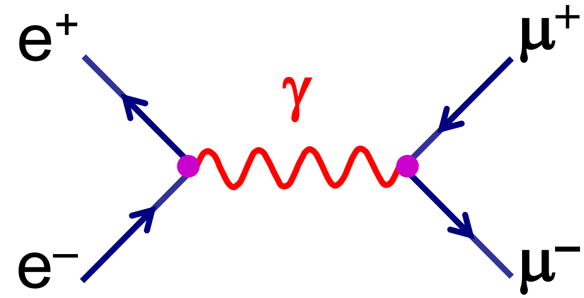
Feynman Diagrams

★ **Particle interactions described in terms of Feynman diagrams**

e.g. scattering

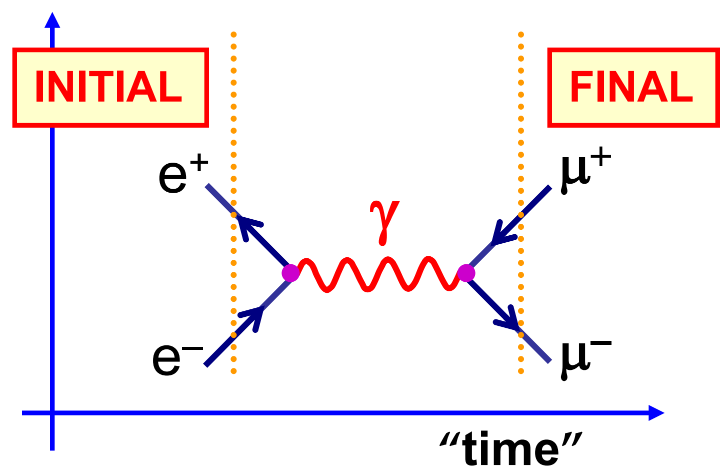


e.g. annihilation



★ **IMPORTANT POINTS TO REMEMBER:**

- **"time" runs from left – right, only in sense that:**
  - **LHS of diagram is initial state**
  - **RHS of diagram is final state**
  - **Middle is "how it might have happened"**
- **anti-particle arrows in –ve "time" direction**
- **Energy, momentum, angular momentum, etc. conserved at all interaction vertices**
- **All intermediate particles are "virtual"** i.e. $E^2 \neq |\vec{p}|^2 + m^2$ **(handout 3)**



Special Relativity and 4-Vector Notation

- Will use 4-vector notation with p^0 as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\}$$ (contravariant)

$$p_\mu = g_{\mu\nu} p^\nu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\}$$ (covariant)

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, usually deal with relativistic particles. <u>**Require**</u> all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.

$$p^\mu p_\mu = E^2 - p^2 = m^2$$ **Invariant mass**

$$x^\mu p_\mu = Et - \vec{p}.\vec{r}$$ **Phase**

- A few words on **NOTATION**

Four vectors written as either: p^μ or p

Four vector scalar product: $p^\mu q_\mu$ or $p.q$

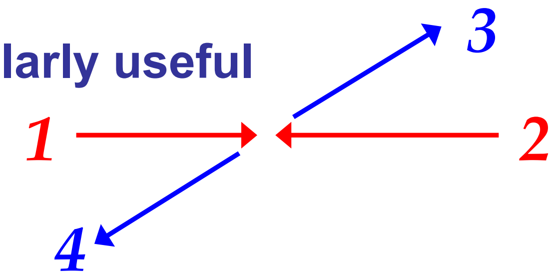
Three vectors written as: $\vec{p}$

Quantities evaluated in the **centre of mass frame**: $\vec{p}^*, p^*$ etc.

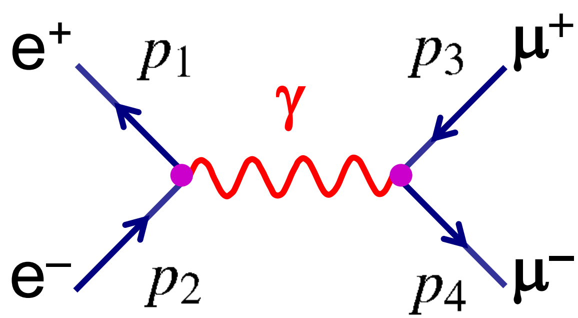
Mandelstam s, t and u

★ In particle scattering/annihilation there are three particularly useful **Lorentz Invariant** quantities: **s**, **t** and **u**

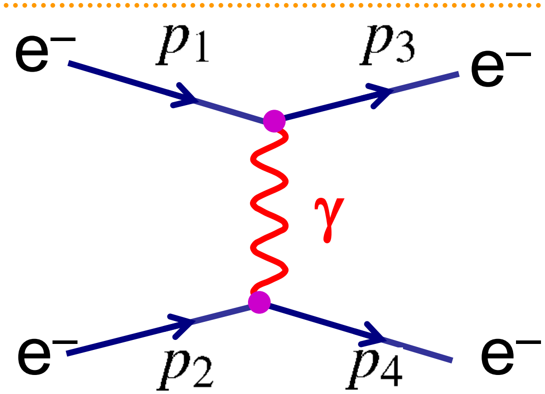
★ Consider the scattering process $1 + 2 \rightarrow 3 + 4$



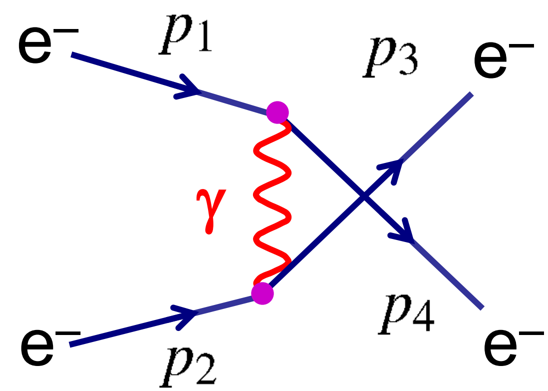
★ (Simple) Feynman diagrams can be categorised according to the four-momentum of the exchanged particle



s-channel



t-channel



u-channel

•Can define **three** kinematic variables: **s**, **t** and **u** from the following four vector scalar products **(squared four-momentum of exchanged particle)**

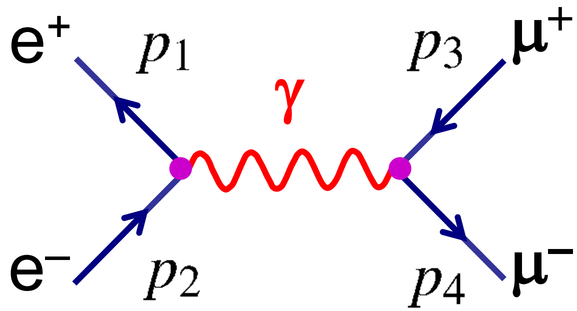
$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

Example: Mandelstam s, t and u

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

Note: $$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$ **(Question 1)**

★ **e.g. Centre-of-mass energy, s:**



$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

- **This is a scalar product of two four-vectors ➡ Lorentz Invariant**
- **Since this is a L.I. quantity, can evaluate in any frame. Choose the most convenient, i.e. the centre-of-mass frame:**

$$p_1^* = (E_1^*, \vec{p}^*) \quad p_2^* = (E_2^*, -\vec{p}^*)$$

➡ $$s = (E_1^* + E_2^*)^2$$

★ **Hence $\sqrt{s}$ is the total energy of collision in the centre-of-mass frame**

From Feynman diagrams to Physics

Particle Physics = Precision Physics

★ Particle physics is about building fundamental theories and testing their predictions against precise experimental data

- Dealing with fundamental particles and can make very precise theoretical predictions – not complicated by dealing with many-body systems
- Many beautiful experimental measurements
  ➡ precise theoretical predictions challenged by precise measurements
- For all its flaws, the Standard Model describes all experimental data !
  This is a (the?) remarkable achievement of late 20th century physics.

Requires understanding of theory and experimental data

★ Part II : Feynman diagrams mainly used to describe how particles interact

★ Part III:
- ♦ will use Feynman diagrams and associated Feynman rules to perform calculations for many processes
- ♦ hopefully gain a fairly deep understanding of the Standard Model and how it explains all current data

Before we can start, need calculations for:

- Interaction cross sections;
- Particle decay rates;

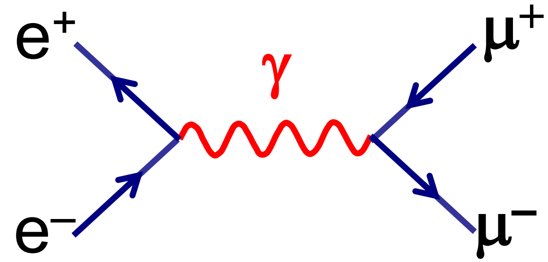
The first five lectures

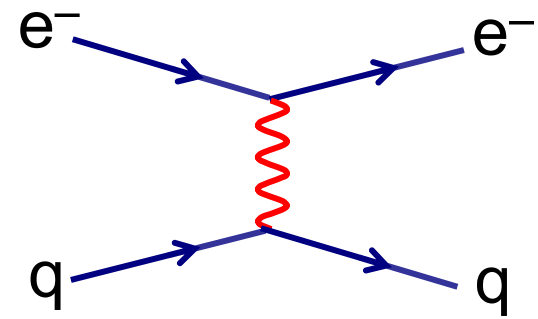
★ **Aiming towards a proper calculation of decay and scattering processes**

Will concentrate on:

- $e^+e^- \rightarrow \mu^+\mu^-$
- $e^- q \rightarrow e^- q$

($e^- q \rightarrow e^- q$ to probe proton structure)





⅄ **Need <u>relativistic</u> calculations of particle decay rates and cross sections:**

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

⅄ **Need <u>relativistic</u> treatment of spin-half particles:**

Dirac Equation

⅄ **Need <u>relativistic</u> calculation of interaction Matrix Element:**

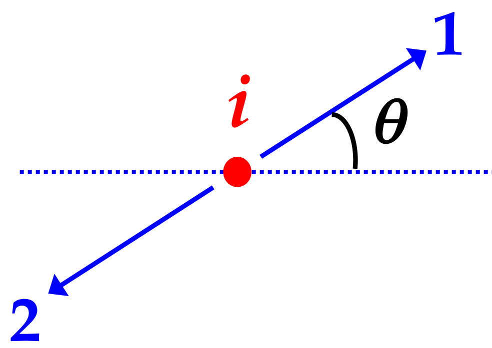
Interaction by particle exchange and Feynman rules

+ and a few mathematical tricks along, e.g. the Dirac Delta Function

Start with single particle decay rate and work up

- Consider the two-body decay

$$i \rightarrow 1 + 2$$



- Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the particles (Born approximation):

$$\psi_1 = N e^{i(\vec{p}.\vec{r} - Et)} \qquad (\vec{k}.\vec{r} = \vec{p}.\vec{r} \quad \text{as} \quad \hbar = 1)$$
$$= N e^{-ip.x}$$

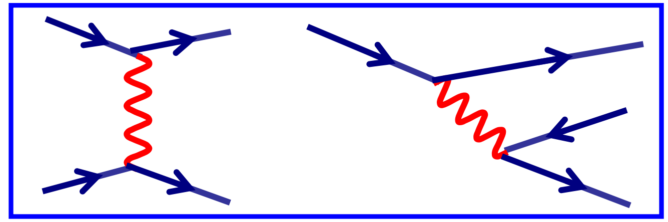
where N is the normalisation and $p.x = p^\mu x_\mu$

<u>For decay rate calculation need to know:</u>

- Wave-function normalisation
- Transition matrix element from perturbation theory
- Expression for the density of states

} All in a Lorentz Invariant form

Cross Sections and Decay Rates



- In particle physics we are mainly concerned with particle <u>interactions</u> and <u>decays</u>, i.e. **transitions between states**
  - these are the experimental observables of particle physics
- Calculate <u>transition rates</u> from **Fermi's Golden Rule**

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

Form assumes one particle per unit volume and $\int \psi^* \psi \mathrm{d}V = 1$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – **not Lorentz Invariant !**

T_{fi} is Transition **Matrix Element**

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j \neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

★ Rates depend on **MATRIX ELEMENT** and **DENSITY OF STATES**

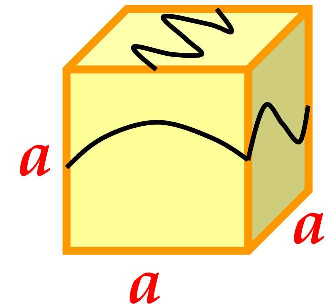
the ME contains the fundamental particle physics | just kinematics

Non-relativistic Phase Space (revision)

- Apply boundary conditions ($\vec{p} = \hbar\vec{k}$):
- Wave-function vanishing at box boundaries

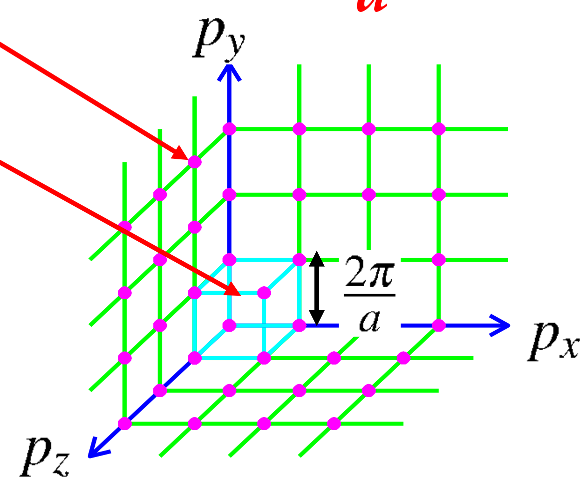
  ➡ quantised particle momenta:

$$p_x = \frac{2\pi n_x}{a};\ p_y = \frac{2\pi n_y}{a};\ p_z = \frac{2\pi n_z}{a}$$



- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$



- Normalising to **one particle/unit volume** gives **number of states** in element: $\mathrm{d}^3\vec{p} = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$

$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{\frac{(2\pi)^3}{V}} \times \frac{1}{V} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

- Therefore density of states in Golden rule:

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{dE}\right|_{E_f} = \left|\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|}\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}E}\right|_{E_f}$$

with $p = \beta E$

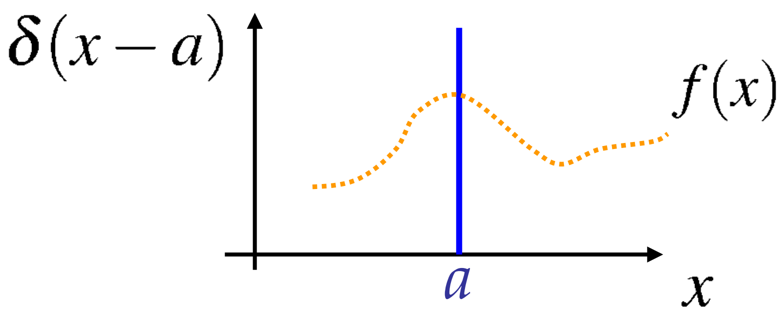
- Integrating over an elemental shell in momentum-space gives

  $(\mathrm{d}^3\vec{p} = 4\pi p^2 \mathrm{d}p)$

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \beta$$

Dirac δ Function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: *"infinitely narrow spike of unit area"*



$$\int_{-\infty}^{+\infty} \delta(x-a)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

- Any function with the above properties can represent $\delta(x)$

e.g. $$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$ (an infinitesimally narrow Gaussian)

- In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \to 1+2$

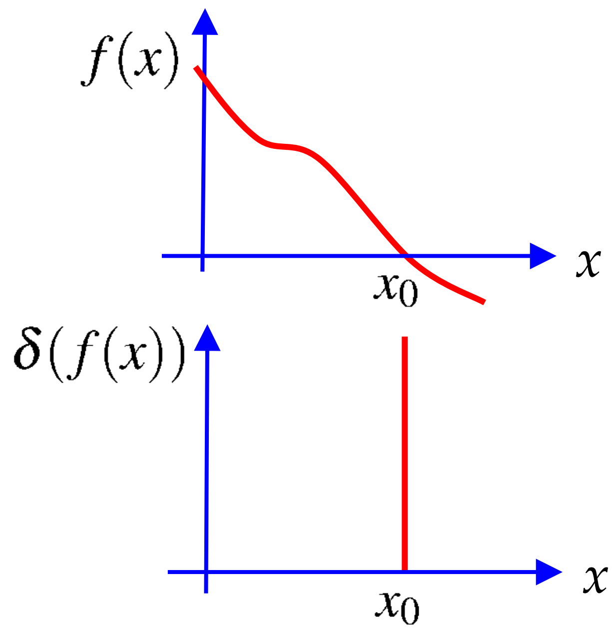
$$\int \ldots \delta(E_a - E_1 - E_2)\,\mathrm{d}E \quad \text{and} \quad \int \ldots \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\,\mathrm{d}^3\vec{p}$$

express energy and momentum conservation

★ **We will soon need an expression for the delta function of a function** $\delta(f(x))$

- **Start from the definition of a delta function**

$$\int_{y_1}^{y_2} \delta(y)\mathrm{d}y = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$



- **Now express in terms of** $y = f(x)$ **where** $f(x_0) = 0$ **and then change variables**

$$\int_{x_1}^{x_2} \delta(f(x))\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- **From properties of the delta function (i.e. here only non-zero at** x_0 **)**

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} \int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- **Rearranging and expressing the RHS as a delta function**

$$\int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \frac{1}{|\mathrm{d}f/\mathrm{d}x|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0)\mathrm{d}x$$

$$\Rightarrow \quad \boxed{\delta(f(x)) = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}^{-1} \delta(x - x_0)} \qquad (1)$$

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

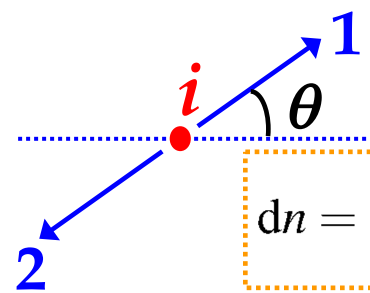
$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \int \frac{\mathrm{d}n}{\mathrm{d}E}\delta(E - E_i)\mathrm{d}E \qquad \text{since } E_f = E_i$$

Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\mathrm{d}n$

the integral is over **all** "allowed" final states of any energy

- For $\mathrm{d}n$ in a two-body decay, only need to consider one particle : mom. conservation fixes the other



$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

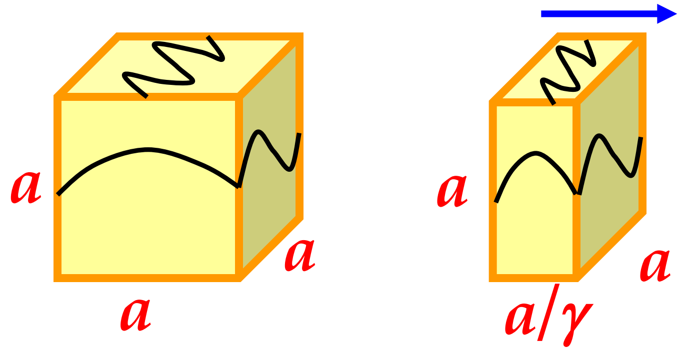
$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2)\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}$$

- However, can include momentum conservation explicitly by integrating over the momenta of both particles and using another δ-fn

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}}\underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}}\underbrace{\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

Lorentz Invariant Phase Space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi \mathrm{d}V = 1$
- When considering relativistic effects, volume contracts by $\gamma = E/m$



- Particle density therefore increases by $\gamma = E/m$
  - ★ Conclude that a relativistic invariant wave-function normalisation needs to be **proportional to E particles per unit volume**
- Usual convention: **Normalise to 2E particles/unit volume** $\int \psi'^* \psi' \mathrm{d}V = 2E$
- Previously used ψ normalised to 1 particle per unit volume $\int \psi^* \psi \mathrm{d}V = 1$
- Hence $\psi' = (2E)^{1/2}\psi$ is normalised to $2E$ per unit volume
- **Define** Lorentz Invariant Matrix Element, M_{fi}, in terms of the wave-functions normalised to $2E$ particles per unit volume

$$M_{fi} = \langle \psi'_1 . \psi'_2 ... | \hat{H} | ... \psi'_{n-1} \psi'_n \rangle = (2E_1 . 2E_2 . 2E_3 .... 2E_n)^{1/2} T_{fi}$$

- For the two body decay

$$i \rightarrow 1+2$$

$$\begin{aligned} M_{fi} &= \langle \psi_1' \psi_2' | \hat{H}' | \psi_i' \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H}' | \psi_i \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} T_{fi} \end{aligned}$$

★ Now expressing T_{fi} in terms of M_{fi} gives

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

Note:

- M_{fi} uses relativistically normalised wave-functions. It is **Lorentz Invariant**
- $\frac{d^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz Invariant Phase Space for each final state particle; the factor of $2E$ arises from the wave-function normalisation

  (prove this in Question 2)
- This form of Γ_{fi} is simply a rearrangement of the original equation **but** the integral is now frame independent (i.e. L.I.)
- Γ_{fi} is inversely proportional to E_i, the energy of the decaying particle. This is exactly what one would expect from time dilation ($E_i = \gamma m$).
- Energy and momentum conservation in the delta functions

Decay Rate Calculations

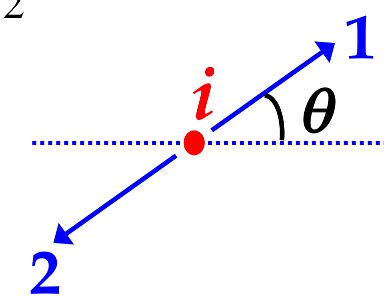
$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}$$

★ **Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient**

- **In the C.o.M. frame** $E_i = m_i$ **and** $\vec{p}_i = 0$ ⇨

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2)\frac{\mathrm{d}^3\vec{p}_1}{2E_1}\frac{\mathrm{d}^3\vec{p}_2}{2E_2}$$

- **Integrating over** $\vec{p}_2$ **using the δ-function:**



$$\Rightarrow \quad \Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\frac{\mathrm{d}^3\vec{p}_1}{4E_1 E_2}$$

now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ **since the δ-function imposes** $\vec{p}_2 = -\vec{p}_1$

- **Writing** $\mathrm{d}^3\vec{p}_1 = p_1^2 \mathrm{d}p_1 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega$

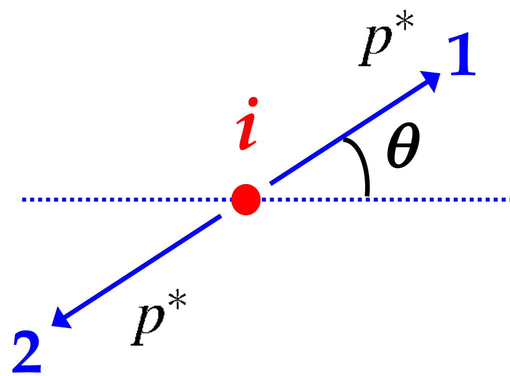
For convenience, here $|\vec{p}_1|$ **is written as** p_1

$$\Rightarrow \quad \Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right)\frac{p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega}{E_1 E_2}$$

- Which can be written in the form $$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega \qquad (2)$$

where $g(p_1) = p_1^2/(E_1E_2) = p_1^2(m_1^2+p_1^2)^{-1/2}(m_2^2+p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2+p_1^2)^{1/2} - (m_2^2+p_1^2)^{1/2}$



Note:
- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1)=0$ determines the C.o.M momenta of the two decay products

  i.e. $f(p_1)=0$ for $p_1 = p^*$

★ Eq. (2) can be integrated using the property of δ – function derived earlier (eq. (1))

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}\int g(p_1)\delta(p_1-p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where p^* is the value for which $f(p^*)=0$

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{(m_1^2+p_1^2)^{1/2}} - \frac{p_1}{(m_2^2+p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2}$$

giving:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{E_1E_2}{p_1(E_1+E_2)}\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} d\Omega$$

$$= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{p_1}{E_1+E_2}\right|_{p_1=p^*} d\Omega$$

- But from $f(p_1)=0$, i.e. energy conservation: $E_1+E_2=m_i$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 d\Omega$$

In the particle's rest frame $E_i = m_i$

$$\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 d\Omega \qquad (3)$$

VALID FOR ALL TWO-BODY DECAYS !

- p^* can be obtained from $f(p_1)=0$

$$(m_1^2+p^{*2})^{1/2}+(m_2^2+p^{*2})^{1/2}=m_i \qquad \text{(Question 3)}$$

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$ (now try Questions 4 & 5)

Cross section definition

$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the <u>effective</u> cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption

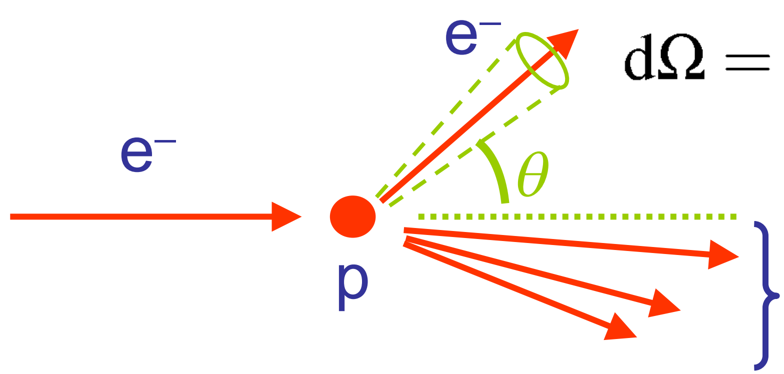
here σ is the projective area of nucleus

Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

or generally

$$\frac{d\sigma}{d\ldots}$$



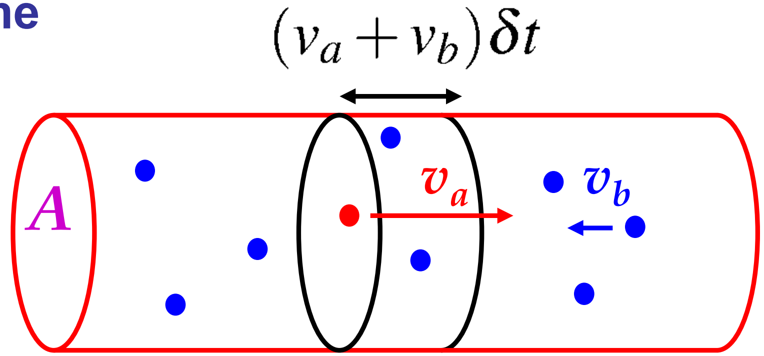
$$d\Omega = d(\cos\theta)d\phi$$

integrate over all other particles

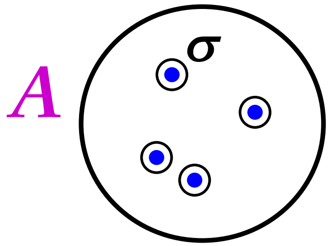
with $$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

example

- Consider a single particle of type a with velocity, v_a, traversing a region of area A containing n_b particles of type b per unit volume



In time δt a particle of type a traverses region containing $n_b(v_a + v_b)A\delta t$ particles of type b



★ Interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a + v_b)A\delta t$ particles of type b

- Interaction Probability = $\dfrac{n_b(v_a + v_b)A\delta t\sigma}{A} = n_b v \delta t \sigma$ $\quad [v = v_a + v_b]$

➡ Rate per particle of type a = $n_b v \sigma$

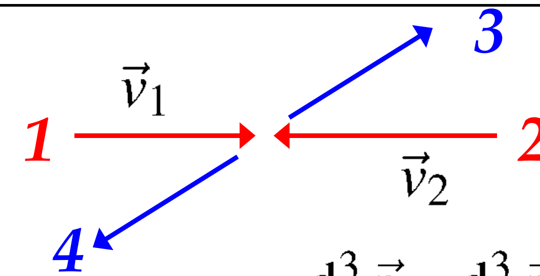
- Consider volume V, total reaction rate = $(n_b v \sigma).(n_a V) = (n_b V)(n_a v)\sigma$
  $= N_b \phi_a \sigma$

- As anticipated: **Rate = Flux x Number of targets x cross section**

Cross Section Calculations

- Consider scattering process

$$1+2\rightarrow 3+4$$



- Start from Fermi's Golden Rule:

$$\Gamma_{fi}=(2\pi)^4\int|T_{fi}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{d^3\vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a normalisation of 1/unit volume

- Now

$$\begin{aligned}\text{Rate/Volume} &= (\text{flux of 1})\times(\text{number density of 2})\times\sigma\\ &= n_1(v_1+v_2)\times n_2\times\sigma\end{aligned}$$

- For 1 target particle of each species per unit volume $\text{Rate/Volume}=(v_1+v_2)\sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_1+v_2)}$$

$$\sigma=\frac{(2\pi)^4}{v_1+v_2}\int|T_{fi}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{d^3\vec{p}_4}{(2\pi)^3}$$

the parts are not Lorentz Invariant

- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element $M_{fi} = (2E_1 2E_2 2E_3 2E_4)^{1/2} T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2(v_1+v_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant (the Lorentz Inv. Flux)

$$F = 4\left[(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

(see appendix I)

- Consequently cross section is a Lorentz Invariant quantity

Two special cases of Lorentz Invariant Flux:

- Centre-of-Mass Frame

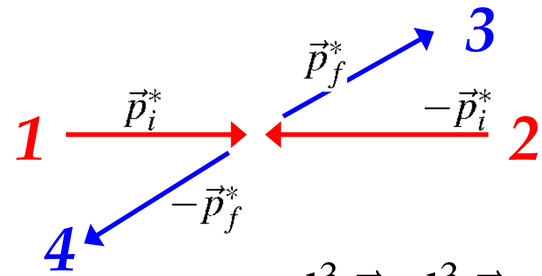
$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2(|\vec{p}^*|/E_1 + |\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*|(E_1+E_2) \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

- Target (particle 2) at rest

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1 m_2 v_1 \\ &= 4E_1 m_2(|\vec{p}_1|/E_1) \\ &= 4m_2|\vec{p}_1| \end{aligned}$$

2→2 Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame



- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

- Here $\vec{p}_1+\vec{p}_2 = 0$ and $E_1+E_2=\sqrt{s}$

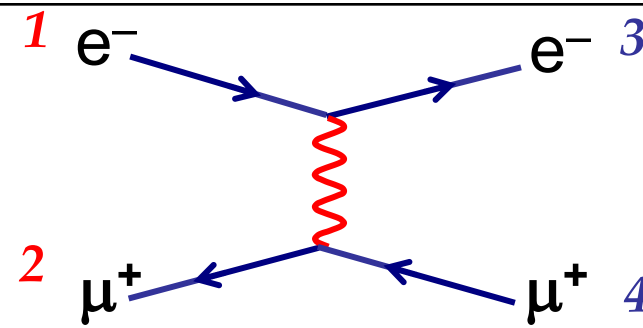
$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

★ The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*}$$

- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$



$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}|M_{fi}|^2 \mathrm{d}\Omega^*$$

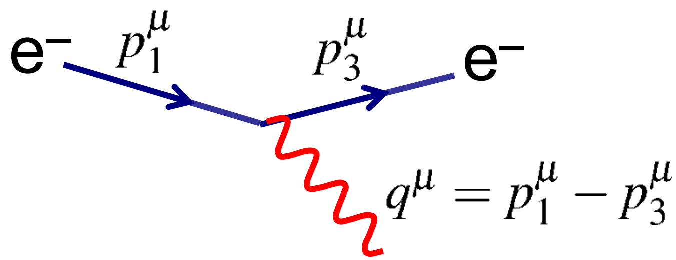
because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$ refer to the C.o.M frame

- For the last calculation in this section, we need to find a **L.I.** expression for $\mathrm{d}\sigma$

★ Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam *t* i.e. the square of the four-momentum transfer

$$t = q^2 = (p_1 - p_3)^2$$

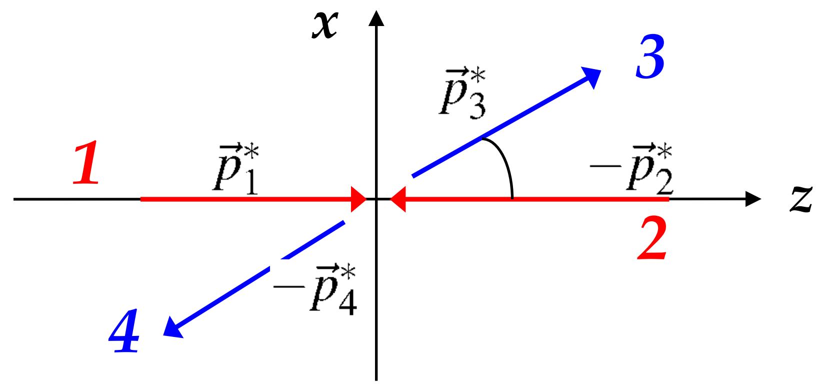
Product of four-vectors therefore L.I.



- **Want to express** $\mathrm{d}\Omega^*$ **in terms of Lorentz Invariant** $\mathrm{d}t$

  where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$

- **In C.o.M. frame:**



$$\begin{aligned} p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\ p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\ p_1^{\mu} p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\ t &= m_1^2 + m_3^3 - 2E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \end{aligned}$$

giving $$\mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*)$$

therefore $$\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \frac{\mathrm{d}t\,\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$$

hence $$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2\mathrm{d}\Omega^* = \frac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t$$

- **Finally, integrating over** $\mathrm{d}\phi^*$ **(assuming no** ϕ^* **dependence of** $|M_{fi}|^2$ **) gives:**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

Lorentz Invariant differential cross section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$

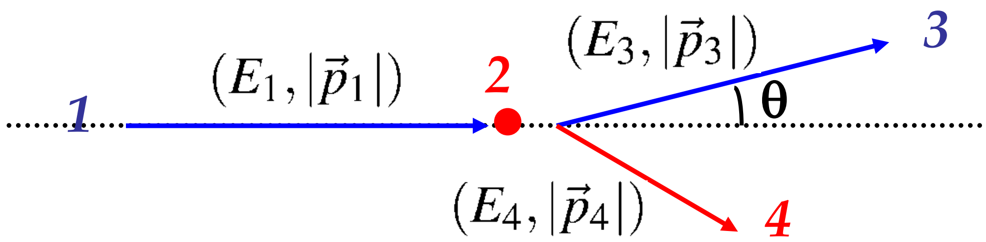
e.g. electron or neutrino scattering

In this limit $$|\vec{p}_i^*|^2 = \frac{(s-m_2)^2}{4s}$$

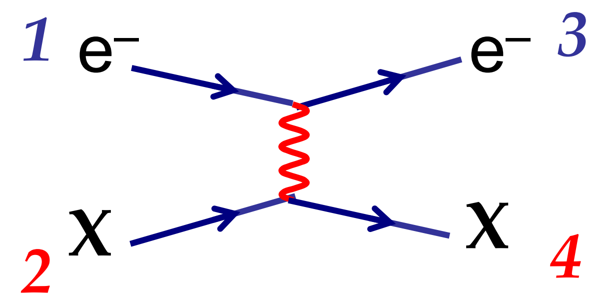
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi(s-m_2^2)^2}|M_{fi}|^2} \qquad (m_1 = 0)$$

2→2 Body Scattering in Lab. Frame

- **The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)**
- **First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected:** $m_1 = m_3 = 0, \quad m_2 = m_4 = M$



e.g.



- **Wish to express the cross section in terms of scattering angle of the e⁻**

$$d\Omega = 2\pi d(\cos\theta)$$

therefore $$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

Integrating over $d\phi$

- **The rest is some rather tedious algebra…. start from four-momenta**

$$p_1 = (E_1, 0, 0, E_1),\ \ p_2 = (M, 0, 0, 0),\ \ p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta),\ \ p_4 = (E_4, \vec{p}_4)$$

so here $$t = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1-\cos\theta)$$

But from (E,p) conservation $p_1 + p_2 = p_3 + p_4$

and, therefore, can also express t in terms of particles 2 and 4

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

- **Equating the two expressions for t gives**

$$\boxed{E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}}$$

so $$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s-M^2)^2}|M_{fi}|^2$$

using $$s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$$

Particle 1 massless ➡ $(p_1^2 = 0)$

gives $$(s - M^2) = 2ME_1$$

➡ $$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2}$$

In limit $m_1 \to 0$

In this equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}$$

giving

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad (m_1 = 0)$$

General form for 2→2 Body Scattering in Lab. Frame

★ The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more "physics" (see appendix II). It gives:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{m_2|\vec{p}_1|}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2) - E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2$$

Again there is **only one independent variable**, θ, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of θ

Summary

★ Used a Lorentz invariant formulation of Fermi′s Golden Rule to derive decay rates and cross-sections in terms of the **Lorentz Invariant Matrix Element** (wave-functions normalised to 2E/Volume)

Main Results:

★ Particle decay:

$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

★ Scattering cross section in C.o.M. frame:

$$\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

★ Invariant differential cross section (valid in all frames):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

Summary cont.

★ **Differential cross section in the lab. frame ($m_1 = 0$)**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \quad \Longleftrightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2$$

★ **Differential cross section in the lab. frame ($m_1 \neq 0$)**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{|m_2||\vec{p}_1|} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$

with $E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2}$

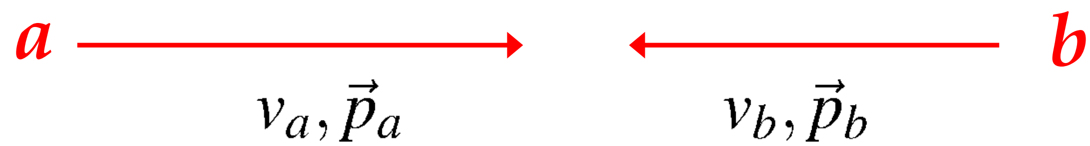
Summary of the summary:

★ **Have now dealt with kinematics of particle decays and cross sections**
★ **The fundamental particle physics is in the matrix element**
★ **The above equations are the basis for all calculations that follow**

Appendix I : Lorentz Invariant Flux

NON-EXAMINABLE

- **Collinear collision:**



$$\begin{aligned}
F = 2E_a 2E_b (v_a + v_b) &= 4E_a E_b \left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b} \right) \\
&= 4(|\vec{p}_a|E_b + |\vec{p}_b|E_a)
\end{aligned}$$

To show this is Lorentz invariant, first consider

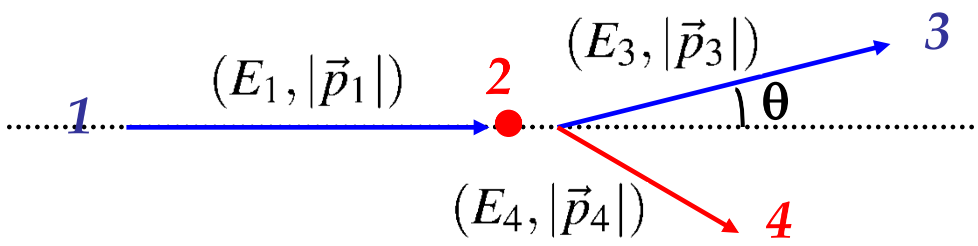
$$p_a.p_b = p_a^\mu p_{b\mu} = E_a E_b - \vec{p}_a.\vec{p}_b = E_a E_b + |\vec{p}_a||\vec{p}_b|$$

Giving

$$\begin{aligned}
F^2/16 - (p_a^\mu p_{b\mu})^2 &= (|\vec{p}_a|E_b + |\vec{p}_b|E_a)^2 - (E_a E_b + |\vec{p}_a||\vec{p}_b|)^2 \\
&= |\vec{p}_a|^2 (E_b^2 - |\vec{p}_b|^2) + E_a^2 (|\vec{p}_b|^2 - E_b^2) \\
&= |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 \\
&= -m_a^2 m_b^2 \\
F &= 4 \left[(p_a^\mu p_{b\mu})^2 - m_a^2 m_b^2 \right]^{1/2}
\end{aligned}$$

Appendix II : general 2→2 Body Scattering in lab frame

NON-EXAMINABLE



$$p_1 = (E_1, 0, 0, |\vec{p}_1|), \quad p_2 = (M_2, 0, 0, 0), \quad p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta), \quad p_4 = (E_4, \vec{p}_4)$$

again $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

But now the invariant quantity *t*:

$$\begin{aligned} t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2.p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\ &= m_2^2 + m_4^2 - 2m_2(E_1 + m_2 - E_3) \end{aligned}$$

➡ $$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2m_2 \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

Which gives $$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{d\sigma}{dt}$$

To determine $dE_3/d(\cos\theta)$**, first differentiate** $E_3^2 - |\vec{p}_3|^2 = m_3^2$

$$2E_3\frac{dE_3}{d(\cos\theta)} = 2|\vec{p}_3|\frac{d|\vec{p}_3|}{d(\cos\theta)} \qquad \text{(AII.1)}$$

Then equate $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ **to give**

$$m_1^2 + m_3^2 - 2(E_1E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2)\frac{dE_3}{d\cos\theta} - |\vec{p}_1|\cos\theta\frac{d|\vec{p}_3|}{d\cos\theta} = |\vec{p}_1||\vec{p}_3|$$

Using (AII.1) ➡ $$\frac{dE_3}{d(\cos\theta)} = \frac{|\vec{p}_1||\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \qquad \text{(AII.2)}$$

$$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

It is easy to show $|\vec{p}_i^*|\sqrt{s} = m_2|\vec{p}_1|$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{m_2}{64\pi^2 m_2^2 |\vec{p}_1|^2} |M_{fi}|^2$$

and using (AII.2) obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{m_2|\vec{p}_1|} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$