# Particle Physics

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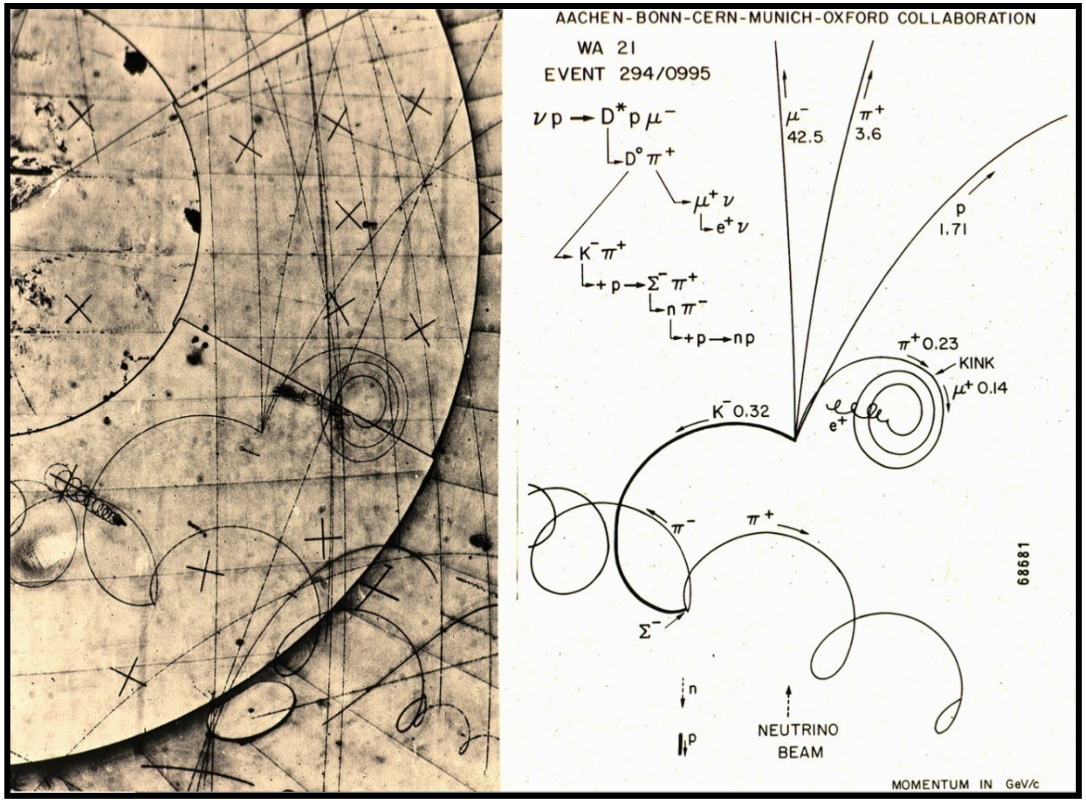



# Handout 7 : Symmetries and the Quark Model

# Introduction/Aims

★ Symmetries play a central role in particle physics; one aim of particle physics is to discover the fundamental symmetries of our universe

★ In this handout will apply the idea of symmetry to the quark model with the aim of :

- Deriving hadron wave-functions
- Providing an introduction to the more abstract ideas of colour and QCD (handout 8)
- Ultimately explaining why hadrons only exist as $\bar{q}q$ (mesons) $qqq$ (baryons) or $\bar{q}\bar{q}\bar{q}$ (anti-baryons)

+ introduce the ideas of the SU(2) and SU(3) symmetry groups which play a major role in particle physics (see handout 13)

# Symmetries and Conservation Laws

★ Suppose physics is invariant under the transformation

$$\psi \rightarrow \psi' = \hat{U}\psi$$

e.g. rotation of the coordinate axes

• To conserve probability normalisation require

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^{\dagger}\hat{U}|\psi\rangle$$

$$\rightarrow \quad \boxed{\hat{U}^{\dagger}\hat{U} = 1}$$

i.e. $\hat{U}$ has to be **unitary**

• For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{U}^{\dagger}\hat{H}\hat{U}|\psi\rangle$$

i.e. require $\quad \hat{U}^{\dagger}\hat{H}\hat{U} = \hat{H}$

$\times\hat{U} \quad \hat{U}\hat{U}^{\dagger}\hat{H}\hat{U} = \hat{U}\hat{H} \quad \rightarrow \quad \hat{H}\hat{U} = \hat{U}\hat{H}$

therefore $\quad \boxed{[\hat{H},\hat{U}] = 0}$

$\hat{U}$ **commutes with the Hamiltonian**

★ Now consider the infinitesimal transformation ( $\varepsilon$ small )

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

( $\hat{G}$ is called the **generator** of the transformation)

• For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^{\dagger} = (1 + i\varepsilon\hat{G})(1 - i\varepsilon\hat{G}^{\dagger}) = 1 + i\varepsilon(\hat{G} - \hat{G}^{\dagger}) + O(\varepsilon^2)$$

neglecting terms in $\varepsilon^2$ $\quad UU^{\dagger} = 1$ ➡ $\boxed{\hat{G} = \hat{G}^{\dagger}}$

i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity $G$ !

•Furthermore, $[\hat{H}, \hat{U}] = 0 \Rightarrow [\hat{H}, 1 + i\varepsilon\,\hat{G}] = 0 \Rightarrow [\hat{H}, \hat{G}] = 0$

But from QM $\quad \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H}, \hat{G}]\rangle = 0$

i.e. $G$ is a conserved quantity.

**Symmetry ⟷ Conservation Law**

★ **For each symmetry of nature have an observable conserved quantity**

**Example:** Infinitesimal spatial translation $x \rightarrow x + \varepsilon$

i.e. expect physics to be invariant under $\psi(x) \rightarrow \psi' = \psi(x + \varepsilon)$

$$\psi'(x) = \psi(x + \varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1 + \varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

but $\quad \hat{p}_x = -i\frac{\partial}{\partial x}$ ➡ $\psi'(x) = (1 + i\varepsilon\hat{p}_x)\psi(x)$

The generator of the symmetry transformation is $\hat{p}_x$ ➡ $p_x$ is conserved

•**Translational invariance** of physics implies **momentum conservation !**

- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\varepsilon}.\vec{G}$$

**For example** for an infinitesimal 3D linear translation : $\vec{r} \rightarrow \vec{r} + \vec{\varepsilon}$

$$\longrightarrow \quad \hat{U} = 1 + i\vec{\varepsilon}.\vec{p} \qquad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n\rightarrow\infty}\left(1 + i\frac{\vec{\alpha}}{n}.\vec{G}\right)^n = e^{i\vec{\alpha}.\vec{G}}$$

**Example:** Finite spatial translation in 1D: $x \rightarrow x + x_0$ with $\hat{U}(x_0) = e^{ix_0\hat{p}_x}$

$$\psi'(x) = \psi(x + x_0) = \hat{U}\psi(x) = \exp\left(x_0\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \qquad \left(p_x = -i\frac{\partial}{\partial x}\right)$$

$$= \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x)$$

$$= \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots$$

i.e. obtain the expected Taylor expansion

# Symmetries in Particle Physics : Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- To reflect this symmetry, Heisenberg (1932) proposed that if you could "switch off" the electric charge of the proton

**There would be no way to distinguish between a proton and neutron**

- Proposed that the neutron and proton should be considered as two states of a single entity; the **nucleon**

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

★ Analogous to the spin-up/spin-down states of a spin-½ particle

**ISOSPIN**

★ Expect physics to be invariant under rotations in this space

- The neutron and proton form an isospin doublet with total isospin $I = \frac{1}{2}$ and third component $I_3 = \pm\frac{1}{2}$

# Flavour Symmetry of the Strong Interaction

**We can extend this idea to the quarks:**

★ **Assume the strong interaction treats all quark flavours equally (it does)**

• **Because $m_u \approx m_d$:**

**The strong interaction possesses an approximate flavour symmetry i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and *vice versa*.**

• **Choose the basis**

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

• **Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under "rotations" in an abstract isospin space**

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

**The 2x2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters But there are four constraints from $\hat{U}^\dagger \hat{U} = 1$**

➡ **8 – 4 = 4 independent matrices**

• **In the language of group theory the four matrices form the U(2) group**

- One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

- The remaining three matrices form an SU(2) group (special unitary) with $\det U = 1$
- For an infinitesimal transformation, in terms of the Hermitian generators $\hat{G}$

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

- $\det U = 1 \quad \Rightarrow \quad Tr(\hat{G}) = 0$
- A linearly independent choice for $\hat{G}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The proposed flavour symmetry of the strong interaction has the same transformation properties as SPIN !
- Define ISOSPIN: $\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$
- Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2}i\vec{\varepsilon}.\vec{\sigma} \quad = \quad 1 + \frac{i}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) = \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2}i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \qquad \det U = 1 + O(\varepsilon^2)$$

# Properties of Isopin

- Isospin has the exactly the same properties as spin

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

$$[T^2, T_3] = 0 \qquad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, $T_1, T_2, T_3$ and even though all three correspond to observables, can't know them simultaneously. So label states in terms of **total isospin** $I$ and the third component of isospin $I_3$

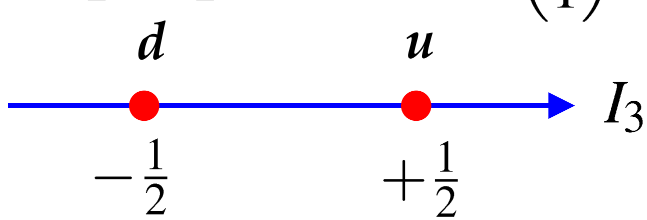
**NOTE: isospin has nothing to do with spin – just the same mathematics**

- The eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s, m\rangle \rightarrow |I, I_3\rangle$

  with $T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle$

- In terms of isospin:

$$u = \begin{pmatrix}1\\0\end{pmatrix} = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \qquad d = \begin{pmatrix}0\\1\end{pmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$



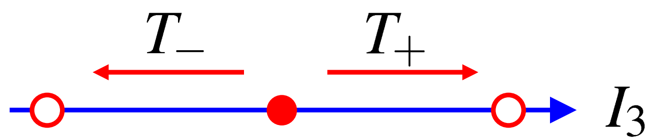

$$I = \tfrac{1}{2}, \quad I_3 = \pm\tfrac{1}{2}$$

- In general $I_3 = \frac{1}{2}(N_u - N_d)$

- Can define isospin ladder operators – analogous to spin ladder operators

$T_- \equiv T_1 - iT_2$

u → d




$T_+ \equiv T_1 + iT_2$

d → u

$$T_+|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}\,|I,I_3+1\rangle$$

$$T_-|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}\,|I,I_3-1\rangle$$

Step up/down in $I_3$ until reach end of multiplet $T_+|I,+I\rangle = 0 \quad T_-|I,-I\rangle = 0$

$$T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0$$

- Ladder operators turn $u \rightarrow d$ and $d \rightarrow u$

★ Combination of isospin: **e.g.** what is the isospin of a system of two **d** quarks, is exactly analogous to combination of spin (i.e. angular momentum)

$$|I^{(1)}, I_3^{(1)}\rangle |I^{(2)}, I_3^{(2)}\rangle \rightarrow |I, I_3\rangle$$

- $I_3$ additive : $I_3 = I_3^{(1)} + I_3^{(2)}$
- $I$ in integer steps from $|I^{(1)} - I^{(2)}|$ to $|I^{(1)} + I^{(2)}|$

★ Assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantites

- In strong interactions $I_3$ and $I$ are conserved, analogous to conservation of $J_z$ and $J$ for angular momentum

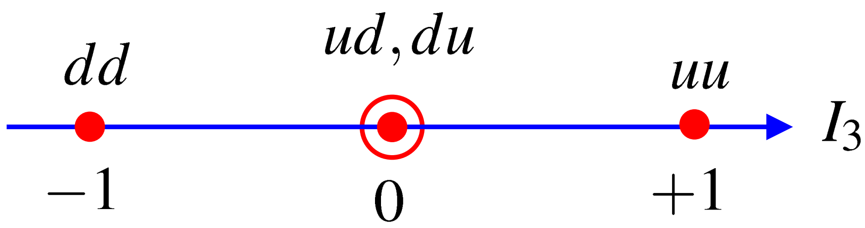
# Combining Quarks

**Goal:** derive proton wave-function

- First combine two quarks, then combine the third
- Use requirement that **fermion** wave-functions are **anti-symmetric**

Isospin starts to become useful in defining states of more than one quark.

e.g. two quarks, here we have four possible combinations:




**Note:** ◉ represents two states with the same value of $I_3$

- We can immediately identify the extremes ( $I_3$ additive)

$$uu \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,+1\rangle \qquad dd \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle$$

To obtain the $|1,0\rangle$ state use ladder operators

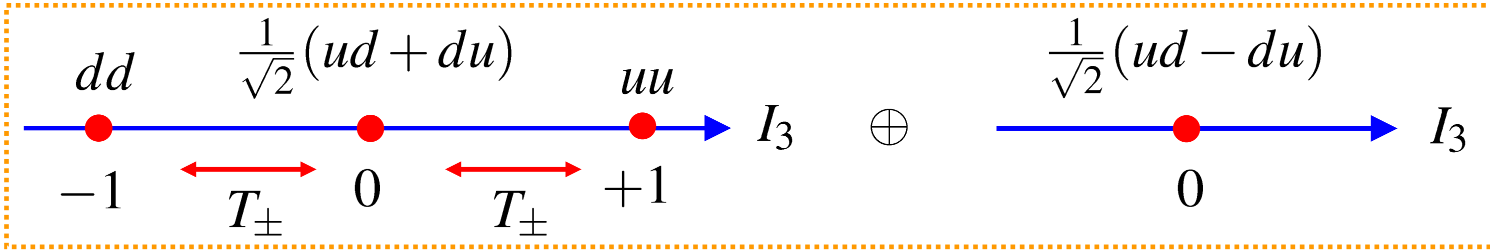
$$T_-|1,+1\rangle = \sqrt{2}|1,0\rangle = T_-(uu) = ud + du$$

$$\Rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}(ud + du)$$

The final state, $|0,0\rangle$, can be found from orthogonality with $|1,0\rangle$

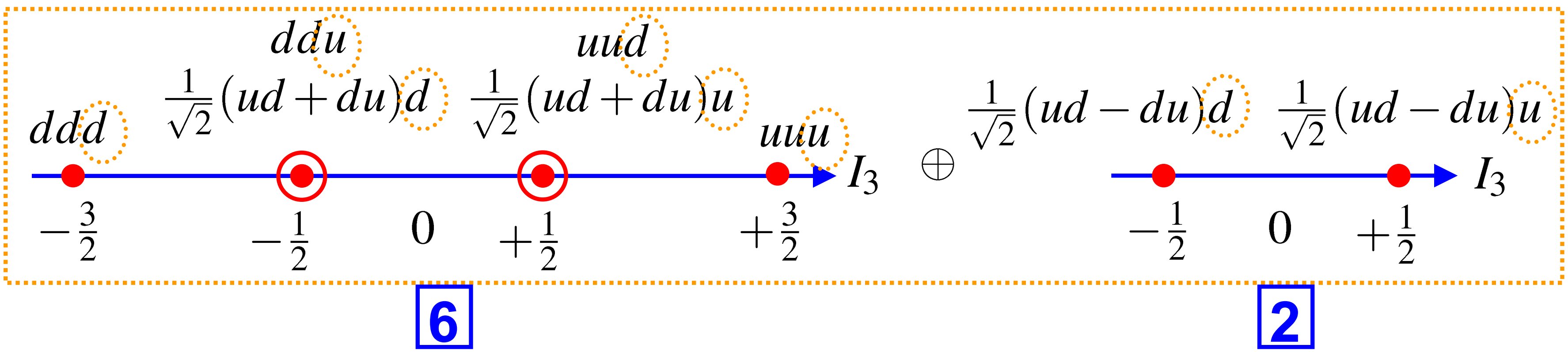
$$\Rightarrow \quad |0,0\rangle = \frac{1}{\sqrt{2}}(ud - du)$$

- From four possible combinations of isospin doublets obtain a **triplet** of **isospin 1** states and a **singlet** **isospin 0** state $2 \otimes 2 = 3 \oplus 1$



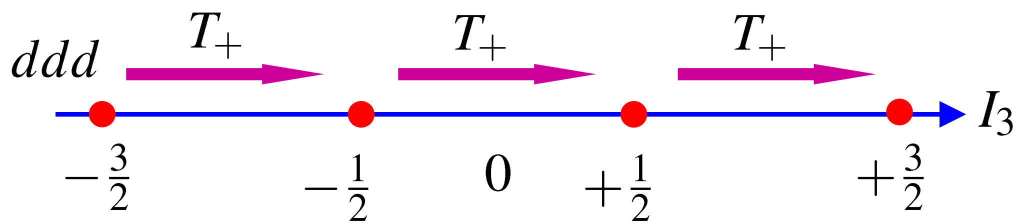

- Can move around <u>within</u> **multiplets** using ladder operators
- note, as anticipated $I_3 = \frac{1}{2}(N_u - N_d)$
- States with different **total isospin** are physically different – the **isospin 1 triplet** is **symmetric** under interchange of quarks 1 and 2 whereas **singlet** is **anti-symmetric**

★ Now add an additional up or down quark. From **each of the above 4 states** get two new isospin states with $I_3' = I_3 \pm \frac{1}{2}$




- Use ladder operators and orthogonality to group the 6 states into isospin multiplets, e.g. to obtain the $I = \frac{3}{2}$ states, step up from $ddd$

★ **Derive the $I=\frac{3}{2}$ states from $ddd \equiv |\frac{3}{2},-\frac{3}{2}\rangle$**




$$
\begin{aligned}
T_+|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= T_+(ddd)=(T_+d)dd+d(T_+d)d+dd(T_+)d\\
\sqrt{3}|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= udd+dud+ddu\\
\boxed{|\tfrac{3}{2},-\tfrac{1}{2}\rangle} &= \boxed{\tfrac{1}{\sqrt{3}}(udd+dud+ddu)}\\
T_+|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(udd+dud+ddu)\\
2|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud+udu+uud+duu+udu+duu)\\
\boxed{|\tfrac{3}{2},+\tfrac{1}{2}\rangle} &= \boxed{\tfrac{1}{\sqrt{3}}(uud+udu+duu)}\\
T_+|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(uud+udu+duu)\\
\sqrt{3}|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uuu+uuu+uuu)\\
\boxed{|\tfrac{3}{2},+\tfrac{3}{2}\rangle} &= \boxed{uuu}
\end{aligned}
$$

★ **From the $\boxed{6}$ states on previous page, use orthoganality to find $|\frac{1}{2},\pm\frac{1}{2}\rangle$ states**

★ **The $\boxed{2}$ states on the previous page give another $|\frac{1}{2},\pm\frac{1}{2}\rangle$ doublet**

★ **The eight states** $uuu,\ uud,\ udu,\ udd,\ duu,\ dud,\ ddu,\ ddd$
**are grouped into an isospin quadruplet and two isospin doublets**

$$2\otimes 2\otimes 2 = 2\otimes(3\oplus 1) = (2\otimes 3)\oplus(2\otimes 1) = 4\oplus 2\oplus 2$$

- **Different multiplets have different symmetry properties**

$$|\tfrac{3}{2},+\tfrac{3}{2}\rangle = uuu$$
$$|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(uud+udu+duu)$$
$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(ddu+dud+udd)$$
$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = ddd$$

**S**

**A quadruplet of states which are symmetric under the interchange of any two quarks**

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\tfrac{1}{\sqrt{6}}(2ddu-udd-dud)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}(2uud-udu-duu)$$

$\mathbf{M_S}$

**Mixed symmetry. Symmetric for 1 ⟷ 2**

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(udd-dud)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(udu-duu)$$

$\mathbf{M_A}$

**Mixed symmetry. Anti-symmetric for 1 ⟷ 2**

- **Mixed symmetry states have no definite symmetry under interchange of quarks** $1\leftrightarrow 3$ **etc.**

# Combining Spin

- Can apply exactly the same mathematics to determine the possible spin wave-functions for a combination of 3 spin-half particles

$$|\tfrac{3}{2},+\tfrac{3}{2}\rangle = \uparrow\uparrow\uparrow$$
$$|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$
$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)$$
$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = \downarrow\downarrow\downarrow$$

**S** — A quadruplet of states which are symmetric under the interchange of any two quarks

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$M_S$ — Mixed symmetry. Symmetric for 1 ⟷ 2

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$M_A$ — Mixed symmetry. Anti-symmetric for 1 ⟷ 2

- Can now form total wave-functions for combination of three quarks

# Baryon Wave-functions (ud)

★ Quarks are fermions so require that the total wave-function is **anti-symmetric** under the interchange of any two quarks

★ the total wave-function can be expressed in terms of:

$$\psi = \phi_{\rm flavour}\chi_{\rm spin}\xi_{\rm colour}\eta_{\rm space}$$

★ The colour wave-function for all bound qqq states is **anti-symmetric** (see handout 8)

- Here we will only consider the lowest mass, **ground state**, baryons where there is no internal orbital angular momentum.
- For **L=0** the spatial wave-function is **symmetric** $(-1)^L$.

➡ $\xi_{\rm colour}\eta_{\rm space}$ — **anti-symmetric**

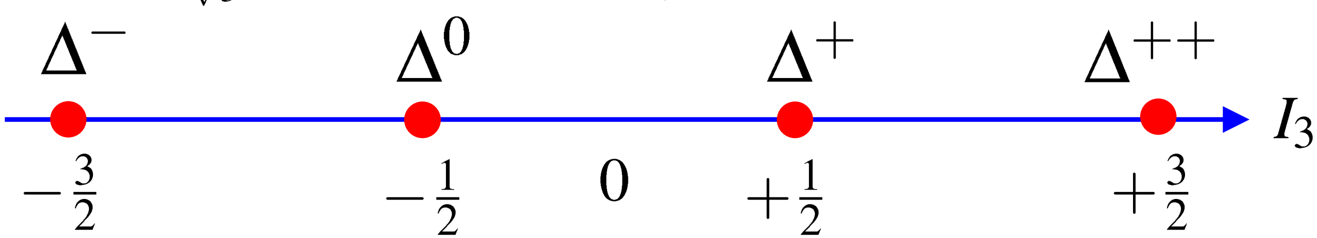
➡ $\phi_{\rm flavour}\chi_{\rm spin}$ — **symmetric**

} **Overall anti-symmetric**

★ **Two ways** to form a totally symmetric wave-function from spin and isospin states:

❶ combine totally symmetric spin and isospin wave-functions $\phi(S)\chi(S)$

$ddd \qquad \frac{1}{\sqrt{3}}(ddu+dud+udd) \qquad \frac{1}{\sqrt{3}}(uud+udu+duu) \qquad uuu$



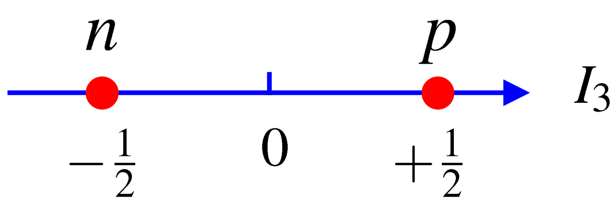

**Spin 3/2**
**Isospin 3/2**

❷ **combine mixed symmetry spin and mixed symmetry isospin states**

- Both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
- Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \ldots$
- However, it is not difficult to show that the (normalised) linear combination:

$$\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)$$

is **totally symmetric** (i.e. symmetric under $1 \leftrightarrow 2;\ \ 1 \leftrightarrow 3;\ \ 2 \leftrightarrow 3$ )




**Spin 1/2**
**Isospin 1/2**

- The spin-up proton wave-function is therefore:

$$|p\uparrow\rangle = \frac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{1}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

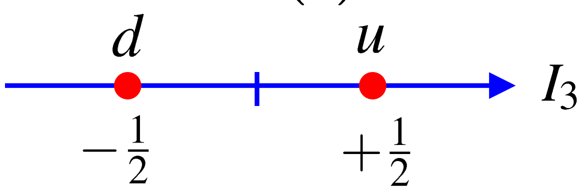
$$\begin{aligned} |p\uparrow\rangle = \frac{1}{\sqrt{18}}( \ & 2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \\ & 2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow + \\ & 2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\downarrow) \end{aligned}$$

**NOTE:** not always necessary to use the fully symmetrised proton wave-function, **e.g.** the first 3 terms are sufficient for calculating the proton magnetic moment

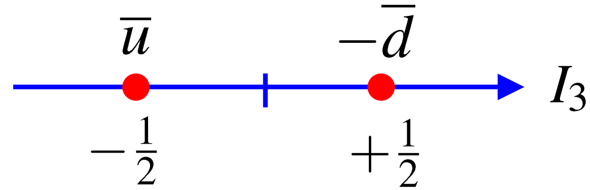
# Anti-quarks and Mesons (u and d)

★ The **u, d** quarks and **ū, d̄** anti-quarks are represented as isospin doublets

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$




$$\overline{q} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix}$$




$$\overline{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \overline{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

• **<u>Subtle point:</u>** The ordering and the minus sign in the anti-quark doublet ensures that anti-quarks and quarks transform in the same way (see Appendix I). This <u>is</u> necessary if we want physical predictions to be invariant under $u \leftrightarrow d;\ \overline{u} \leftrightarrow \overline{d}$

• Consider the effect of ladder operators on the anti-quark isospin states

**e.g** $$T_+\overline{u} = T_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\overline{d}$$

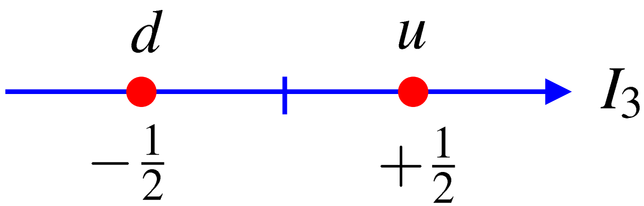
• The effect of the ladder operators on anti-particle isospin states are:

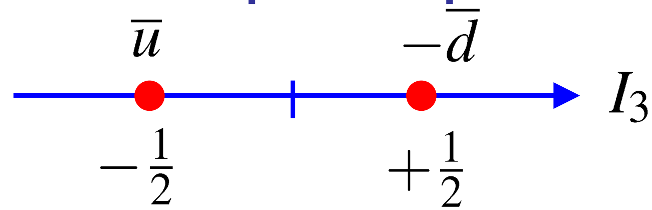
$$T_+\overline{u} = -\overline{d} \qquad T_+\overline{d} = 0 \qquad T_-\overline{u} = 0 \qquad T_-\overline{d} = -\overline{u}$$

Compare with $$T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0$$

# Light ud Mesons

★ **Can now construct meson states from combinations of up/down quarks**




• **Consider the $q\overline{q}$ combinations in terms of isospin**

$$|1,+1\rangle = |\tfrac{1}{2},+\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},+\tfrac{1}{2}\rangle} = -u\overline{d}$$

$$|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},-\tfrac{1}{2}\rangle} = d\overline{u}$$

**The bar indicates this is the isospin representation of an anti-quark**

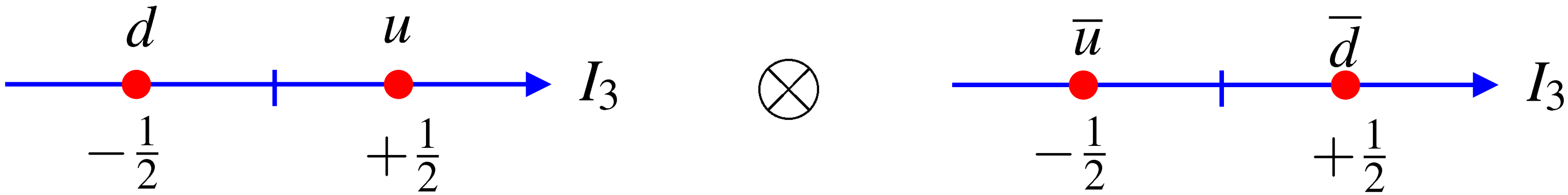
**To obtain the $I_3 = 0$ states use ladder operators and orthogonality**

$$\begin{aligned} T_-|1,+1\rangle &= T_-\left[-u\overline{d}\right] \\ \sqrt{2}|1,0\rangle &= -T_-\left[u\right]\overline{d} - uT_-\left[\overline{d}\right] \\ &= -d\overline{d} + u\overline{u} \end{aligned}$$

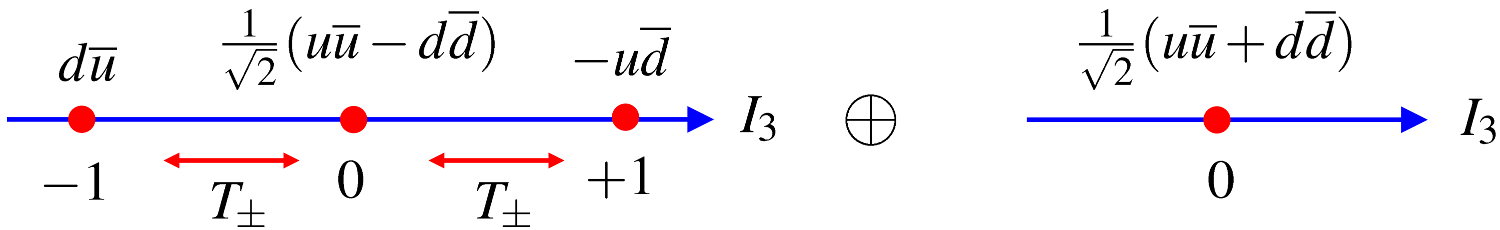
$$\Rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} - d\overline{d}\right)$$

• **Orthogonality gives:** $$|0,0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} + d\overline{d}\right)$$

★ **To summarise:**




**Triplet of $I=1$ states and a singlet $I=0$ state**




- **You will see this written as** $2\otimes\overline{2}=3\oplus 1$

**Quark doublet** (→ $2$)   **Anti-quark doublet** (→ $\overline{2}$)

- **To show the state obtained from orthogonality with** $|1,0\rangle$ **is a singlet use ladder operators**

$$T_+|0,0\rangle = T_+\frac{1}{\sqrt{2}}(u\overline{u}+d\overline{d}) = \frac{1}{\sqrt{2}}\left(-u\overline{d}+u\overline{d}\right)=0$$

**similarly** $T_-|0,0\rangle = 0$

★ **A singlet state is a "dead-end" from the point of view of ladder operators**

# SU(3) Flavour

★ Extend these ideas to include the strange quark. Since $m_s > m_u, m_d$ don't have an **exact symmetry**. But $m_s$ not so very different from $m_u, m_d$ and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- **NOTE:** any results obtained from this assumption are only approximate as the symmetry is not exact.
- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$

⇒ Can form 18 – 9 = 9 linearly independent matrices

**These 9 matrices form a U(3) group.**

- As before, one matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining 8 matrices have $\det U = 1$ and form an SU(3) group
- The eight matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\hat{U} = e^{i\vec{\alpha}.\vec{T}}$

★ **In SU(3) flavour, the three quark states are represented by:**

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

★ **In SU(3) uds flavour symmetry contains SU(2) ud flavour symmetry which allows us to write the first three matrices:**

$$\lambda_1 = \begin{pmatrix} \sigma_1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

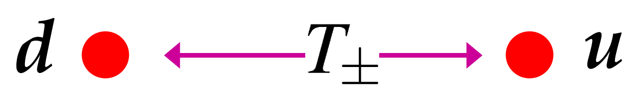
**i.e.** **u ↔ d**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- **The third component of isospin is now written** $I_3 = \frac{1}{2}\lambda_3$

  **with** $I_3 u = +\frac{1}{2}u \quad I_3 d = -\frac{1}{2}d \quad I_3 s = 0$

- $I_3$ **"counts the number of up quarks – number of down quarks in a state**
- **As before, ladder operators** $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$



- Now consider the matrices corresponding to the **u ↔ s and d ↔ s**

**u ↔ s**

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

**d ↔ s**

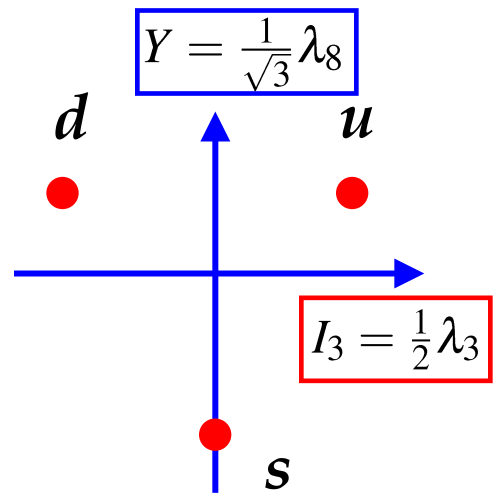
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- Hence in addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have two other traceless diagonal matrices

- However the three diagonal matrices are not be independent.
- Define the eighth matrix, $\lambda_8$, as the linear combination:

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

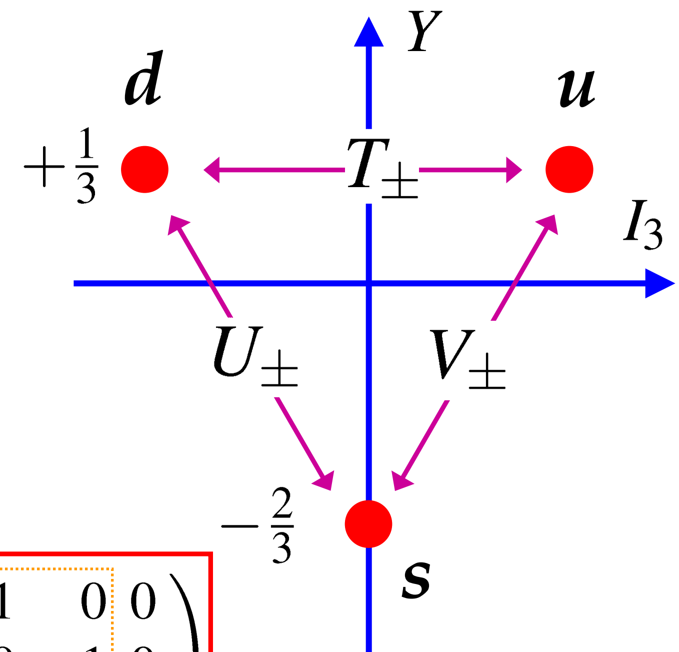
which specifies the "vertical position" in the 2D plane




"Only need two axes (quantum numbers) to specify a state in the 2D plane": **($I_3$,Y)**

★ **The other six matrices form six ladder operators which step between the states**

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$




**with** $I_3 = \tfrac{1}{2}\lambda_3$ $\quad Y = \frac{1}{\sqrt{3}}\lambda_8$

**and the eight Gell-Mann matrices**

**u ↔ d**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**u ↔ s**

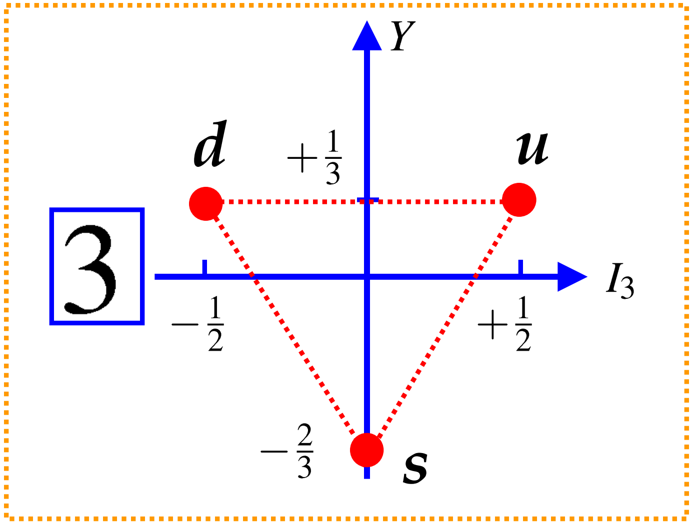
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

**d ↔ s**

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

# Quarks and anti-quarks in SU(3) Flavour

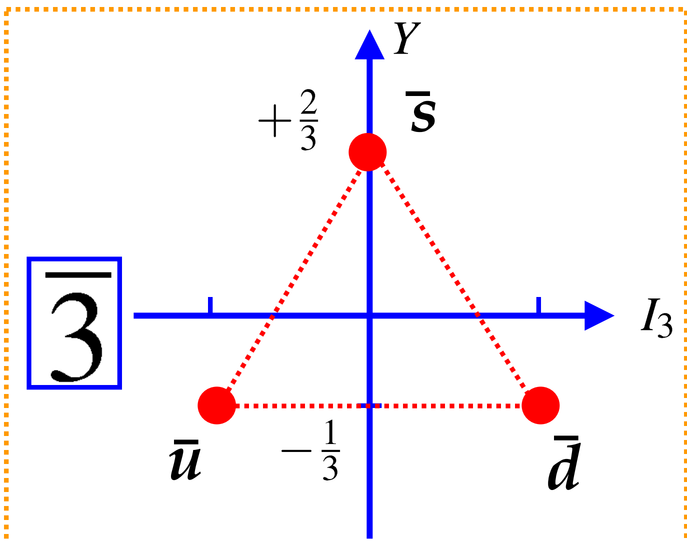



**Quarks**

$$I_3 u = +\tfrac{1}{2}u; \quad I_3 d = -\tfrac{1}{2}d; \quad I_3 s = 0$$

$$Yu = +\tfrac{1}{3}u; \quad Yd = +\tfrac{1}{3}d; \quad Ys = -\tfrac{2}{3}s$$

- The anti-quarks have opposite **SU(3)** flavour quantum numbers




**Anti-Quarks**

$$I_3 \overline{u} = -\tfrac{1}{2}\overline{u}; \quad I_3 \overline{d} = +\tfrac{1}{2}\overline{d}; \quad I_3 \overline{s} = 0$$

$$Y\overline{u} = -\tfrac{1}{3}\overline{u}; \quad Y\overline{d} = -\tfrac{1}{3}\overline{d}; \quad Y\overline{s} = +\tfrac{2}{3}\overline{s}$$

# SU(3) Ladder Operators

- **SU(3)** ***uds*** **flavour symmetry contains** ***ud*****,** ***us*** **and** ***ds*** **SU(2) symmetries**
- **Consider the** $u \leftrightarrow s$ **symmetry "V-spin" which has the associated** $s \rightarrow u$ **ladder operator**

$$V_+ = \tfrac{1}{2}(\lambda_4 + i\lambda_5) = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \tfrac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**with**

$$V_+ s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = +u$$

★ **The effects of the six ladder operators are:**

| | |
|---|---|
| $T_+ d = u; \quad T_- u = d;$ | $T_+ \bar{u} = -\bar{d}; \quad T_- \bar{d} = -\bar{u}$ |
| $V_+ s = u; \quad V_- u = s;$ | $V_+ \bar{u} = -\bar{s}; \quad V_- \bar{s} = -\bar{u}$ |
| $U_+ s = d; \quad U_- d = s;$ | $U_+ \bar{d} = -\bar{s}; \quad U_- \bar{s} = -\bar{d}$ |

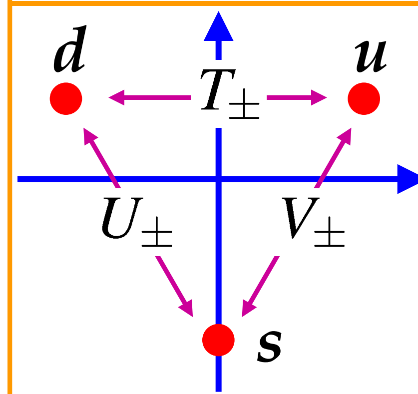
**all other combinations give zero**

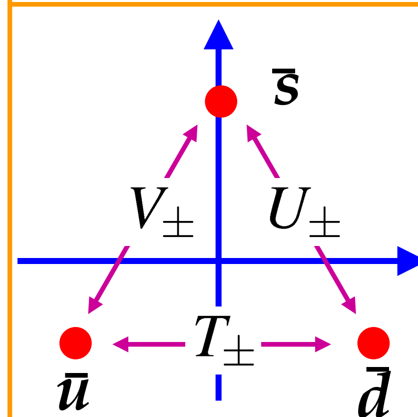
**SU(3) LADDER OPERATORS**

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$

$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$

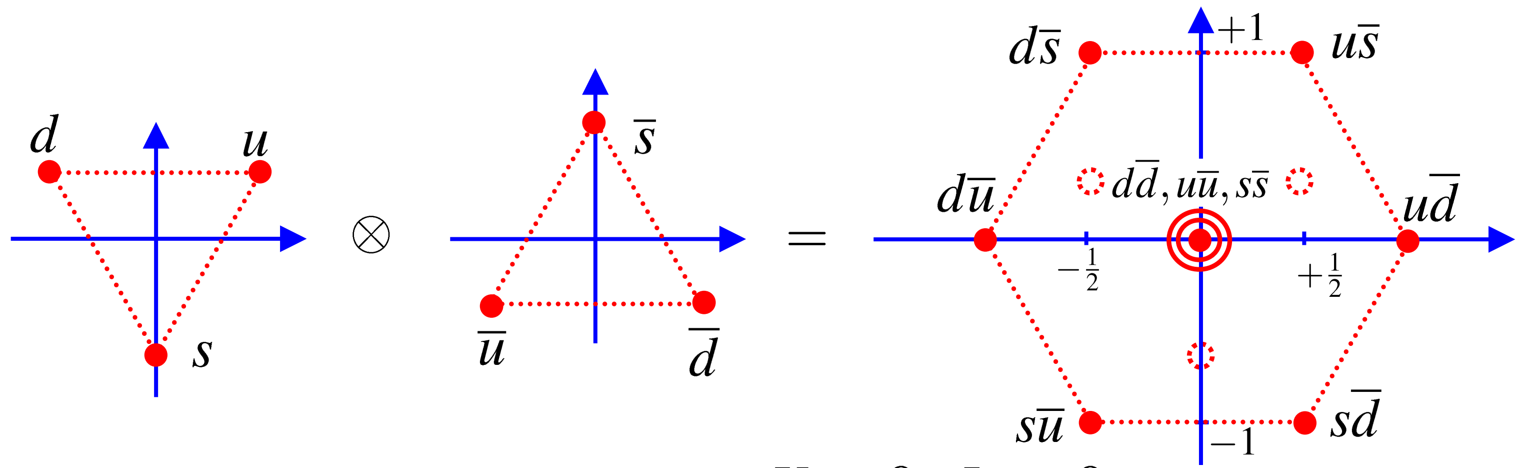
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



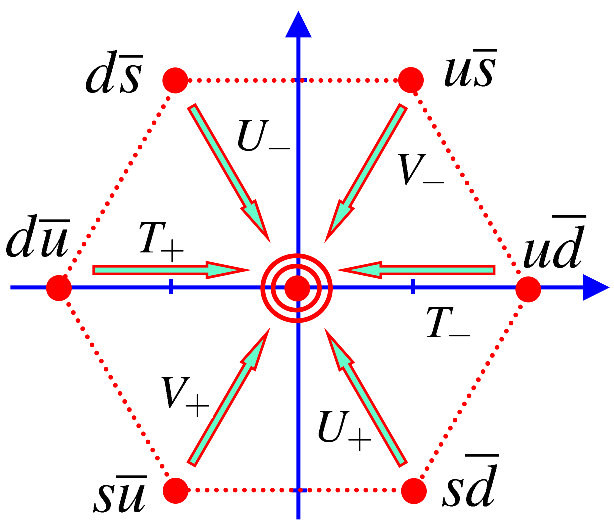



# Light (uds) Mesons

- Use ladder operators to construct **uds** mesons from the nine possible $q\bar{q}$ states




- The three central states, all of which have $Y = 0;\ I_3 = 0$ can be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways




$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \qquad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$$

$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \qquad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle$$

$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \qquad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle$$

- Only two of these six states are linearly independent.
- But there are three states with $Y = 0;\ I_3 = 0$
- Therefore one state is not part of the same multiplet, i.e. cannot be reached with ladder ops.

- First form two linearly independent orthogonal states from:

$$|u\bar{u}\rangle - |d\bar{d}\rangle \qquad |u\bar{u}\rangle - |s\bar{s}\rangle \qquad |d\bar{d}\rangle - |s\bar{s}\rangle$$

★ If the SU(3) flavour symmetry were exact, the choice of states wouldn't matter. However, $m_s > m_{u,d}$ and the symmetry is only approximate.

- **Experimentally** observe three light mesons with m~140 MeV: $\pi^+, \pi^0, \pi^-$
- Identify one state (the $\pi^0$) with the isospin triplet (derived previously)

$$\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

- The second state can be obtained by taking the linear combination of the other two states which is orthogonal to the $\pi^0$

$$\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |s\bar{s}\rangle)$$

with orthonormality: $\langle\psi_1|\psi_2\rangle = 0; \quad \langle\psi_2|\psi_2\rangle = 1$

$$\Rightarrow \quad \psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

- The final state (which is not part of the same multiplet) can be obtained by requiring it to be orthogonal to $\psi_1$ and $\psi_2$

$$\Rightarrow \quad \psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

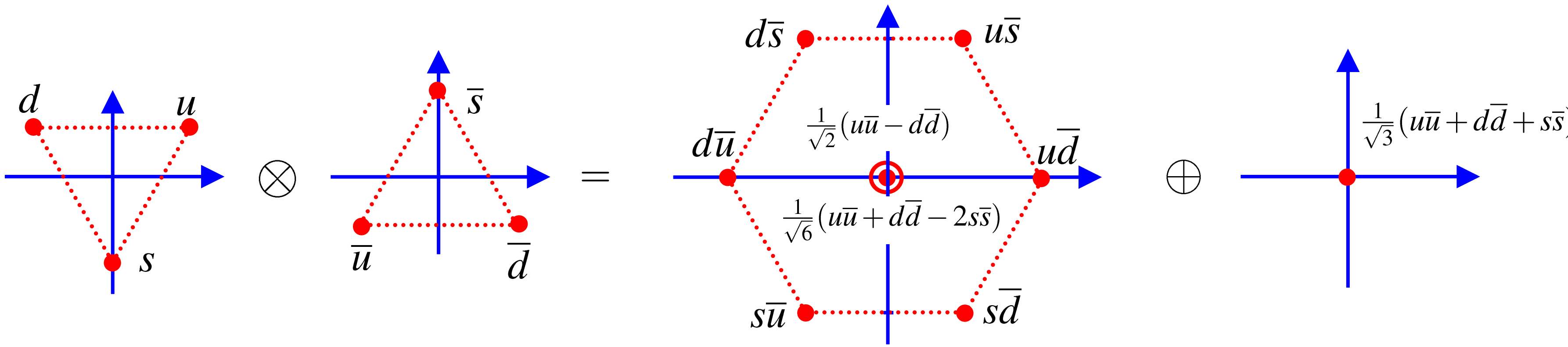
**SINGLET**

★It is easy to check that $\psi_3$ is a singlet state using ladder operators

$$T_+\psi_3 = T_-\psi_3 = U_+\psi_3 = U_-\psi_3 = V_+\psi_3 = V_-\psi_3 = 0$$

which confirms that $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ is a "flavourless" singlet

- Therefore the combination of a quark and anti-quark yields nine states which breakdown into an OCTET and a SINGLET




- In the language of group theory: $3 \otimes \bar{3} = 8 \oplus 1$

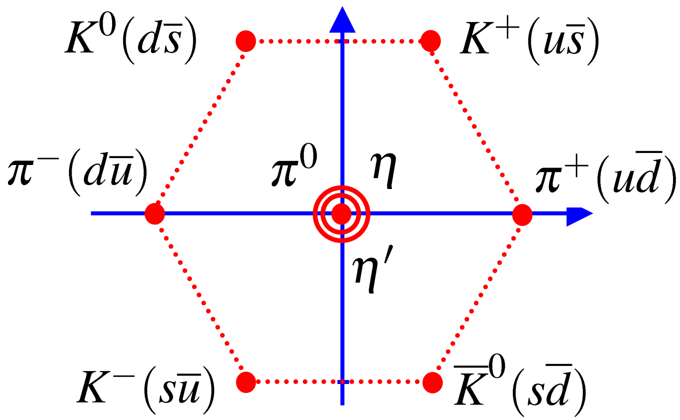
★ **Compare with combination of two spin-half particles** $2 \otimes 2 = 3 \oplus 1$

TRIPLET of spin-1 states: $|1,-1\rangle,\ |1,0\rangle,\ |1,+1\rangle$

spin-0 SINGLET: $|0,0\rangle$

- These spin triplet states are connected by ladder operators just as the meson ***uds*** octet states are connected by **SU(3)** flavour ladder operators
- The singlet state carries no angular momentum – in this sense the **SU(3)** flavour singlet is "flavourless"

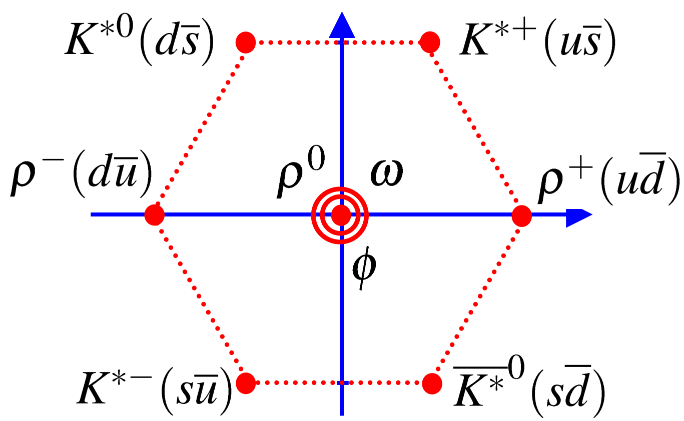
## PSEUDOSCALAR MESONS (L=0, S=0, J=0, P= –1 )




- Because SU(3) flavour is only approximate the physical states with $I_3 = 0,\ Y = 0$ can be mixtures of the octet and singlet states. **Empirically find:**

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \leftarrow \text{singlet}$$

## VECTOR MESONS (L=0, S=1, J=1, P= –1 )




- For the vector mesons the physical states are found to be approximately **"ideally mixed"**:

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

## MASSES

| | |
|---|---|
| $\pi^\pm : 140\,\text{MeV}$ | $\pi^0 : 135\,\text{MeV}$ |
| $K^\pm : 494\,\text{MeV}$ | $K^0/\overline{K}^0 : 498\,\text{MeV}$ |
| $\eta : 549\,\text{MeV}$ | $\eta' : 958\,\text{MeV}$ |

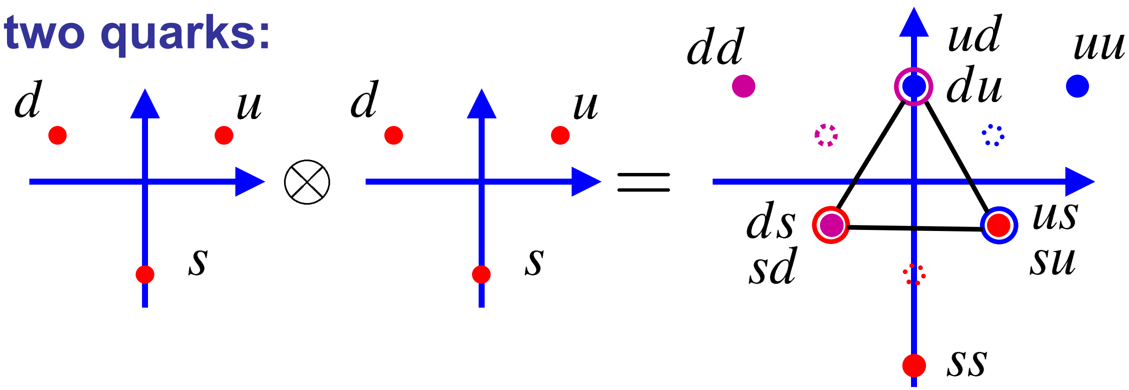
| | |
|---|---|
| $\rho^\pm : 770\,\text{MeV}$ | $\rho^0 : 770\,\text{MeV}$ |
| $K^{*\pm} : 892\,\text{MeV}$ | $K^{*0}/\overline{K^*}^0 : 896\,\text{MeV}$ |
| $\omega : 782\,\text{MeV}$ | $\phi : 1020\,\text{MeV}$ |

# Combining uds Quarks to form Baryons

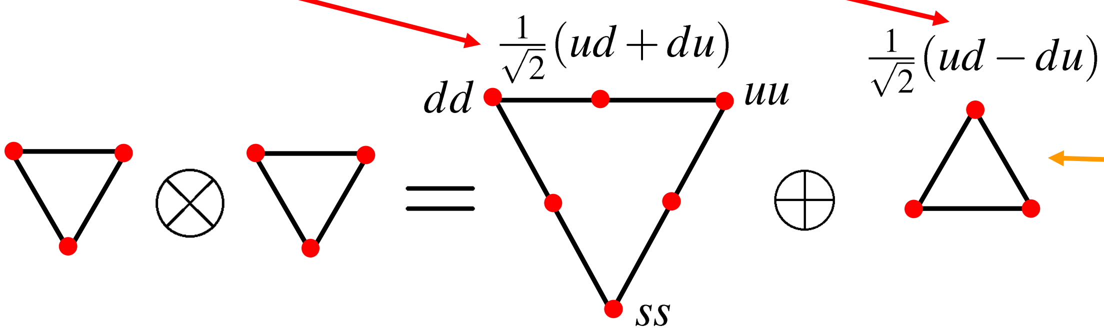
★ Have already seen that constructing Baryon states is a fairly tedious process when we derived the proton wave-function. Concentrate on multiplet structure rather than deriving all the wave-functions.

★ **Everything we do here is relevant to the treatment of colour**

- First combine two quarks:



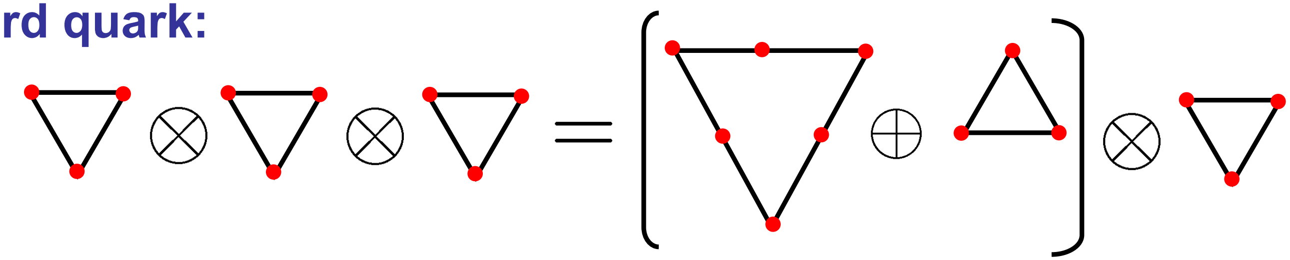

★Yields a symmetric sextet and anti-symmetric triplet: $3 \otimes 3 = 6 \oplus \overline{3}$




Same "pattern" as the anti-quark representation

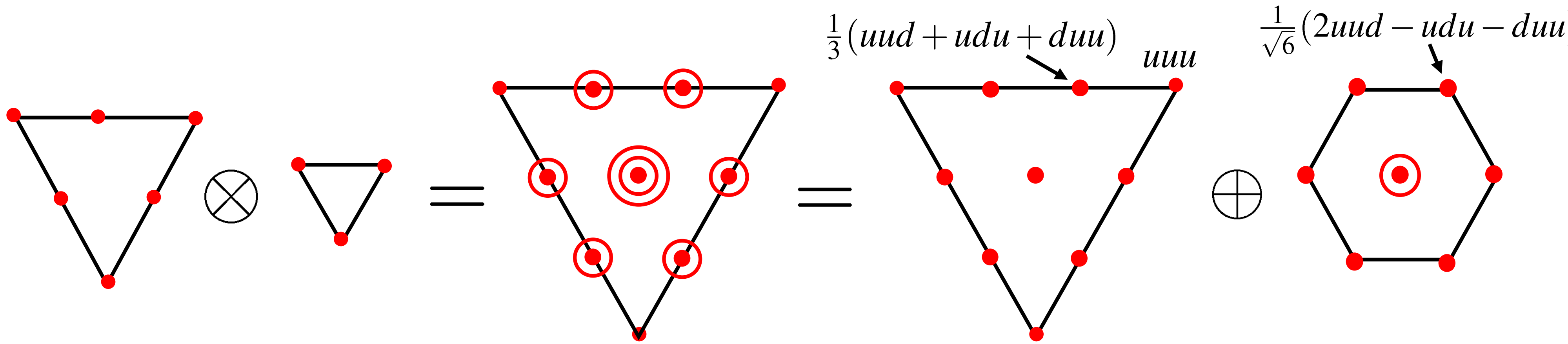
**SYMMETRIC** **ANTI-SYMMETRIC**

•Now add the third quark:



•Best considered in two parts, building on the **sextet** and **triplet**. Again concentrate on the multiplet structure (for the wave-functions refer to the discussion of proton wave-function).

❶ **Building on the sextet:** $3 \otimes 6 = 10 \oplus 8$

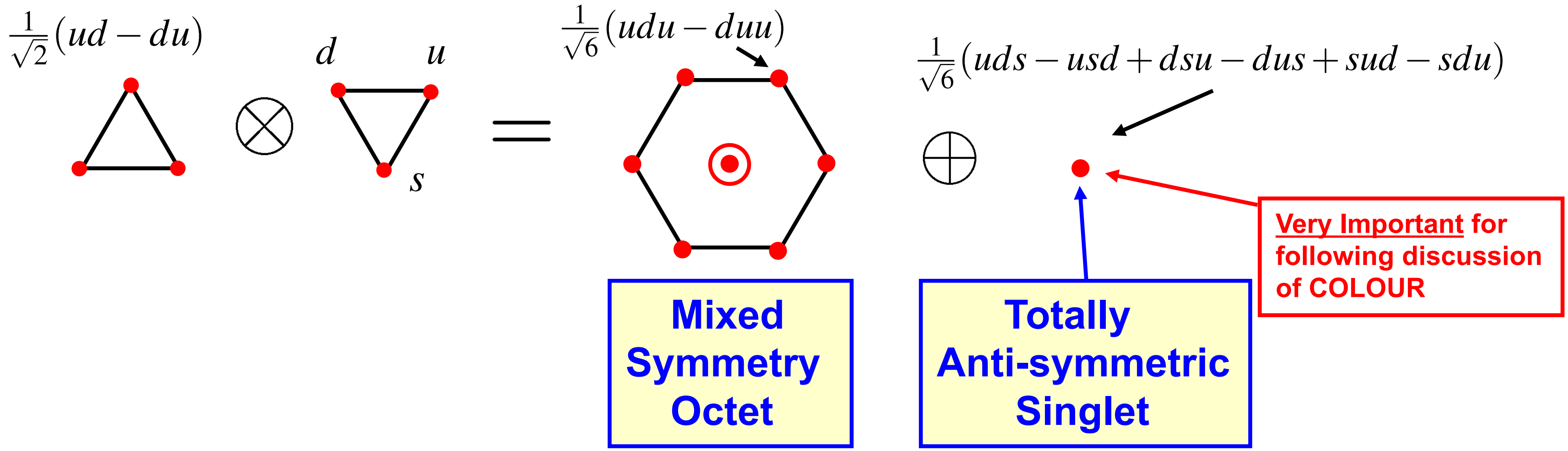



**Symmetric Decuplet**

**Mixed Symmetry Octet**

## ❷ Building on the triplet:

- Just as in the case of uds mesons we are combining $\bar{3} \times 3$ and again obtain an octet and a singlet




- Can verify the wave-function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$ is a singlet by using ladder operators, e.g.

$$T_+ \psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus - usu + usu - uus + suu - suu) = 0$$

★ In summary, the combination of three ***uds*** quarks decomposes into

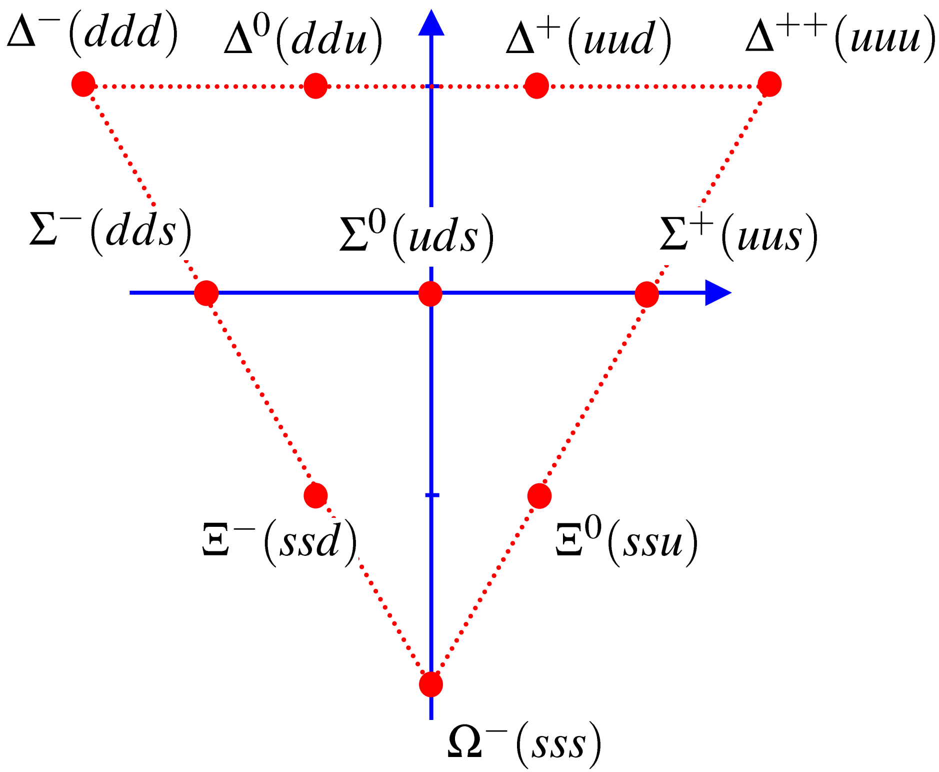
$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1$$

# Baryon Decuplet

★ The baryon states (L=0) are:

- the **spin 3/2 decuplet** of symmetric flavour and symmetric spin wave-functions $\phi(S)\chi(S)$

**BARYON DECUPLET** (L=0, S=3/2, J=3/2, P= +1 )




Mass in MeV

Δ(1232)

Σ(1318)

Ξ(1384)

Ω(1672)

★ If SU(3) flavour were an exact symmetry all masses would be the same (broken symmetry)

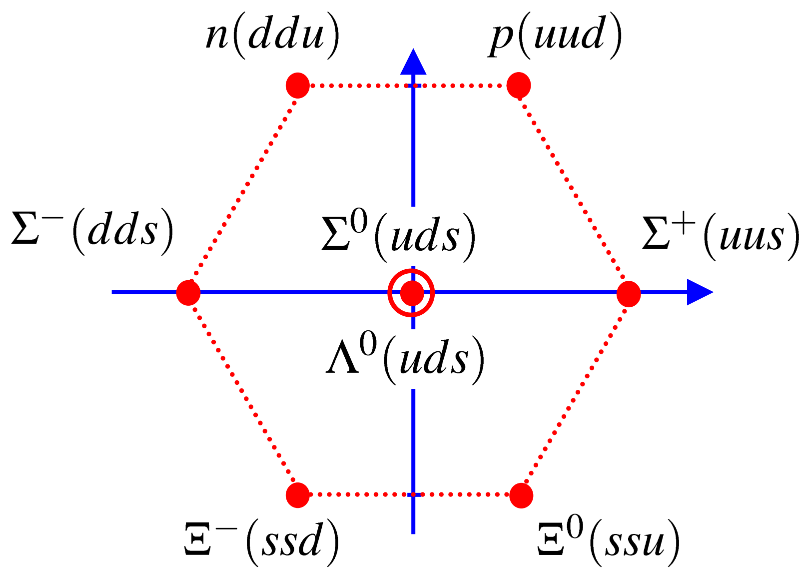
# Baryon Octet

★ The **spin 1/2 octet** is formed from mixed symmetry flavour and mixed symmetry spin wave-functions

$$\alpha\phi(M_S)\chi(M_S)+\beta\phi(M_A)\chi(M_A)$$

See previous discussion proton for how to obtain wave-functions

**BARYON OCTET** (L=0, S=1/2, J=1/2, P= +1 )




**Mass in MeV**

**939**

**Σ(1193)**
**Λ(1116)**

**Ξ(1318)**

★ **NOTE:** Cannot form a totally symmetric wave-function based on the anti-symmetric flavour singlet as there no totally anti-symmetric spin wave-function for 3 quarks

# Summary

★ Considered SU(2) **ud** and SU(3) **uds** flavour symmetries

★ Although these flavour symmetries are only approximate can still be used to explain observed multiplet structure for mesons/baryons

★ In case of SU(3) flavour symmetry results, e.g. predicted wave-functions should be treated with a pinch of salt as $m_s \neq m_{u/d}$

★ Introduced idea of singlet states being "spinless" or "flavourless"

★ In the next handout apply these ideas to colour and QCD…

# Appendix: the SU(2) anti-quark representation

**Non-examinable**

- Define anti-quark doublet $\overline{q} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$

- The quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ transforms as $q' = Uq$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix} \xrightarrow[\text{conjugate}]{\text{Complex}} \begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = U^* \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

- Express in terms of anti-quark doublet

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}' = U^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}$$

- Hence $\overline{q}$ transforms as

$$\overline{q}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}$$

- A special 2x2 unitary matrix can always be written in the form

$$U = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix}$$

… provided that $|c_{11}|^2+|c_{12}|^2=1$. This gives:

$$\begin{aligned} \overline{q}' &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ -c_{12} & c_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q} \\ &= \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix} \\ &= U\overline{q} \end{aligned}$$

- Therefore the anti-quark doublet $\overline{q} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix}$

transforms in the same way as the quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$

★ **NOTE:** this is a special property of SU(2) and for SU(3) there is no analogous representation of the anti-quarks