Some basic properties of Lie groups, algebras and their representations.

<u>def:</u> Group G : a set $\{g\}$ of elements, with one additional operation "multiplication":

$$g_1 \circ g_2 = g_3 \quad \text{where} \quad g_1\, g_2\, g_3 \in G.$$

additional requirements:

1) Identity: there $\exists$ unit element $\underset{\in G}{\mathbb{1}}$: $g\mathbb{1} = \mathbb{1}g = g$

2) An inverse exists: g^{-1} : $gg^{-1} = g^{-1}g = \mathbb{1}$

3) $g_1(g_2 g_3) = (g_1 g_2) g_3$

(*) Groups can be finite or $\underbrace{\text{continuous}}$ → in topological sense.

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let's merge abstract groups with analysis : Lie groups:

1) Parameterize the group elements:

$$g(\underbrace{\alpha_1 \ldots \alpha_n})$$ → some real numbers. (for now this is all we know)

$$\Rightarrow \mathbb{1} = g(0, \ldots, 0)$$

Idea: by changing smoothly the parameters $\alpha_1 \ldots \alpha_n$ we can cover the whole group!

The requirement of group "multiplication" adds important constraints:

$$g(\alpha) \circ g(\beta) = g(\gamma)$$

however $\gamma = f(\alpha, \beta)$ as implied by the "$\circ$" operation

identity: $\underbrace{\gamma = f(\gamma, 0) = f(0, \gamma)}$

Associativity: $f(\alpha, f(\beta, \gamma)) = f(f(\alpha, \beta), \gamma)$

If the functions $f(\alpha)$ are ∞ differentiable we call such groups Lie groups (after Sophus Lie 19th century math'n.)

Let's study the behaviour of the group close to $\mathbb{1}$, i.e. for parameters $\alpha \approx 0$.

$$g(\alpha) = \underset{\underset{\mathbb{1}}{\parallel}}{g(0)} + \sum_{k=1}^{n} \alpha_k \left(\frac{\partial g}{\partial \alpha_k}\right)_{\alpha=0} + \dots O(\alpha^2) =$$

$$= \mathbb{1} + \sum_{k=1}^{n} \alpha_k \cancel{X} A_k + O(\alpha^2)$$

$$\boxed{A_k \equiv \left.\frac{\partial g}{\partial \alpha_k}\right|_{\alpha=0} \equiv \text{group generators!}}$$

$$\boxed{\mathcal{A} = \{A_k\}}$$

As we will see, A_k, $k=1 \dots n$ generate what is known as the Lie algebra $\mathcal{A}$ associated with the Lie group G.

Let's expand to $O(\alpha^3)$:

$$g(\alpha) = \mathbb{1} + \sum_{k=1}^{n} \alpha_k A_k + \frac{1}{2} \sum_{k,l=1}^{n} \alpha_k \alpha_l A_k A_l + O(\alpha^3)$$

$$g^{-1}(\alpha) = \mathbb{1} - (\text{—}\,''\,\text{—}) + (\text{—}\,''\,\text{—}) + O(\alpha^3)$$

Def: A commutator of two group elements:

$$g^{-1}(\beta)\, g^{-1}(\alpha)\, g(\beta)\, g(\alpha) = g(\gamma) \in G$$

some element of G

expand

$$\|$$

$$\mathbb{1} + \sum_{k,l=}^{n} \beta_k \alpha_l [A_k, A_l] + O(\alpha^2, \beta^2)$$

However $g(\gamma) = \mathbb{1} + \sum \gamma_k A_k +$

And both must be equal!

$$\Rightarrow \boxed{[A_k, A_l] = C^m_{kl} A_m}, \quad \text{where } \alpha_m = C^m_{kl} \beta_k \gamma_l$$

$\hookrightarrow$ "structure constants" of the Lie group

They are antisymmetric and satisfy Jacobi id:

$$\underbrace{C^m_{kl} = -C^m_{lk}}\,, \quad \underbrace{C^n_{kl} C^p_{mn} + C^n_{lm} C^p_{kn} + C^n_{mk} C^p_{ln} = 0}$$

since $[[A_k, A_l], A_m] + \dots = 0$

A set $\{A_k\}$ with constants C that satisfies is said to define a Lie algebra.

This can be extended to any α's (not small)

$$\boxed{g(\alpha) = e^{\sum \alpha_k A_k} = \mathbb{1} + \sum \alpha_k A_k + \dots}$$

Note: in the lectures we define $A \to iA$

If elements commute: Abelian group

otherwise: non-abelian.

Example: Matrix group $SO(2)$

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad SO(2) \text{ is } 1\text{-dim.}$$

Multiplication is matrix multiplication:

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} c' & -s' \\ s' & c' \end{pmatrix} = \begin{pmatrix} \cos(\theta+\theta') & -\sin(\quad) \\ \sin(\quad) & \cos(\quad) \end{pmatrix}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } \theta = 0$$

$$A = \left.\frac{\partial g}{\partial \theta}\right|_{\theta=0} = \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

group generator.

(if we define $A \to iA$ the generator becomes $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\sigma^2$)

Example 2: group $SU(2)$. $\text{Dim} = 3$ $\Big(\underbrace{2\times2\times2}_{\text{matrix elements}} - \underbrace{4}_{\text{unitary}} - \underbrace{1}_{\det=1}\Big)$

Generators: Pauli matrices

Structure constants $\sim \varepsilon_{kem}$

Representations of Lie groups

Rep: a map: $G \longrightarrow R(V \to V)$

R: maps; V: some vector space

i.e. we use the group elements to "label" all maps that map some Vector space into itself.

Dim of representation $= \dim(V) \neq \dim(G)$

(*) Usually R are matrices! that act on a vector space.

They satisfy: $\boxed{R(g_1)\,R(g_2) = R(g_1 g_2)}$

this is a requirement: since $g_1 \circ g_2 \longrightarrow g_1 g_2$

Therefore $R_{g_1} \circ R_{g_2} \longrightarrow R(g_1 g_2)$

If a matrix representation is:

$$\left(\begin{array}{c|c} A^{(N_1)} & B \\ \hline 0 & A^{(N_2)} \end{array}\right) \longrightarrow \text{reducible (total dim} = N_1 + N_2)$$

$$\left(\begin{array}{c|c} A^{N_1} & 0 \\ \hline 0 & A^{N_2} \end{array}\right) \longrightarrow \text{fully reducible.}$$

$$\left(\begin{array}{c|c} \neq 0 & \neq 0 \\ \hline \neq 0 & \neq 0 \end{array}\right) - \text{irreducible.}$$

A direct product of irreducible reps is a reducible, and sum over irreps:

$$\boxed{R_1 \otimes R_2 = \bigoplus_i R_i}$$ ← details are group specific.

Every matrix group has at least 2 representations:

1) Fundamental: R is G itself. ($\dim R = \dim G$)
2) Adjoint: R is $\mathcal{A}$ i.e. the Lie algebra of G.

→ Both are very important in non-abelian theories:

say QCD: $G = SU(3)$, $\dim = 8$

quarks transform in fundamental (i.e. 3×3 matrices)

gluons are adjoint ⇒ matrix elements are the structure constants!

$$(A^a)_{bc} \sim f^a{}_{bc} \quad (8 \times 8)$$

i.e. the adjoint representation has dimension equal to the dim of the group.

Gell-Mann matrices

$$\Rightarrow \boxed{\begin{aligned} \dim(\text{fundam}) &= 3 \\ \dim(\text{adjoint}) &= 8 \\ \dim G = \dim(\mathcal{A}) &= 8 \end{aligned}}$$

<u>Note</u>: All unitary (and orthogonal) groups are compact!