NST Part III Experimental and Theoretical Physics — Michaelmas 2018

Dr. Alexander Mitov

Particle Physics Major Option

EXAMPLES SHEET 1

SPECIAL RELATIVITY

1. a) Draw the two leading-order Feynman diagrams for $\mathrm{e^+e^-} \to \mathrm{e^+e^-}$ involving single photon exchange, and write q, the 4-momentum of the exchanged virtual photon, in terms of the 4-momenta of the initial and/or final state particles. By evaluating q^2 in the centre of mass frame, or otherwise, determine whether q is *timelike* ($q^2 > 0$) or *spacelike* ($q^2 < 0$) in each case.

   b) The *Mandelstam variables* s, t, u in the scattering process $a + b \to 1 + 2$ are defined in terms of the particle 4-vectors as

   $$s = (p_a + p_b)^2, \qquad t = (p_a - p_1)^2, \qquad u = (p_a - p_2)^2 \ .$$

   Show that $s + t + u = {m_a}^2 + {m_b}^2 + {m_1}^2 + {m_2}^2$.

   c) Show that $\sqrt{s}$ is the total energy of the collision in the centre of mass frame.

   d) At the HERA accelerator in Hamburg, 27.5 GeV electrons are brought into head-on collision with 820 GeV protons. Calculate the centre of mass energy, $\sqrt{s}$, of $\mathrm{e^-p}$ collisions at HERA, and determine the beam energy that would be needed to produce $\mathrm{e^-p}$ collisions with this value of $\sqrt{s}$ using electrons incident on a stationary proton target.

   e) Show that, in the laboratory frame with particle X at rest, the reaction $\nu + X \to \ell + Y$ can only proceed if the incoming neutrino has an energy above a threshold given by

   $$E_\nu > \frac{(m_l + m_Y)^2 - m_X^2}{2m_X} \ .$$

2. a) For a particle of four-momentum $p^\mu = (E, p_x, p_y, p_z)$, show that the scalar product

   $$p^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

   is Lorentz invariant by explicitly transforming the four components of p^μ.

   b) Use the Lorentz transformations to show that the volume element d^3p/E in momentum space is Lorentz invariant, *i.e.* that

   $$\frac{\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z}{E} = \frac{\mathrm{d}p'_x \mathrm{d}p'_y \mathrm{d}p'_z}{E'} \ .$$

3. In a 2-body decay, $a \to 1 + 2$, show that the three-momentum of the final state particles in the centre of mass frame has magnitude

   $$p^* = \frac{1}{2m_a}\sqrt{[m_a^2 - (m_1 + m_2)^2]\,[m_a^2 - (m_1 - m_2)^2]} \ .$$

TWO BODY DECAY

4. According to the hypothesis of SU(3) symmetry (*i.e.* uds flavour independence) of invariant matrix elements, the two-body decay processes $\rho \to \pi\pi$ and $\mathrm{K}^* \to \mathrm{K}\pi$ have invariant matrix elements of the form

$$M_{\mathrm{fi}} = C p_\pi$$

where $C_\rho / C_{K^*} = 2/\sqrt{3}$ and p_π is the final state centre of mass momentum. Show that the predicted ratio of decay rates agrees with experiment to within about 15%.

[Use the result of Question 3 to obtain p_π. Take the π, ρ, K and K^* meson masses to be 139, 770, 494 and 892 MeV respectively. The measured widths are $\Gamma(\rho \to \pi\pi) = 153 \pm 2\,\mathrm{MeV}$ and $\Gamma(\mathrm{K}^* \to \mathrm{K}\pi) = 51.3 \pm 0.8\,\mathrm{MeV}$.]

5. The π^+ meson decays almost entirely via the two body decay process $\pi^+ \to \mu^+ \nu_\mu$, with an invariant matrix element given by

$$|M_{\mathrm{fi}}|^2 = 2 G_{\mathrm{F}}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

where $G_{\mathrm{F}} = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi constant, and f_π is related to the size of the pion wavefunction (the pion being a composite object).

a) Obtain a formula for the $\pi^+ \to \mu^+ \nu_\mu$ decay rate. Assuming $f_\pi \sim m_\pi$, calculate the pion lifetime in natural units and in seconds, and compare to measurement.

[$m_\pi = 139.6\,\mathrm{MeV}$, $m_\mu = 105.7\,\mathrm{MeV}$.]

b) By replacing m_μ by m_e, show that the rate of $\pi^+ \to \mathrm{e}^+ \nu_{\mathrm{e}}$ decay is 1.28×10^{-4} times smaller than the corresponding decay rate to muons. Show also that, on the basis of phase space alone (*i.e.* neglecting the factor $|M_{\mathrm{fi}}|^2$), the decay rate to electrons would be expected to be *greater* than the rate to muons.

THE DIRAC EQUATION

6. Write down a simplified form of the Dirac equation for a spinor $\psi(t)$ describing a particle of mass m at rest. For the standard Pauli-Dirac representation of the γ matrices, obtain a differential equation for each component ψ_i of the spinor ψ, and hence write down a general solution for the evolution of ψ. Comment on your result and on its relation to the standard plane wave solutions involving $u_1(p)$, $u_2(p)$, $v_1(p)$, $v_2(p)$.

7. a) For the standard Pauli-Dirac representation of the γ matrices, and for an arbitrary pair of spinors ψ and ϕ with components ψ_i and ϕ_i, show that the current $\overline{\psi}\gamma^\mu\phi$ is given by

$$\begin{aligned}\overline{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\ \overline{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\ \overline{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\ \overline{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2\end{aligned}$$

   b) For a particle or antiparticle with four-momentum $p^\mu = (E, p_x, p_y, p_z)$, show that

$$\overline{u}_1\gamma^\mu u_1 = \overline{u}_2\gamma^\mu u_2 = \overline{v}_1\gamma^\mu v_1 = \overline{v}_2\gamma^\mu v_2 = 2p^\mu$$

   and that

$$\overline{u}_1\gamma^\mu u_2 = \overline{u}_2\gamma^\mu u_1 = \overline{v}_1\gamma^\mu v_2 = \overline{v}_2\gamma^\mu v_1 = 0\ .$$

   c) Hence show that the current $j^\mu = \overline{\psi}(p)\gamma^\mu\psi(p)$ corresponding to a general free particle spinor $\psi(p) = u(p)e^{i(\boldsymbol{p}\cdot\boldsymbol{r}-Et)}$ or antiparticle spinor $\psi(p) = v(p)e^{-i(\boldsymbol{p}\cdot\boldsymbol{r}-Et)}$ is given by $j^\mu = 2p^\mu$. Write down the particle density and flux represented by j^μ.

8. a) For a particle with 4-momentum $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, show that the spinors $(1+\gamma^5)u_1$ and $(1+\gamma^5)u_2$ are not in general proportional to $u_\uparrow$ but become so in the relativistic limit $E \gg m$.

   b) Define the terms *helicity* and *chirality*. How are chirality and helicity related to the spinors and result described in part (a) ?

   c) What would be the equivalent result to that described in (a) for the corresponding antiparticle spinors $(1+\gamma^5)v_1$ and $(1+\gamma^5)v_2$?

9. a) Without resorting to an explicit representation of the Dirac gamma matrices, show that the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ has the following properties:

$$(\gamma^5)^2 = 1, \qquad \gamma^{5\dagger} = \gamma^5, \qquad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5\ .$$

   b) Show that the adjoint spinors $\overline{\psi_{\mathrm{L}}}$ and $\overline{\psi_{\mathrm{R}}}$ corresponding to the left-handed and right-handed components $\psi_{\mathrm{L}} \equiv \frac{1}{2}(1-\gamma^5)\psi$ and $\psi_{\mathrm{R}} \equiv \frac{1}{2}(1+\gamma^5)\psi$ are:

$$\begin{aligned}\overline{\psi_{\mathrm{L}}} &= \overline{\psi}\tfrac{1}{2}(1+\gamma^5) \\ \overline{\psi_{\mathrm{R}}} &= \overline{\psi}\tfrac{1}{2}(1-\gamma^5)\ .\end{aligned}$$

   c) Show that $\overline{\psi_{\mathrm{L}}}\gamma^\mu\psi_{\mathrm{R}} = \overline{\psi_{\mathrm{R}}}\gamma^\mu\psi_{\mathrm{L}} = 0$, and that the current $\overline{\psi}\gamma^\mu\psi$ can be decomposed as

$$\overline{\psi}\gamma^\mu\psi = \overline{\psi_{\mathrm{L}}}\gamma^\mu\psi_{\mathrm{L}} + \overline{\psi_{\mathrm{R}}}\gamma^\mu\psi_{\mathrm{R}}\ .$$

ELECTRON-MUON ELASTIC SCATTERING

10. a) Show that the matrix element for $e^-\mu^- \to e^-\mu^-$ scattering via single photon exchange is

$$M_{\rm fi} = -\frac{e^2}{(p_1-p_3)^2} g_{\mu\nu}\left[\overline{u}(p_3)\gamma^\mu u(p_1)\right]\left[\overline{u}(p_4)\gamma^\nu u(p_2)\right]$$

where p_1 and p_3 are the initial and final e^- four-momenta and p_2 and p_4 are the initial and final μ^- four-momenta.

b) Show that, for scattering in the centre of mass frame with incoming and outgoing e^- four-momenta $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$, the electron current for the various possible electron spin combinations is

$$\begin{aligned}
\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1c, ps, -ips, pc)\\
\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0)\\
\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1c, ps, ips, pc)\\
\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= -2(ms, 0, 0, 0)
\end{aligned}$$

where m is the electron mass and $s \equiv \sin\theta/2$, $c \equiv \cos\theta/2$.

c) Write down the incoming and outgoing muon 4-momenta p_2 and p_4, and the helicity eigenstate spinors $u_\uparrow(p_2)$, $u_\downarrow(p_2)$, $u_\uparrow(p_4)$ and $u_\downarrow(p_4)$. [Take the muon mass to be M and the muon energy to be E_2]. By comparing the forms of the muon and electron spinors, explain how the muon currents

$$\begin{aligned}
\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2c, -ps, -ips, -pc)\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0)\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2c, -ps, ips, -pc)\\
\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0)
\end{aligned}$$

can be written down without any further calculation.

d) Explain why some of the above currents vanish in the relativistic limit where the electron mass and muon mass can be neglected. Sketch the spin configurations which are allowed in this limit.

e) Show that, in the relativistic limit, the matrix element squared $|M_{\rm LL}|^2$ for the case where the incoming e^- and incoming μ^- are both left-handed is given by

$$|M_{\rm LL}|^2 = \frac{4e^4s^2}{(p_1-p_3)^4}$$

where $s = (p_1+p_2)^2$. Why is the numerator of $|M_{\rm LL}|^2$ independent of θ ?

f) Find a similar expression for the matrix element $|M_{\rm RL}|^2$ for a right-handed incoming e^- and a left-handed incoming μ^-, and explain why $|M_{\rm RL}|^2$ vanishes when $\theta = \pi$. Write down the corresponding results for $|M_{\rm RR}|^2$ and $|M_{\rm LR}|^2$.

g) Show that, in the relativistic limit, the differential cross section for unpolarised $e^-\mu^- \to e^-\mu^-$ scattering in the centre of mass frame is

$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2}{s}\cdot\frac{1+\frac{1}{4}(1+\cos\theta)^2}{(1-\cos\theta)^2}\,.$$

h) Show that the spin-averaged matrix element squared can be expressed in Lorentz-invariant form as

$$\left\langle |M_{\mathrm{fi}}|^2 \right\rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) \right] ,$$

and that a Lorentz invariant form for the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2}{q^4} \left[1 + \left(1 + \frac{q^2}{s} \right)^2 \right]$$

where $q^2 = (p_1 - p_3)^2$.

The remainder of this question involves the derivation of a general expression for $\langle |M_{\mathrm{fi}}|^2 \rangle$ for the case of finite electron and muon masses, and is optional:

i) Show that the spin-averaged matrix element squared for unpolarised $\mathrm{e}^-\mu^- \to \mathrm{e}^-\mu^-$ scattering can be written in the form

$$\left\langle |M_{\mathrm{fi}}|^2 \right\rangle = \frac{1}{4} \sum_{\text{spins}} |M_{\mathrm{fi}}|^2 = \frac{1}{4} \frac{e^4}{(p_1 - p_3)^4} L^{\mu\nu} W_{\mu\nu}$$

where the electron and muon tensors $L^{\mu\nu}$ and $W^{\mu\nu}$ are given by

$$L^{\mu\nu} \equiv \sum_{\text{spins}} \left[\overline{u}(p_3) \gamma^\mu u(p_1) \right] \left[\overline{u}(p_3) \gamma^\nu u(p_1) \right]^*$$

$$W_{\mu\nu} \equiv \sum_{\text{spins}} \left[\overline{u}(p_4) \gamma_\mu u(p_2) \right] \left[\overline{u}(p_4) \gamma_\nu u(p_2) \right]^*$$

j) Using the electron currents from part b) above, show that the components of the electron tensor $L^{\mu\nu}$ are

$$\begin{pmatrix} L^{00} & L^{01} & L^{02} & L^{03} \\ L^{10} & L^{11} & L^{12} & L^{13} \\ L^{20} & L^{21} & L^{22} & L^{23} \\ L^{30} & L^{31} & L^{32} & L^{33} \end{pmatrix} = 8 \begin{pmatrix} E_1^2 c^2 + m^2 s^2 & E_1 psc & 0 & E_1 pc^2 \\ E_1 psc & p^2 s^2 & 0 & p^2 sc \\ 0 & 0 & p^2 s^2 & 0 \\ E_1 pc^2 & p^2 sc & 0 & p^2 c^2 \end{pmatrix} ,$$

and hence verify that $L^{\mu\nu}$ has the Lorentz invariant form

$$L^{\mu\nu} = 4 \left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu} \left(m^2 - p_1.p_3 \right) \right] .$$

k) Write down the corresponding expression for $W^{\mu\nu}$ and hence show that

$$\langle |M_{\mathrm{fi}}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_2.p_4)m^2 + 2m^2M^2 \right]$$

NUMERICAL ANSWERS

1. d) $\sqrt{s} = 300\,\mathrm{GeV}$; $E = 48000\,\mathrm{GeV}$

4. $\Gamma(\rho \to \pi\pi)/\Gamma(\mathrm{K}^* \to \mathrm{K}\pi) = 3.46$; expt $= 2.98$

5. a) $\tau_\pi = 3.0 \times 10^{16}\,\mathrm{GeV}^{-1} = 1.97 \times 10^{-8}\,\mathrm{s}$; expt $= 2.6 \times 10^{-8}\,\mathrm{s}$

   b) from phase space alone: $\Gamma(\pi^+ \to \mathrm{e}^+\nu_\mathrm{e})/\Gamma(\pi^+ \to \mu^+\nu_\mu) = 2.34$