5.2 Scalar field quantization

There exist two popular formalisms for QFT. Each has its advantages and disadvantages. Here we follow the approach of canonical quantization. Its great advantage, for our purposes, is that it is rather close to what you have already done in QM. Its great disadvantage is that it is not well-suited to gauge field theories. We shall circumvent this hurdle by studying only simple examples of QFTs, which are suited to canonical quantization, to begin with, and by using these examples to motivate the form of the Feynman rules for more complex theories.

Those of you who view this course as the beginning of your career in physics (rather than the end) would be well advised to consult the literature for how to do canonical quantization properly and for the other, path integral, approach.

We begin with a real, scalar field. We have already encountered the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial^{\mu} \phi \partial_{\mu} \phi - m^2 \phi^2).$$

(5.16)

The point of departure from QM is that we shall try to quantize the field $\phi$, rather than the position $x$.$^{12}$ Thus, we compute the momentum conjugate to the field $\phi$, namely $\pi \equiv \frac{\delta \mathcal{L}}{\delta \phi}$ and impose the equal time commutation relations

$$[\phi(x^i, t), \pi(x^j, t)] = i \delta^3(x^i - x^j),$$

(5.17)

$$[\phi(x^i, t), \phi(x^j, t)] = [\pi(x^i, t), \pi(x^j, t)] = 0.$$  

(5.18)

The $\delta$ function simply accounts for the fact that the fields at different space points are considered to be independent. $^{13}$ Notice that, since the operators $\phi$ and $\pi$ depend on time, we are working in the Heisenberg picture of QM, rather than the Schrödinger picture (in the latter, operators are constant in time and states have all the time dependence). We’ll have more to say about this later on.

The basic goal in QM is to find the spectrum of energies and eigenstates of the Hamiltonian. This looks like a hard problem for our field theory, for which the Hamiltonian (density) is given by

$$\mathcal{H}(\phi, \pi) \equiv \pi \dot{\phi} - \mathcal{L} = \frac{1}{2} \left( \pi^2 + (\nabla \phi)^2 + m^2 \phi^2 \right).$$

(5.19)

Thankfully, it is rendered almost trivial if we make the Fourier transform

$$\phi(x, t) = \int \frac{d^3p}{(2\pi)^3 2E} \left( a_p e^{-iEt+i\mathbf{p} \cdot \mathbf{x}} + a_p^\dagger e^{+iEt-i\mathbf{p} \cdot \mathbf{x}} \right),$$

(5.20)

with $E \equiv +\sqrt{\mathbf{p}^2 + m^2}$. Note that we have forced $\phi$ to be real (or rather Hermitian, since it is now to be interpreted as an operator). Note also that we have normalized using the Lorentz-invariant integration measure $\frac{d^3p}{(2\pi)^3 2E}$. $^{14}$

$^{12}$ Such a dramatic change makes it hard to imagine how QM can be recovered as a limit of QFT; we shall have to go through some acrobatics later on to do so.

$^{13}$ This statement can be generalized to independence on any space-like separation of which the above is a particular case.

$^{14}$ This is Lorentz invariant, because it can also be written as $\frac{1}{(2\pi)^3} \int d^4p \delta(p^2 - m^2)$. 

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With this transformation, one may show (recall that \( \int d^3p e^{ipx} = (2\pi)^3 \delta^3(x) \)) that the commutation relations (5.17) can be reproduced by

\[
[a_p, a_{p'}^\dagger] = (2\pi)^3 2E \delta^3(p - p'), \tag{5.21}
\]
\[
[a_p, a_p] = [a_p^\dagger, a_{p'}^\dagger] = 0. \tag{5.22}
\]

This is encouraging, since (apart from a normalization factor) these are the usual commutation relations for the ladder operators \( a \) and \( a^\dagger \) of the simple harmonic oscillator, with one oscillator for each \( p \). The delta function expresses the fact that the different oscillators are independent. Even better, the various contributions to the Hamiltonian (not the Hamiltonian density, for once) may be written as (note that \( E = E' \) when \( p' = -p \), etc)

\[
\frac{1}{2} \int d^3x \, m^2 \phi^2 = \frac{1}{(2\pi)^3 8E^2} \int d^3p \, m^2 \left( a_p a_{-p} e^{-2iEt} + a_{-p}^\dagger a_p e^{2iEt} + a_p a_{p}^\dagger + a_{p}^\dagger a_p \right) \tag{5.23}
\]

\[
\frac{1}{2} \int d^3x \, (\nabla \phi)^2 = \frac{1}{(2\pi)^3 8E^2} \int d^3p \, p^2 \left( a_p a_{-p} e^{-2iEt} + a_{-p}^\dagger a_p e^{2iEt} + a_p a_{p}^\dagger + a_{p}^\dagger a_p \right) \tag{5.24}
\]

\[
\frac{1}{2} \int d^3x \, \pi^2 = \frac{1}{(2\pi)^3 8E^2} \int d^3p \, E^2 \left( -a_p a_{-p} e^{-2iEt} - a_{-p}^\dagger a_p e^{2iEt} + a_p a_{p}^\dagger + a_{p}^\dagger a_p \right). \tag{5.25}
\]

All in all, we end up with

\[
H = \int \frac{d^3p}{(2\pi)^3 2E} \, E \left( a_p a_p^\dagger + a_p^\dagger a_p \right). \tag{5.26}
\]

Again, this is nothing other than the Hamiltonian of a set of independent simple harmonic oscillators\footnote{Recall that the SHO Hamiltonian may be written as \( \omega (a^\dagger a + \frac{1}{2}) \equiv \frac{E}{\hbar} (a^\dagger a + aa^\dagger) \).} (one for each \( p \)) of frequency \( \omega = E \), summed over \( p \) with the density of states factor. It is then simple to figure out the spectrum. Define the vacuum (a.k.a. the ground state) to be the state \( |0\rangle \) annihilated by all of the \textit{annihilation operators}, \( a_p \), viz. \( a_p |0\rangle = 0 |\rangle p \). Then, acting on the vacuum with a single \textit{creation operator}, \( a_p^\dagger \), one produces a state \( |p\rangle \equiv a_p^\dagger |0\rangle \) of momentum \( p \) and energy \( E \). (To show this explicitly, one should act on the state \( a_p^\dagger |0\rangle \) with the Hamiltonian \( H \) and with the momentum \( P \), where \( P \) here is not the field momentum \( \pi \), but rather is the operator corresponding to the generator of spatial translations. We shall do this later on.) In QM we call this the first excited state, but in QFT we interpret it as a state with a single particle of momentum \( p \). A two-particle state would be given by \( |p, p'\rangle \equiv a_p^\dagger a_{p'}^\dagger |0\rangle \), where the particles have momenta \( p \) and \( p' \), and so on. Note how the commutation relation \( [a_p^\dagger, a_{p'}^\dagger] = 0 \) implies immediately that a multiparticle wavefunction is symmetric under the interchange of any two particles: \( \ldots a_p^\dagger a_{p'}^\dagger \ldots |0\rangle = \ldots a_{p'}^\dagger a_p^\dagger \ldots |0\rangle \). Thus, quantum field theory predicts that spinless excitations of the Klein-Gordon field obey Bose-Einstein statistics. Amazing.

The simple harmonic oscillator number operator \( a_p^\dagger a_p \) is now interpreted as counting the number of particles that are present with momentum \( p \) (this is easy to check by acting...
on any state of the above type). Note that the total number of particles is measured by the operator

\[ N = \int \frac{d^3p}{(2\pi)^3} 2E a_p^\dagger a_p \]  

(5.27)

which is not a conserved quantity for the real Klein-Gordon field (it does not correspond to a symmetry of the action). So the total number of particles, unlike in QM, is not fixed.

Notice also that the problem of negative energy solutions has gone away. Indeed, the negative frequency modes in the superposition (5.20) now have a different interpretation: they accompany the annihilation operators \( a_p \) and reflect the fact that annihilating a particle of energy \( E \) causes the total energy stored in the field to decrease by \( E \).

In its place, a different problem appears. Let us try to calculate the energy of the vacuum state \(|0\rangle\). It is

\[ \langle 0 | H | 0 \rangle = \int d^3p \, \delta^3(0) \frac{E}{2}. \]  

(5.28)

The first disturbing thing about this expression is that it contains \( \delta(0) \). This in fact just corresponds to the volume of space: since \( \int d^3x \, e^{ip \cdot x} = (2\pi)^3 \delta^3(p) \), we may write \( V \equiv \int d^3x = (2\pi)^3 \delta^3(0) \). But even the Hamiltonian density is divergent, because it is a sum over all momentum modes of the SHO zero point energy \( \frac{E}{2} \). At least if we forget about gravity, we can sidestep this problem by observing that we are only able to measure energy differences in experiment. Thus we can simply re-define the Hamiltonian to be \( H = \langle 0 | H | 0 \rangle \).

Effectively, this can be implemented by ensuring that we always put operators in normal order, by which we mean that annihilation operators always appear to the right of creation operators. This guarantees that a normally-ordered operator will vanish when acting on the vacuum state. A normally-ordered operator is denoted by enclosing it in a pair of colons. The normally-ordered Hamiltonian, for example, is given by

\[ :H: \equiv \int \frac{d^3p}{(2\pi)^3} 2E a_p^\dagger a_p. \]  

(5.29)

This problem of the vacuum energy is only the first of many peccadillos that appear in quantum field theory. In this case, it seems relatively benign. The other peccadillos (which confused the founding fathers for decades) are now well understood. But this first problem of the vacuum energy reappears when we consider coupling quantum field theory to gravity, giving rise to the cosmological constant problem. It is arguably among the greatest unsolved problems in the Universe today.

### 5.3 Multiple scalar fields

Quantization of more than one scalar field is trivial, but it is helpful to point out one or two conceptual issues. Consider \( n \) real, scalar fields, \( \phi_i \). If we allow a maximum of two derivatives and two fields in each term, we claim that the Lagrangian can be written, without loss of generality, as

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi_i \partial^\mu \phi_i - m_i^2 \phi_i^2). \]  

(5.30)
Why? The most general kinetic term (the one involving the derivatives) could be written as $Z_{ij} \partial_\mu \phi_i \partial^\mu \phi_j$, but the matrix $Z_{ij}$ may be diagonalized by an orthogonal transformation of the fields $\phi_i$. An independent rescaling of the fields $\phi_i$ can then make each of the eigenvalues equal to $\pm 1$. An eigenvalue of $-1$ would result in an inconsistent theory, since the kinetic energy would be unbounded below. So the kinetic term can always be written in the canonical form $\delta_{ij} \partial_\mu \phi_i \partial^\mu \phi_j$. Now, this kinetic term (which must be present in order to have a consistent theory) has a global $O(n)$ symmetry,\footnote{$O(n)$ just means the group of $n \times n$ orthogonal matrices. We’ll say more about it later on.} corresponding to orthogonal rotations of the fields $\phi_i$. This then is the largest possible symmetry that a theory based on $n$ real scalar fields can have, since the kinetic term must always be present for a dynamical field. This observation will be important when we come to consider gauge theories, since the name of the game there will be to promote a subgroup of this to a local symmetry.

As for the mass term, this too could be an arbitrary symmetric matrix, in the basis in which the kinetic term is canonical. This too can be diagonalized by an orthogonal transformation, without changing the form of the kinetic term. Hence we arrive at the Lagrangian written above. Note that the mass terms break the $O(n)$ symmetry, unless we force all the $m_i$ to be equal.

A particularly interesting example is $n = 2$, with $m_1 = m_2 \equiv m$. This theory has $SO(2)$ symmetry, which you may know is (locally) equivalent to a $U(1)$ symmetry.\footnote{Again, if you don’t know what $SO(2)$ and $U(1)$ mean yet, don’t panic: I’ll say more about them later on. For now, $SO(2)$ is the group of $2 \times 2$, orthogonal matrices with unit determinant and $U(1)$ is the group of $1 \times 1$, unitary matrices, a.k.a complex numbers of the form $e^{i\theta}$.} One possibility is to simply quantize the two fields, $\phi_1$ and $\phi_2$, independently, as we did in the last section. Evidently there are two types of ‘particle’, related somehow by the $SO(2)$ symmetry. More illuminating is to define a complex scalar field, $\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, in terms of which the Lagrangian may be written as

$$\mathcal{L} = (\partial_\mu \phi^\ast \partial^\mu \phi - m^2 |\phi|)^2. \quad (5.31)$$

This can be quantized via the mode expansion

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3 2E} \left( a_p e^{-iEt + ip \cdot x} + b_p^\dagger e^{iEt - ip \cdot x} \right), \quad (5.32)$$

with

$$[a_p, a_{p'}^\dagger] = (2\pi)^3 2E \delta^3(p - p'), \quad (5.33)$$

$$[b_p, b_{p'}^\dagger] = (2\pi)^3 2E \delta^3(p - p'), \quad (5.34)$$

with all other commutators vanishing. It is not surprising that there are now two particle creation operators, since there were two real scalar fields to begin with. In the complex field formalism here, we need two mode operators in the Fourier expansion because $\phi$ is complex. The Hamiltonian is given by

$$H := \int \frac{d^3p}{(2\pi)^3 2E} E \left( a_p^\dagger a_p + b_p^\dagger b_p \right). \quad (5.35)$$
As expected, since the two types of particle have the same mass, they contribute in the same way to the total energy.

What about the $SO(2)$ invariance? In the complex field formalism, it maps to the simple $U(1)$ rephasing: $\phi \rightarrow e^{i\alpha}\phi$. Noether’s theorem tells us that there is a conserved charge and in terms of creation and annihilation operators it is given by

$$Q = \int \frac{d^3p}{(2\pi)^3 2E} \left( a_p^\dagger a_p - b_p^\dagger b_p \right). \quad (5.36)$$

Note, crucially, that it is the number of particles of type $a$ minus the number of particles of type $b$ that is conserved. We call the particles of type $b$ antiparticles. They have the same mass as the particles, but the opposite charge. Recall that when we couple such a field to electromagnetism, we do so precisely by gauging the phase invariance $\phi \rightarrow e^{i\alpha}\phi$, so the charge $Q$ is to be interpreted as the electric charge.

What we achieved up to here is not too complicated technically but is rather deep conceptually. Perhaps this is a good place to pause for a while and reflect on what we just derived.

This leads us naturally on to study charge conjugation. Roughly speaking, this operation is defined as exchanging particles with their antiparticles and is related to complex conjugation; many treatments therefore define it in association with various flips of $i$ to minus $i$ and $e$ to minus $e$, etc.

This, in my view, is deeply confusing, since $i$ and $e$ are supposed to be fixed constants of Nature (indeed, we have known since the old testament that we should only exchange an $i$ for an $i$ . . . ). Much better is to define charge conjugation as a symmetry in exactly the way that we defined other symmetries above: a transformation acting on fields that leaves the action invariant.

We’ll begin with the Klein-Gordon field. The Lagrangian is

$$\mathcal{L} = (\partial_\mu - ieA_\mu)\phi^* (\partial^\mu + ieA^\mu)\phi - m^2|\phi|^2. \quad (5.37)$$

I hope it is obvious that this is invariant under the transformation $A_\mu \rightarrow -A_\mu$ and $\phi \rightarrow \phi^*$. More particularly, the transformation corresponds to the symmetry group $\mathbb{Z}_2$, because transforming twice takes $A_\mu \rightarrow -A_\mu \rightarrow A_\mu$ and $\phi \rightarrow \phi^* \rightarrow \phi$, which is the same as the identity transformation. Because it is a discrete transformation, Noether’s theorem does not imply a conserved charge in this case. Note that the transformation $A_\mu \rightarrow -A_\mu$ is just what we expect for charge conjugation from Maxwell’s equations, which will be unchanged if we also flip the sign of the charge and the current (which in QFT will be generated by field configurations like $\phi$ and $\psi$).

Now let’s do it for the Dirac field. Here it is not so simple to guess what the symmetry transformation is by looking at the Lagrangian, so we’ll find our way along with the help of Simplicio, Salviati, and Sagredo, the three fictional characters of the Galilean trialogue.

The Dirac Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \quad (5.38)$$
with $D_\mu = \partial_\mu + ieA_\mu$. Simplicio knows, from his study of Maxwell’s equations, that the transformation of $A_\mu$ must be $A_\mu \rightarrow -A_\mu$ and he guesses that he can just complex conjugate $\psi$, as he did for the Klein-Gordon field. This doesn’t work well at all. Consider the mass term for example, this transforms as

$$
\bar{\psi}\psi \rightarrow \psi^T \gamma^0 \psi^* = -\psi^\dagger (\gamma^0)^T \psi = -\psi^\dagger \gamma^0 \psi = -\bar{\psi}\psi.
$$

(5.39)

This argument is a bit subtle: in the second step we have used the fact that the whole quantity is just a number (not a matrix) and therefore equals its transpose. But as we shall see in the next subsection, this theory can only make sense as a QFT if the field anticommutes with itself. Thus, the transpose of a product of two fields is equal to minus the reversed product of the transposed fields. Once we take this into account, we see that charge conjugation cannot just involve complex conjugation of the fields, because the mass term in the Lagrangian would not be invariant. If we wanted the electron to be charged, it would have to be massless, which it is not. Simplicio is stuck.

Now Salviati enters the fray. He realises that complex conjugation is somewhat ambiguously defined for a multi-component spinor, since one could also mix up the different components at the same time. So he says, “Maybe it should be $\psi \rightarrow C\gamma^0 \psi^*$,” for some matrix $C$. Then we’d find

$$
\bar{\psi}\psi \rightarrow \bar{\psi}\psi,
\bar{\psi}\gamma^\mu \psi' \rightarrow -\bar{\psi}\gamma^\mu \psi,
$$

(5.40)

provided $CC^\dagger = 1$ and $C^\dagger \gamma^\mu C = -(\gamma^\mu)^T$.” Note that Salviati carefully wrote the second relation for a bi-linear combination of two different fields $\psi$ and $\psi'$, to stress that they get flipped by $C$.

Only now does Sagredo realise the true genius of Salviati. Sagredo realises that if we set $\psi' = \psi$ in (5.40), we find $\bar{\psi}A_\mu \gamma^\mu \psi \rightarrow \bar{\psi}A_\mu \gamma^\mu \psi$, whereas if we set $\psi' = \partial_\mu \psi$, we find $\bar{\psi}\partial_\mu \gamma^\mu \psi \rightarrow -\bar{\psi}\partial_\mu \gamma^\mu \psi + \bar{\psi}\partial_\mu \gamma^\mu \psi$ (where in the last step we integrated by parts). So all terms in the Lagrangian will be invariant.

Simplicio hasn’t really followed any of this, but he does point out that a suitable $C$ is $i\gamma^2 \gamma^0$. Thus, we can now forget the triologue and remember only that charge conjugation can be implemented on Dirac spinors as $\psi \rightarrow i\gamma^2 \psi^*$. Let me make one last point, which will be important when we study non-Abelian gauge theories. Imagine that $\psi$ carries an extra index $i$ and that $A_\mu$ is really a matrix with indices $i$ and $j$. Then, by an obvious generalization of Salviati’s result, $\bar{\psi}_i \gamma^\mu \psi_j \rightarrow -\bar{\psi}_j \gamma^\mu \psi_i$ and charge conjugation will only be a symmetry of the Lagrangian if we also define $A_\mu^{ij} \rightarrow -A_\mu^{ji}$. So a matrix-valued gauge field must go to minus its transpose under charge conjugation.

### 5.4 Spin-half quantization

We now wish to quantize the Dirac Lagrangian

$$
\mathcal{L} = \bar{\psi}(i\Slash{D} - m)\psi.
$$

(5.41)
To do so, we first derive the Hamiltonian. The field momenta conjugate to the fields $\psi$ and $\bar{\psi}$ are

$$\pi \equiv \frac{\delta L}{\delta \dot{\psi}} = i\psi^\dagger,$$  \hspace{1em} (5.42)

$$\bar{\pi} \equiv \frac{\delta L}{\delta \dot{\bar{\psi}}} = 0,$$  \hspace{1em} (5.43)

whence the Hamiltonian is

$$\mathcal{H} = -\bar{\psi}i\gamma \cdot \nabla \psi + m\bar{\psi}\psi.$$  \hspace{1em} (5.44)

We guess from our experience with the Klein-Gordon system that our best chance at solving this system is to do a Fourier transform. For this, we need a complete set of plane wave solutions to the Dirac equation. For the positive-energy solutions, we write these as $\psi = u_p^s e^{-ip \cdot x}$; plugging into the Dirac equation, we find that they satisfy

$$(\slashed{p} - m)u_p^s = 0.$$  \hspace{1em} (5.45)

There are two solutions (one for each of the two possible spin states), which we label by $s \in \{1, 2\}$. We found explicit expressions for these earlier in the Pauli-Dirac basis, but we do not need them here. Instead we simply note that since the $u$ provide a complete set of states, the combination

$$\sum_s u_p^s \bar{\pi}_p^s$$

must satisfy a completeness relation. Moreover, this must be proportional to $\slashed{p} + m$, since acting on the left with $\slashed{p} - m$ then gives something proportional to $\slashed{p}^2 - m^2 = p^2 - m^2 = 0$. This is as it should be, since $(\slashed{p} - m)u_p^s = 0$. We fix the normalization so that the proportionality constant is unity (this corresponds to $2E$ particles per unit volume, as for the Klein-Gordon field). Thus

$$\sum_s u_p^s \bar{\pi}_p^s = \slashed{p} + m.$$  \hspace{1em} (5.47)

Similarly, for the two negative energy solutions, we write $\psi = v_p^s e^{+ip \cdot x}$; plugging into the Dirac equation, we find that they satisfy

$$(\slashed{p} + m)v_p^s = 0$$

with completeness relation

$$\sum_s v_p^s \bar{\pi}_p^s = \slashed{p} - m.$$  \hspace{1em} (5.49)

Our mode expansion is then

$$\psi = \int \frac{d^3p}{(2\pi)^3 2E} \left( c_p^s u_p^s e^{-ip \cdot x} + d_p^s v_p^s e^{+ip \cdot x} \right).$$  \hspace{1em} (5.50)
where a sum on $s$ is implicit. As for the complex Klein-Gordon case, since $\psi$ is complex we need two operators $c$ and $d$.

So far, we have made no mention of commutation relations, with good reason. To see why, let us compute the form of the conserved charge, $Q \equiv \int d^3x \psi \bar{\psi}$ (corresponding to the re-phasing symmetry $\psi \to e^{i\alpha}\psi$). We find

$$Q = \int \frac{d^3p}{(2\pi)^3(2E)^2} \left( u_p^s \bar{u}_p^{s'} c_p^{s'} \bar{c}_p^s + v_p^s \bar{v}_p^{s'} d_p^{s'} \bar{d}_p^s + v_p^s \bar{v}_p^{s'} c_p^{s'} d_p^s e^{+2iEt} + u_p^s \bar{u}_p^{s'} d_p^{s'} \bar{c}_p^s e^{-2iEt} \right),
$$

or something similar. We can simplify things using our completeness relations. Consider, for example

$$\sum_s u_p^s \bar{u}_p^{s'} = \rho + m.
$$

Multiplying this matrix equation on the right by $\gamma^0$ and then taking the trace, we get

$$\sum_s u_p^s \bar{u}_p^{s'} = \text{tr}[(\rho + m)\gamma^0] = 4E.
$$

But since this corresponds to a sum over two orthogonal spin states, we must have that

$$u_p^s \bar{u}_p^{s'} = 2E \delta_{ss'}.
$$

We similarly derive $v_p^s \bar{v}_p^{s'} = 2E \delta_{ss'}$. To get an expression for $u_p^s \bar{v}_p^{s'}$, which appears in $Q$ above, requires a little more ingenuity. Consider $\sum_s u_p^s \bar{u}_p^{s}$. This must vanish when we act on the left with $\rho - m$ (since $(\rho - m)u_p = 0$), whence it is proportional to $\rho + m$. But it also must vanish when we act on the right with $\rho + m$, so it is proportional to $\rho - m$. Hence it vanishes identically. But the $\bar{\tau}_p$ are proportional to $v_\downarrow - u_\uparrow$ (one may easily check that they both satisfy the same equation). Hence $u_p^s \bar{v}_p^{s'} = 0$. In all, $Q$ simplifies to

$$Q = \int \frac{d^3p}{(2\pi)^3(2E)^2} \left( c_p^s \bar{c}_p^{s'} + d_p^s \bar{d}_p^{s'} \right).
$$

Similarly, one may show that

$$H = \int \frac{d^3p}{(2\pi)^3(2E)^2} E \left( c_p^s \bar{c}_p^{s'} - d_p^s \bar{d}_p^{s'} \right).
$$

Now, if we impose commutation relations on $c$ and $d$, we may simply permute the $d$ with the $d^\dagger$ to get operators into normal order, but we end up with a disaster: not only will the charge count the numbers of both particles and antiparticles, but also the antiparticles will give a negative contribution to the total energy as measured by the Hamiltonian. Now, you may try as you like to insert factors of $i$ to try to patch things up, but nothing will work. What does work is to make the simple but bold step of declaring that for fermions the commutation relations should be replaced by anticommutation relations. Thus,

$$\{c_p^s, c_p^{s'}\} = (2\pi)^3 2E \delta^3(p - p') \delta_{ss'}.$$

$$\{d_p^s, d_p^{s'}\} = (2\pi)^3 2E \delta^3(p - p') \delta_{ss'}.$$

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with other anti-commutators vanishing. Then the charge measures the number of particles minus the number of antiparticles and both particles and antiparticles contribute positively to the energy. Moreover, any $n$-particle state $\ldots c\dagger\ldots c\dagger\ldots|0\rangle$ is manifestly antisymmetric under the interchange of two particles. As Pauli realized, this means that if we try to put two particles into the same state, we find $(c_{p}\dagger)^{2}|0\rangle = 0$. So the Pauli exclusion principle of QM follows from the fact that in QFT, we can only quantize spin-half fields consistently by using anticommutation relations. Amazing.\footnote{Another philosophical discourse: Even if QFTs of both fermions and bosons are mathematically consistent, why did Nature choose to realize them both? One possibility is that consistency of the laws of Nature at a more fundamental level (e.g. including gravity) requires an even larger symmetry, called supersymmetry. If you want to know more, take courses on supersymmetry and string theory.}