NST Part III Experimental and Theoretical Physics — Lent Term 2024

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GAUGE FIELD THEORY

Examples Sheet 3

Electroweak interactions; combining gauge symmetries

12 A. Show that a Lagrangian term $i\overline{\Psi}\gamma^\mu D_\mu \Psi$ for a multiplet Ψ of spin-half fields, with a covariant derivative of the form

$$D^\mu \Psi = \left(\partial^\mu + igT_j A_j^\mu + ig'YB^\mu\right)\Psi \ ,$$

where the T_j are generators of SU(2), is invariant under both SU(2) and U(1) gauge transformations.

[You may assume the results for the transformation properties of the individual SU(2) and U(1) covariant derivatives obtained in the lectures.]

Z boson interactions

13 B. For each pair of quark or lepton fields, the electroweak Standard Model Lagrangian contains the covariant derivative contributions

$$\mathcal{L} = i\overline{\Psi_L}\gamma^\mu D_\mu \Psi_L + i\overline{\psi_{1R}}\gamma^\mu D_\mu \psi_{1R} + i\overline{\psi_{2R}}\gamma^\mu D_\mu \psi_{2R} \ ,$$

where $\Psi_L = (\psi_{1L}, \psi_{2L})^T$ is a an SU(2) doublet of left-handed fields, and the D_μ are appropriate SU(2)×U(1) covariant derivatives.

Show that the interaction between a Z^0 boson and a fermion f is governed by a Lagrangian term of the form

$$\mathcal{L}_Z = -J_Z^\mu Z^\mu$$

where

$$J_Z^\mu = \frac{e}{\sin(2\theta_W)}\overline{\psi}_f\gamma^\mu(g_V - g_A\gamma^5)\psi_f \ ,$$

the vector and axial-vector coupling constants g_V and g_A are given by

$$g_V = (I_3^W)_L - 2Q\sin^2\theta_W \ , \qquad g_A = (I_3^W)_L \ ,$$

$(I_3^W)_L$ is the weak isospin of the left-handed state of the fermion, and Q is the fermion electric charge in units of e.

Spontaneous symmetry breaking

14. An SU(2) gauge theory has Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_j^{\mu\nu}F_{j\mu\nu} + (D^\mu\Phi)^\dagger(D_\mu\Phi) - \mu^2(\Phi^\dagger\Phi) - \lambda(\Phi^\dagger\Phi)^2 \ ,$$

where Φ is a triplet of real scalar fields,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \ ,$$

and the covariant derivative $D^\mu\Phi$ depends on the gauge fields A_j^μ and group generators T_j as $D^\mu\Phi = (\partial^\mu + igA_j^\mu T_j)\Phi$.

(a A) Show that the 3×3 matrices $T_{1,2,3}$ defined by $(T_j)_{kl} = -i\epsilon_{jkl}$ can serve as a suitable representation of the generators T_j .

(b A) For the case $\mu^2 > 0$, identify the physical particles in the theory and determine their masses and interactions.

(c B) Consider the case $\mu^2 < 0$, taking the vacuum expectation value of the triplet field Φ to be

$$\Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \ ,$$

with $v > 0$ a real constant. Identify the physical particles in the theory and determine their masses and interactions. Comment on the number of degrees of freedom in the theory before and after spontaneous symmetry breaking.

(d B) Evaluate $T_j\Phi_0$ for $j = 1, 2, 3$, and hence identify an unbroken gauge symmetry of the vacuum state that explains why one gauge boson remains massless. What happens for other possible choices of Φ_0 ?

Renormalisability

15. (a B) Consider the Lagrangian

$$\mathcal{L}_5 = \frac{g}{M}\left(\Psi_L^T\tau_2\Phi\right)C^\dagger\left(\Phi^T\tau_2\Psi_L\right) + \text{h.c.} \ ,$$

where g is a dimensionless constant, M is a constant mass parameter, Ψ_L is an SU(2) doublet of left-handed lepton fields and Φ is the usual doublet of scalar fields. Show that $\mathcal{L}_5$ is invariant under the SU(2) and U(1) gauge transformations of the Standard Model.

(b B) Show that, after spontaneous symmetry breaking, $\mathcal{L}_5$ generates a Majorana mass term for the neutrino, and identify the form of the resulting interactions between the neutrino and the Higgs boson.

(c A) Explain why adding the contribution $\mathcal{L}_5$ to the Standard Model Lagrangian would lead to a non-renormalisable theory.

(d C) Show that the Lagrangian

$$\mathcal{L}_6 = \frac{g}{M^2}\epsilon_{jkl}\left(d_{jR}^T C^\dagger u_{kR}\right)\left(u_{lR}^T C e_R\right) + \text{h.c.}\ ,$$

where fermion fields are represented by their particle names ($d_{jR} \equiv (\psi_{dj})_R$ *etc.*), and where j,k,l are colour indices (summed over $j,k,l = 1,2,3$), is invariant under the SU(3), SU(2) and U(1) gauge transformations of the Standard Model. Show that this interaction does not conserve baryon or lepton number and explain how the Lagrangian $\mathcal{L}_6$ permits proton decay, and obtain an *estimate* of the resulting proton lifetime for a plausible choice of mass scale M.

[Hint: the determinant of a 3×3 matrix U can be written $\epsilon_{pqr}\det U = \epsilon_{jkl}U_{jp}U_{kq}U_{lr}$.]

Higgs boson decays

16. (a B) Show that the sum over polarization states P of a massive spin-one boson with mass M and four-momentum p^μ can be written as

$$\sum_P \varepsilon_P^\mu \varepsilon_P^{*\nu} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}\ .$$

[Hint: look at it in the boson rest frame.]

Show that the Feynman rules give the matrix element as

$$-iM_{\text{fi}} = \frac{1}{2} i v g^2 g^{\mu\nu}\varepsilon_\mu^*(p)\varepsilon_\nu^*(p')\ .$$

Use this result to calculate the rate for the decay of the Higgs boson into W^+W^-:

$$\Gamma(H\to WW) = \frac{G_{\text{F}} m_{\text{H}}^3}{8\pi\sqrt{2}}\left(1 - 4\frac{m_{\text{W}}^2}{m_{\text{H}}^2} + 12\frac{m_{\text{W}}^4}{m_{\text{H}}^4}\right)\sqrt{1 - 4\frac{m_{\text{W}}^2}{m_{\text{H}}^2}}\ .$$

(b B) Show that, for the decay of the Higgs boson into a fermion-antifermion pair, the Feyman rules give the leading order matrix element

$$-iM_{\text{fi}} = \frac{-im_f}{v}\overline{u}(p_1)v(p_2)\ .$$

Calculate the rate for the decay of the Higgs boson into a fermion-antifermion pair:

$$\Gamma(H\to f\bar{f}) = \frac{C G_{\text{F}} m_f^2 m_{\text{H}}}{4\pi\sqrt{2}}\left(1 - 4\frac{m_f^2}{m_{\text{H}}^2}\right)^{\frac{3}{2}}\ ,$$

where C is a colour factor ($C = 1$ for leptons, 3 for quarks).

$\text{W}^+\text{W}^- \to \text{W}^+\text{W}^-$ **scattering**

17 A. Draw the seven Feynman diagrams which contribute to $\text{W}^+\text{W}^- \to \text{W}^+\text{W}^-$ scattering at leading order.