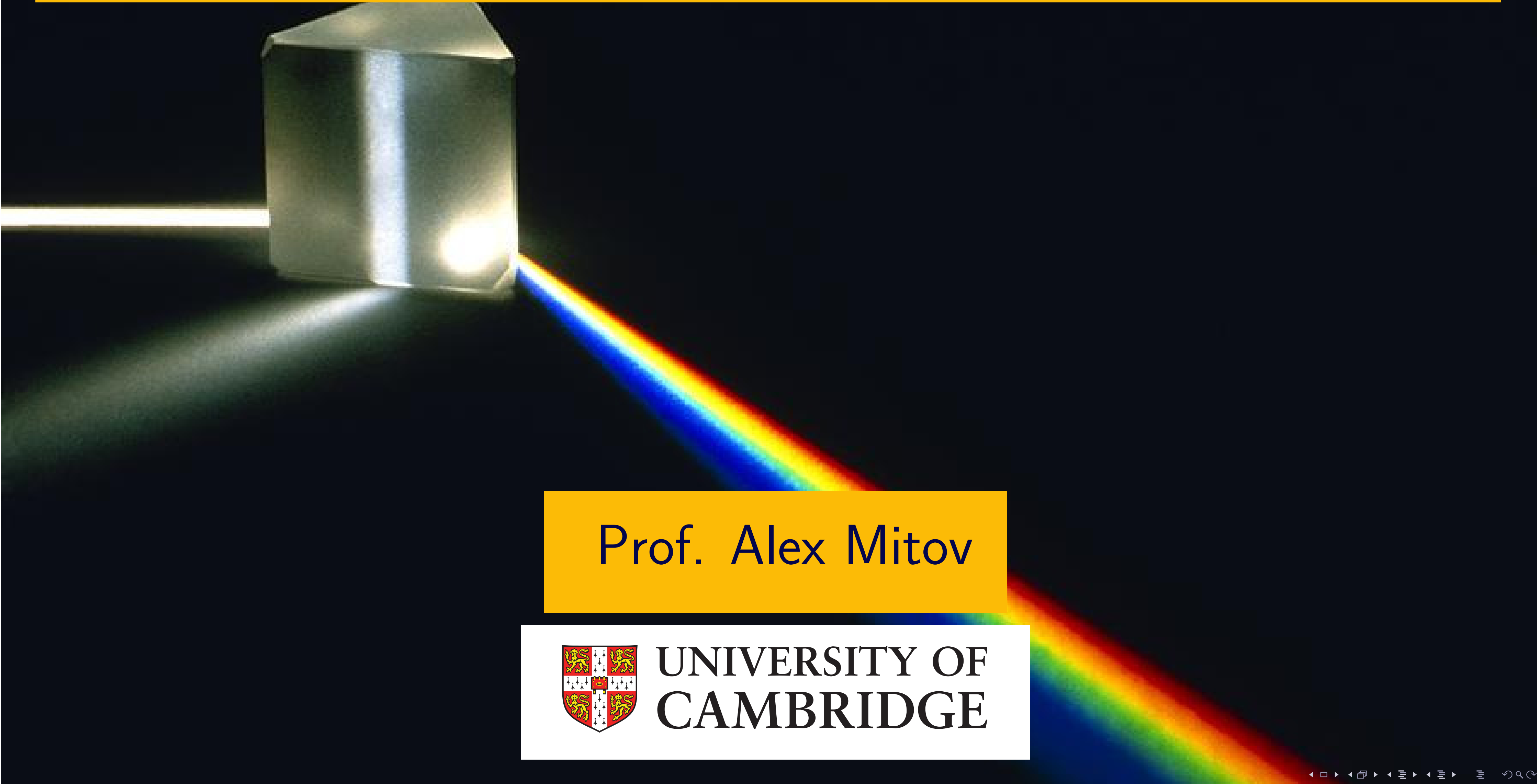
6. QED

Particle and Nuclear Physics



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In this section...

- Gauge invariance
- Allowed vertices + examples
- Scattering
- Experimental tests
- Running of alpha

QED

Quantum Electrodynamics is the gauge theory of electromagnetic interactions.

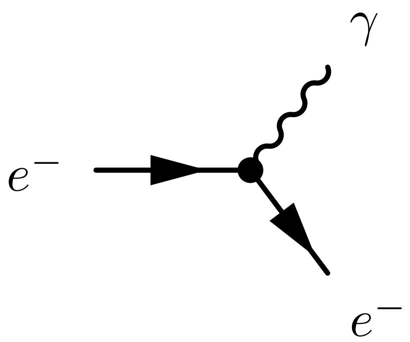
Consider a non-relativistic charged particle in an EM field:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$\vec{E}, \vec{B}$ given in term of vector and scalar potentials $\vec{A}, \varphi$

$$\vec{B} = \vec{\nabla} \times \vec{A}; \quad \vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$ Maxwell's Equations

$$\hat{H} = \frac{1}{2m}(\hat{\vec{p}} - q\vec{A})^2 + q\varphi$$ Classical Hamiltonian



Change in state of e^- requires change in field
$\Rightarrow$ Interaction via virtual γ emission

QED

Schrödinger equation
$$\left[\frac{1}{2m}(\hat{\vec{p}} - q\vec{A})^2 + q\varphi\right]\psi(\vec{r},t) = i\frac{\partial\psi(\vec{r},t)}{\partial t}$$

is invariant under the local gauge transformation $\psi \rightarrow \psi' = \mathrm{e}^{\mathrm{i}q\alpha(\vec{r},t)}\psi$

so long as $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\alpha \; ; \quad \varphi \rightarrow \varphi - \frac{\partial\alpha}{\partial t}$ (See Appendix E)

Local Gauge Invariance requires the existence of a physical Gauge Field (photon) and completely specifies the form of the interaction between the particle and field.

- Photons are massless

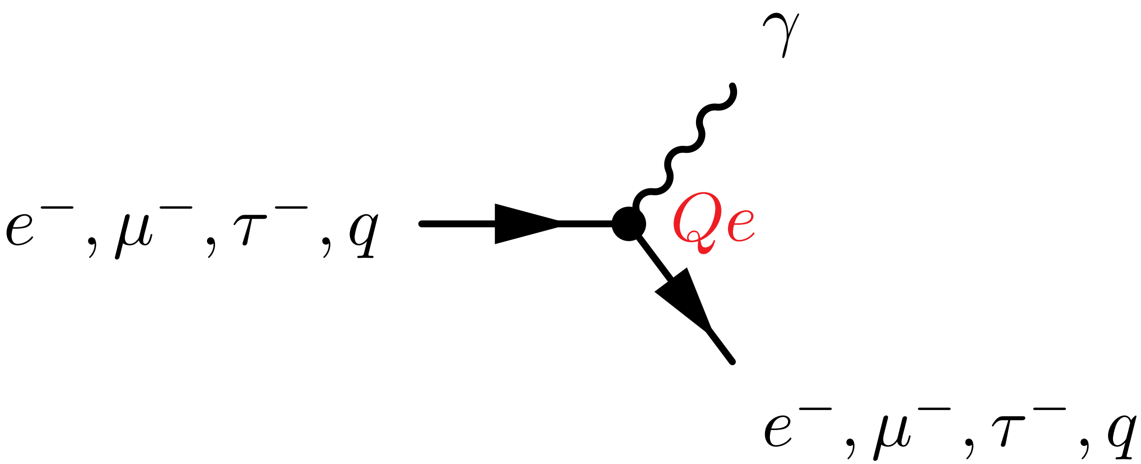
  (in order to cancel phase changes over all space-time, the range of the photon must be infinite)

- Charge is conserved – the charge q which interacts with the field must not change in space or time

QED is a gauge theory

The Electromagnetic Vertex

All electromagnetic interactions can be described by the photon propagator and the EM vertex:



The Standard Model Electromagnetic Vertex

$+$ antiparticles

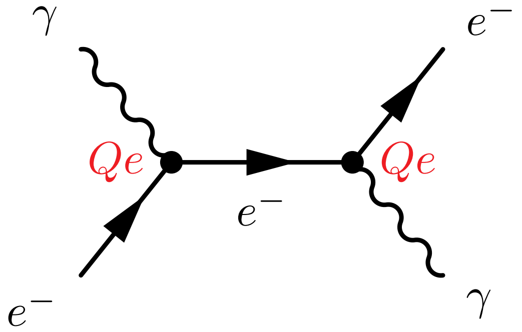
$$\alpha = \frac{e^2}{4\pi}$$

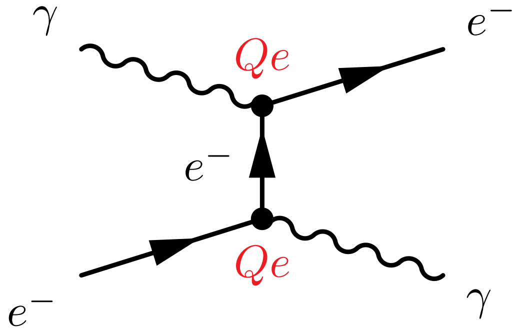
- The coupling constant is proportional to the fermion charge.
- Energy, momentum, angular momentum, parity and charge always conserved.
- QED vertex never changes particle type or flavour
  i.e. $e^- \to e^- \gamma$, but not $e^- \to q\gamma$ or $e^- \to \mu^- \gamma$

Important QED Processes

$$M \sim \frac{g^2}{q^2}, \quad \alpha = \frac{e^2}{4\pi}$$

Compton Scattering ($\gamma e^- \to \gamma e^-$)

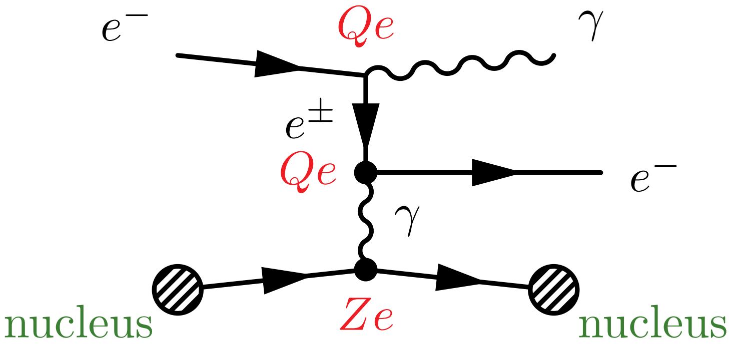




$$M \propto e^2$$

$$\sigma \propto |M|^2 \propto e^4$$
$$\propto (4\pi)^2\alpha^2$$

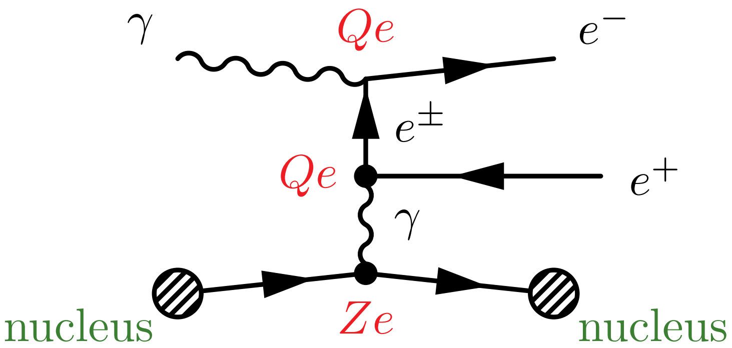
Bremsstrahlung ($e^- \to e^-\gamma$)



$$M \propto Ze^3$$

$$\sigma \propto |M|^2 \propto Z^2e^6$$
$$\propto (4\pi)^3 Z^2\alpha^3$$

Pair Production ($\gamma \to e^+e^-$)



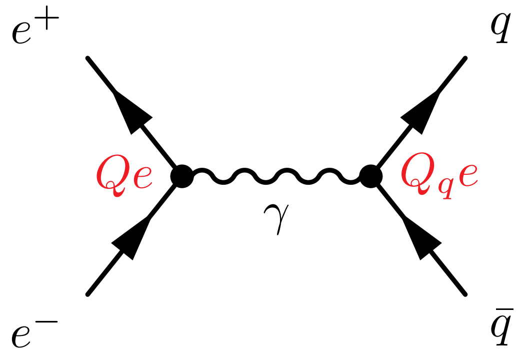
$$M \propto Ze^3$$

$$\sigma \propto |M|^2 \propto Z^2e^6$$
$$\propto (4\pi)^3 Z^2\alpha^3$$

The processes $e^- \to e^-\gamma$ and $\gamma \to e^+e^-$ cannot occur for real e^-, γ due to energy & momentum conservation

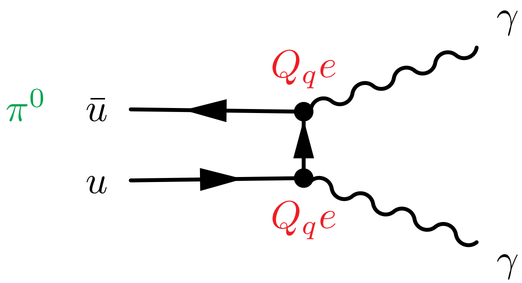
Important QED Processes

Electron-Positron Annihilation ($e^- e^+ \to q\bar{q}$)



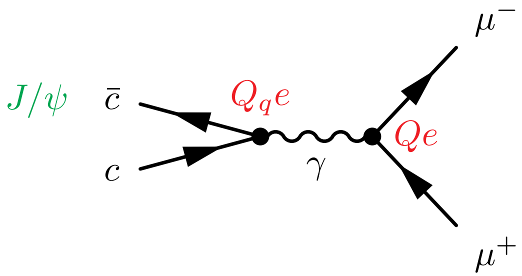
$$M \propto Q_q e^2$$
$$\sigma \propto |M|^2 \propto Q_q^2 e^4$$
$$\propto (4\pi)^2 Q_q^2 \alpha^2$$

Pion Decay ($\pi^0 \to \gamma\gamma$)



$$M \propto Q_u^2 e^2$$
$$\Gamma \propto |M|^2 \propto Q_u^4 e^4$$
$$\propto (4\pi)^2 Q_u^4 \alpha^2$$

J/ψ Decay ($J/\psi \to \mu^+ \mu^-$)



$$M \propto Q_c e^2$$
$$\Gamma \propto |M|^2 \propto Q_c^2 e^4$$
$$\propto (4\pi)^2 Q_c^2 \alpha^2$$

The coupling strength determines "order of magnitude" of the matrix element.

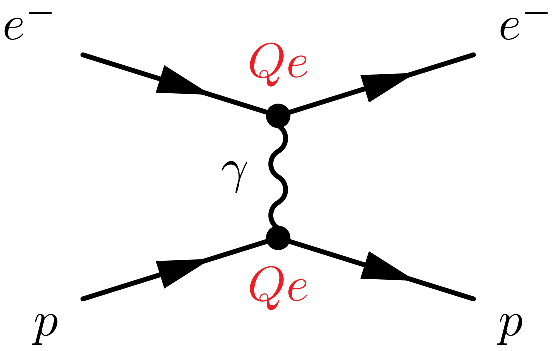
For particles interacting/decaying via EM interaction: typical values for cross-sections/ lifetimes

$$\sigma_{\rm EM} \sim 10^{-2}\ \text{mb};$$
$$\tau_{\rm EM} \sim 10^{-20}\ \text{s}$$

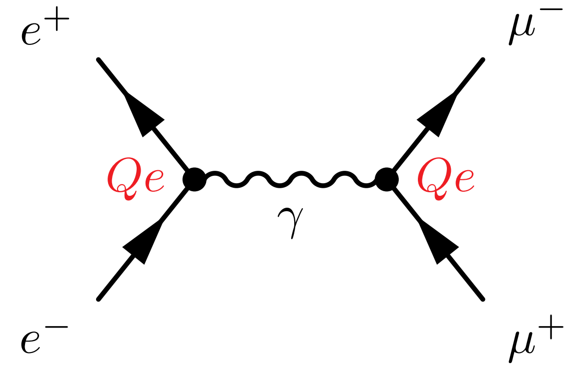
Scattering in QED *Examples*

Calculate the "spin-less" cross-sections for the two processes:

1. Electron-proton scattering



2. Electron-positron annihilation



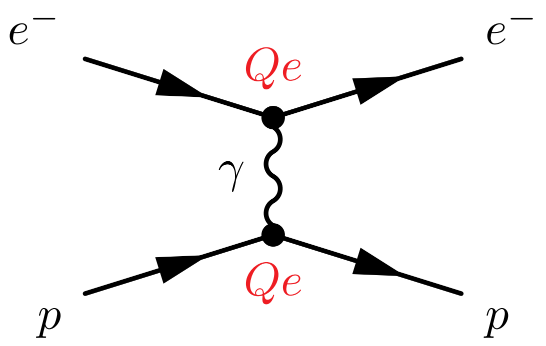
Fermi's Golden rule and Born Approximation

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2}|M|^2$$

For both processes we have the same matrix element (though q^2 is different)

$$M = \frac{e^2}{q^2} = \frac{4\pi\alpha}{q^2}$$

- $e^2 = 4\pi\alpha$ is the strength of the interaction.
- $1/q^2$ measures the probability that the photon carries 4-momentum $q^\mu = (E, \vec{p})$; $q^2 = E^2 - |\vec{p}|^2$ i.e. smaller probability for higher mass.



$$M = \frac{e^2}{q^2} = \frac{4\pi\alpha}{q^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2}|M|^2 = \frac{E^2}{(2\pi)^2}\frac{(4\pi\alpha)^2}{q^4} = \frac{4\alpha^2 E^2}{q^4}$$

q^2 is the four-momentum transfer $q^2 = q^\mu q_\mu = (E_\mathrm{f} - E_\mathrm{i})^2 - (\vec{p}_\mathrm{f} - \vec{p}_\mathrm{i})^2$

$$= E_\mathrm{f}^2 + E_\mathrm{i}^2 - 2E_\mathrm{f}E_\mathrm{i} - \vec{p}_\mathrm{f}^2 - \vec{p}_\mathrm{i}^2 + 2\vec{p}_\mathrm{f}.\vec{p}_\mathrm{i}$$

$$= 2m_e^2 - 2E_\mathrm{f}E_\mathrm{i} + 2|\vec{p}_\mathrm{f}||\vec{p}_\mathrm{i}|\cos\theta$$

Neglecting electron mass: i.e. $m_e = 0$ and $|\vec{p}_\mathrm{f}| = E_\mathrm{f}$

$$q^2 = -2E_\mathrm{f}E_\mathrm{i}(1 - \cos\theta) = -4E_\mathrm{f}E_\mathrm{i}\sin^2\frac{\theta}{2}$$

Therefore, for elastic scattering $E_\mathrm{i} = E_\mathrm{f}$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}$$

Rutherford Scattering

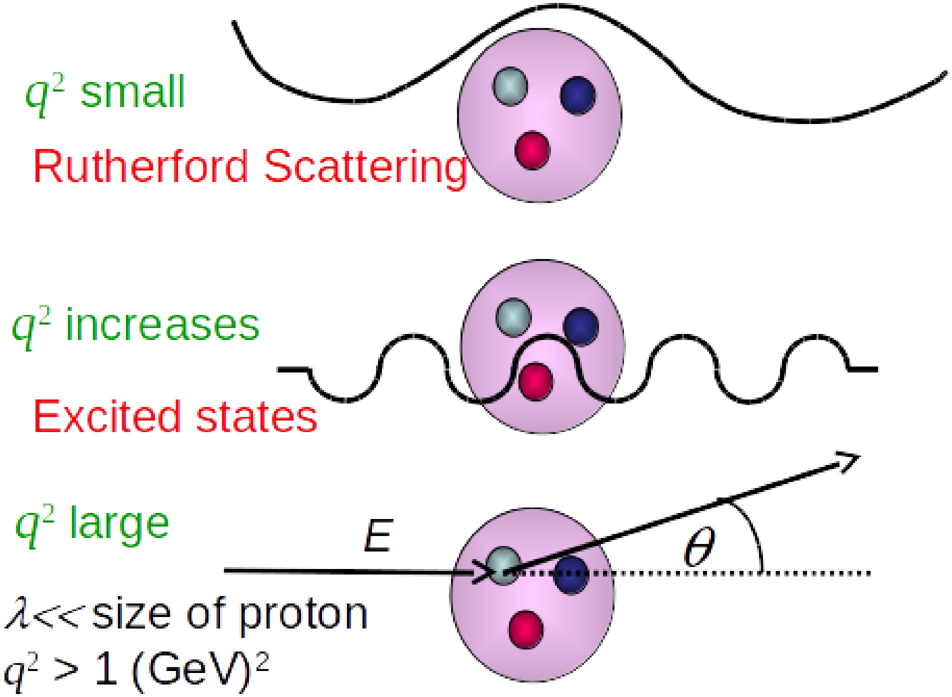
same result from QED as from conventional QM

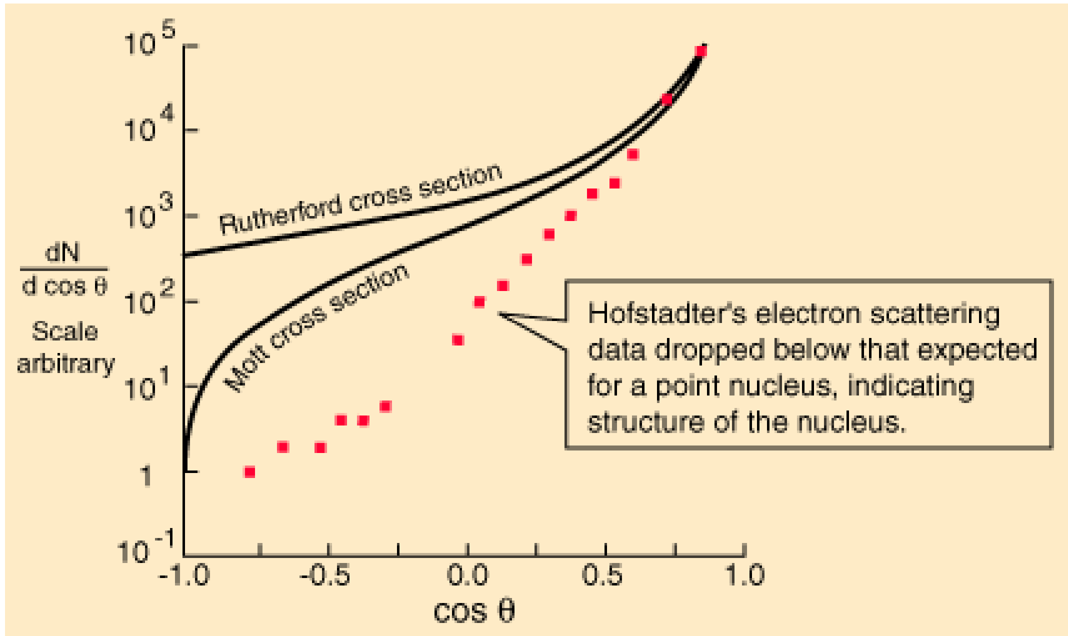
The discovery of quarks

Virtual γ carries 4-momentum $q^\mu = (E, \vec{p})$

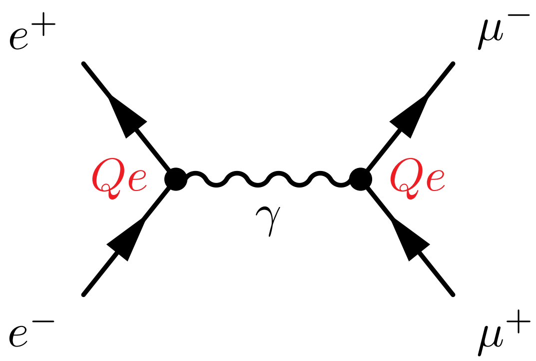
$$\text{Large } q \Rightarrow \text{Large } \vec{p}\text{, small } \lambda \qquad |\vec{p}| = \hbar/\lambda$$
$$\text{Large } E\text{, large } \omega \qquad E = \hbar\omega$$

High q wavefunction oscillates rapidly in space and time
$\Rightarrow$ probes short distances and short time.





Elastic scattering from quarks in proton.



$$M = \frac{e^2}{q^2} = \frac{4\pi\alpha}{q^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2}|M|^2 = \frac{E^2}{(2\pi)^2}\frac{(4\pi\alpha)^2}{q^4} = \frac{4\alpha^2 E^2}{q^4}$$

Same formula, but different four-momentum transfer

$$q^2 = q^\mu q_\mu = (E_{e^+} + E_{e^-})^2 - (\vec{p}_{e^+} + \vec{p}_{e^-})^2$$

assuming we are in the centre-of-mass system, $E_{e^+} = E_{e^-} = E$, $\vec{p}_{e^+} = -\vec{p}_{e^-}$

$$q^2 = q^\mu q_\mu = (2E)^2 = s$$

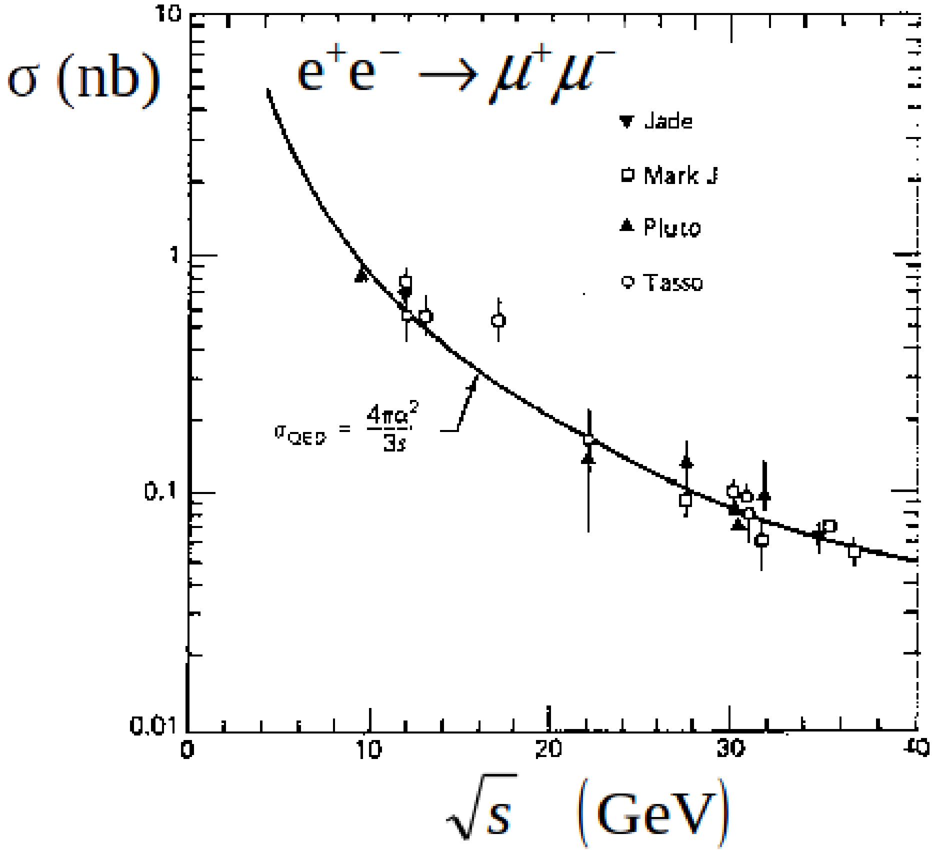
$$\frac{d\sigma}{d\Omega} = \frac{4\alpha^2 E^2}{q^4} = \frac{4\alpha^2 E^2}{16E^4} = \frac{\alpha^2}{s}$$

Integrating gives total cross-section: $\sigma = \dfrac{4\pi\alpha^2}{s}$

… the actual cross-section (using the Dirac equation to take spin into account) is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$



Example: Cross-section at $\sqrt{s} = 22\ \mathrm{GeV}$
(i.e. 11 GeV electrons colliding with 11 GeV positrons)

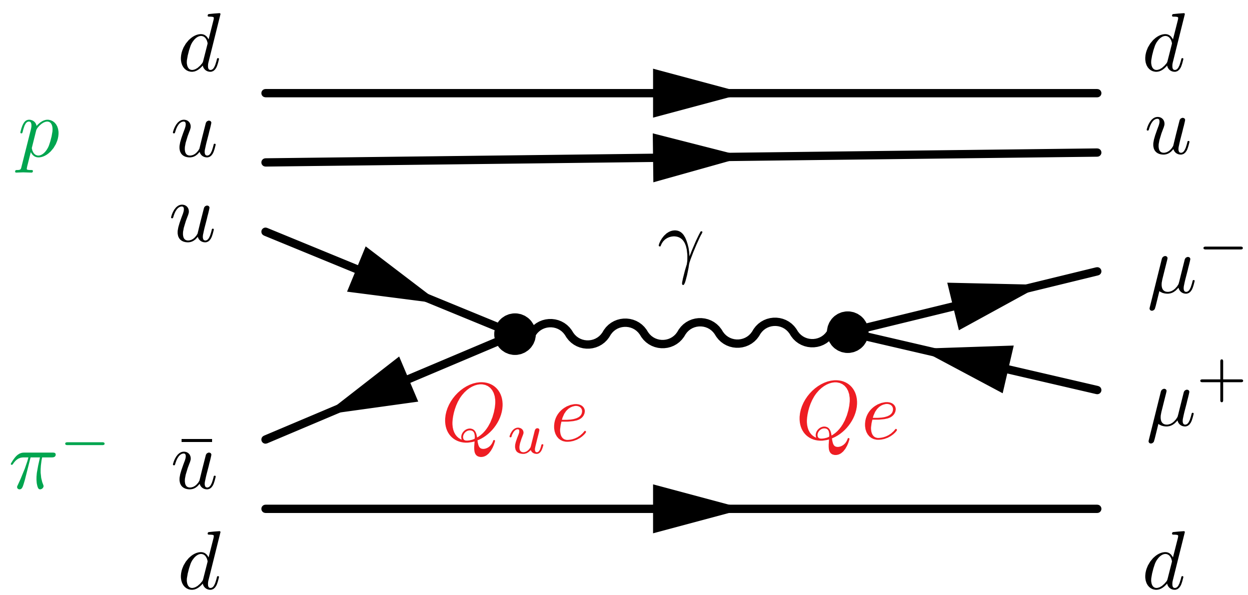
$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s} = \frac{4\pi}{(137)^2}\frac{1}{3\times 22^2}$$

$$= 4.6\times10^{-7}\ \mathrm{GeV}^{-2} = 4.6\times10^{-7}\times(0.197)^2\ \mathrm{fm}^2 = 1.8\times10^{-8}\ \mathrm{fm}^2 = 0.18\ \mathrm{nb}$$

The Drell-Yan Process

Can also annihilate $q\bar{q}$ as in the "Drell-Yan" process.

Example: $\pi^- p \to \mu^+\mu^- +$ hadrons (See problem sheet q.13)



$$\sigma(\pi^- p \to \mu^+\mu^- + \text{ hadrons}) \propto Q_u^2\alpha^2 \propto Q_u^2 e^4$$

(Also need to account for presence of two u quarks in proton)

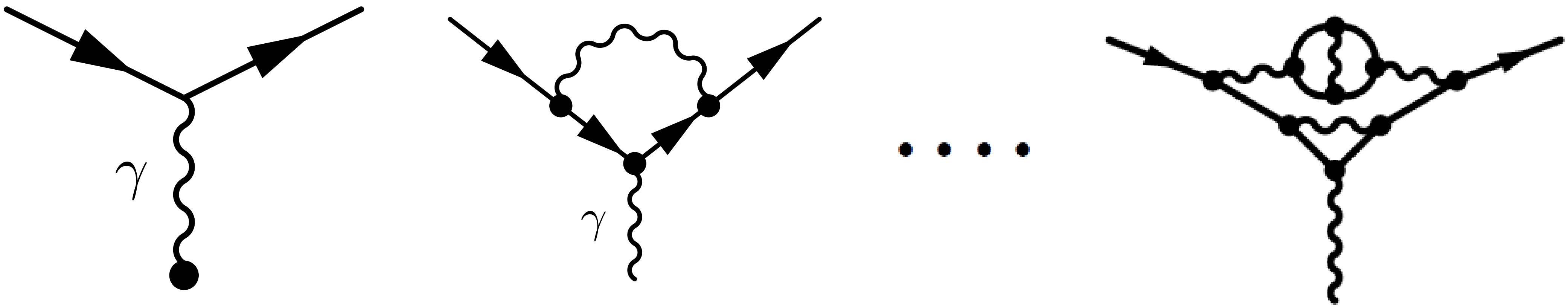
Experimental Tests of QED

QED is an extremely successful theory tested to very high precision.

Example:

- Magnetic moments of $e^{\pm}$, $\mu^{\pm}$: $\vec{\mu} = g\frac{e}{2m}\vec{s}$
- For a point-like spin 1/2 particle: $g = 2$ Dirac Equation

However, higher order terms in QED introduce an anomalous magnetic moment $\Rightarrow g$ is not quite equal to 2.

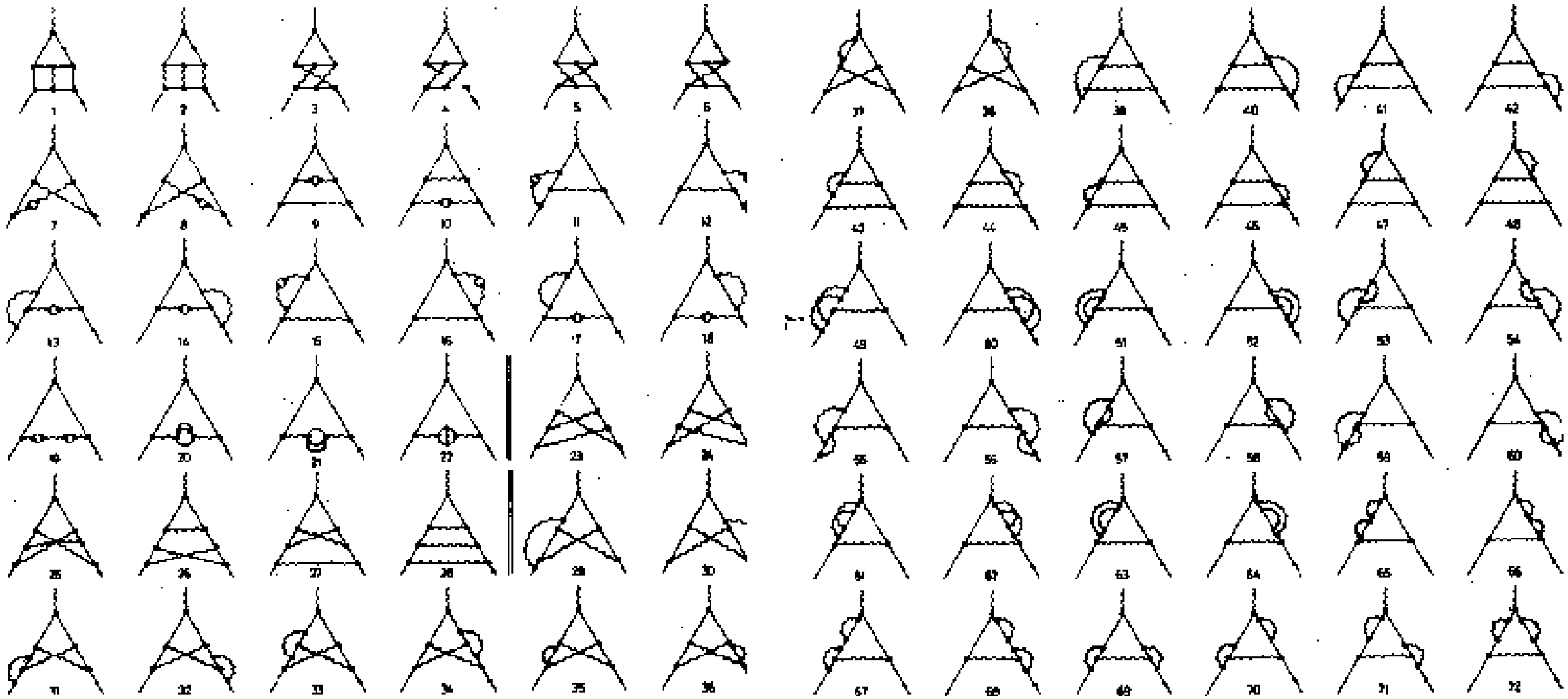


O(1) O(α) O(α^4)

12672 diagrams

Experimental Tests of QED

$O(\alpha^3)$



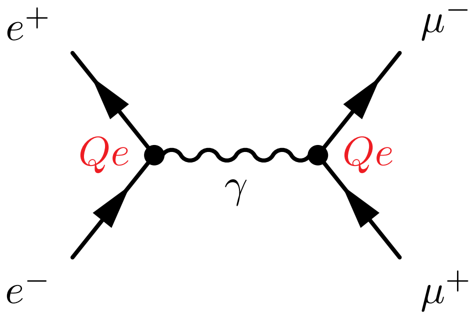
$$\frac{g_e - 2}{2} = 11596521.811 \pm 0.007 \times 10^{-10} \quad \text{Experiment}$$

$$= 11596521.3 \pm 0.3 \times 10^{-10} \quad \text{Theory}$$

- Agreement at the level of 1 in 10^8
- QED provides a remarkably precise description of the electromagnetic interaction!

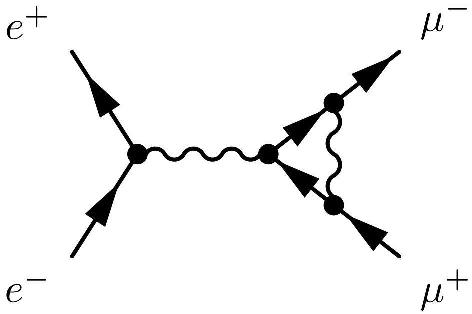
So far only considered lowest order term in the perturbation series.
Higher order terms also contribute (and also interfere with lower orders)

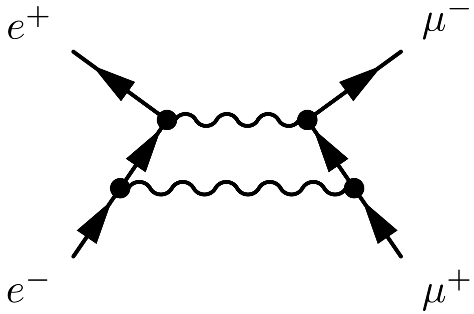
Lowest Order



$$|M|^2 \propto e^4 \propto \alpha^2 \sim \left(\frac{1}{137}\right)^2$$

Second Order

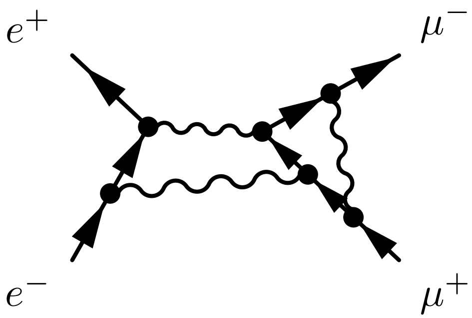


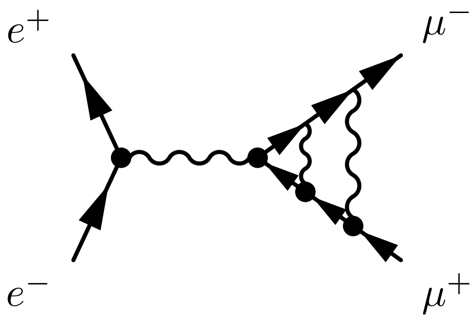


$+ \ldots$

$$|M|^2 \propto \alpha^4 \sim \left(\frac{1}{137}\right)^4$$

Third Order





$+ \ldots$

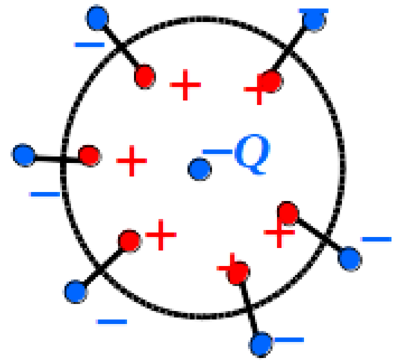
$$|M|^2 \propto \alpha^6 \sim \left(\frac{1}{137}\right)^6$$

Second order suppressed by α^2 relative to first order.
Provided α is small, i.e. perturbation is small, lowest order dominates.

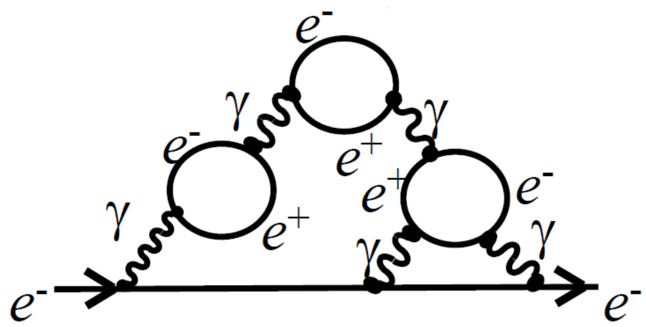
Running of α

- $\alpha = \frac{e^2}{4\pi}$ specifies the strength of the interaction between an electron and a photon.
- But α is not a constant

Consider an electric charge in a dielectric medium.
Charge Q appears screened by a halo of $+$ve charges.
Only see full value of charge Q at small distance.



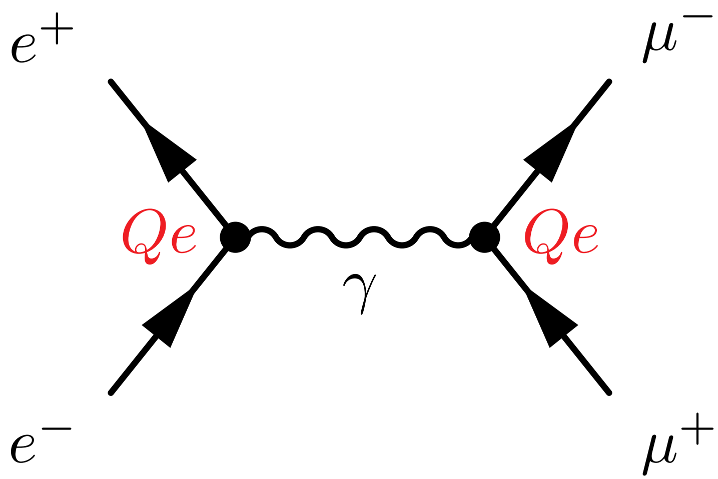
Consider a free electron.
The same effect can happen due to quantum fluctuations that lead to a cloud of virtual e^+e^- pairs.

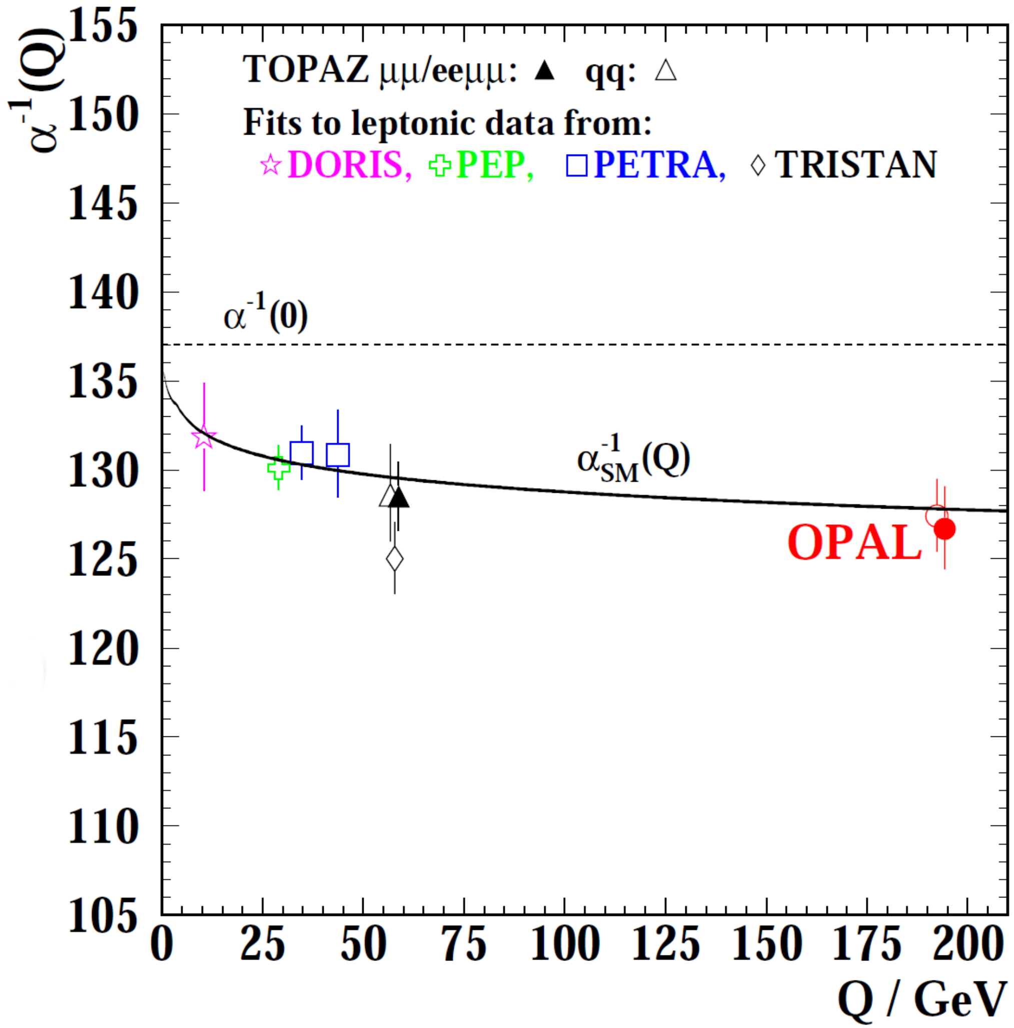


- The vacuum acts like a dielectric medium
- The virtual e^+e^- pairs are therefore polarised
- At large distances the bare electron charge is screened.
- At shorter distances, screening effect reduced and we see a larger effective charge i.e. a larger α.

Running of α

Can measure $\alpha(q^2)$ from $e^+e^- \to \mu^+\mu^-$ etc.

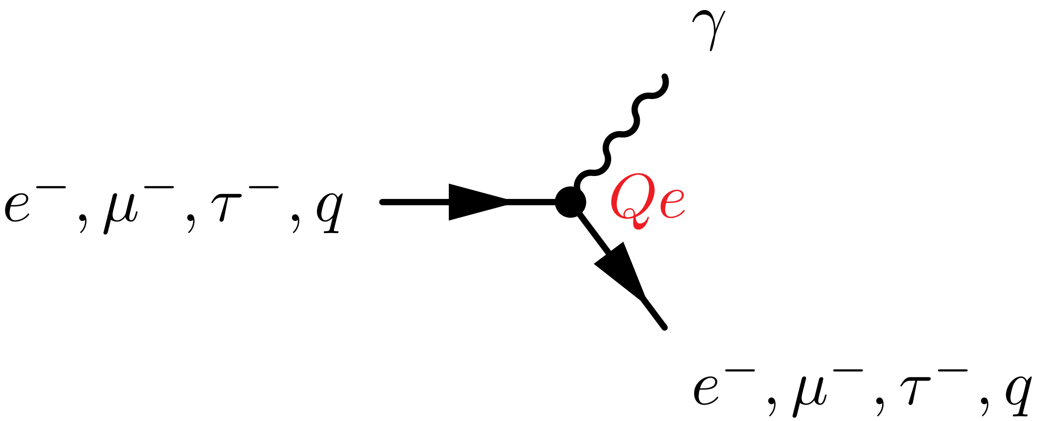




- α increases with increasing q^2 (i.e. closer to the bare charge)
- At $q^2 = 0$: $\alpha \sim 1/137$
- At $q^2 \sim (100\ \mathrm{GeV})^2$: $\alpha \sim 1/128$

Summary

- QED is the physics of the photon + "charged particle" vertex:



$$\alpha = \frac{e^2}{4\pi}$$

- Every EM vertex has:
  - has an arrow going in & out (lepton or quark), and a photon
  - does not change the type of lepton or quark "passing through"
  - conserves charge, energy and momentum
- The dimensionless coupling $\sqrt{\alpha}$ is proportional to the electric charge of the lepton or quark, and it "runs" with energy scale.
- QED has been tested at the level of 1 part in 10^8.

Problem Sheet: q.12-14

Up next... Section 7: QCD