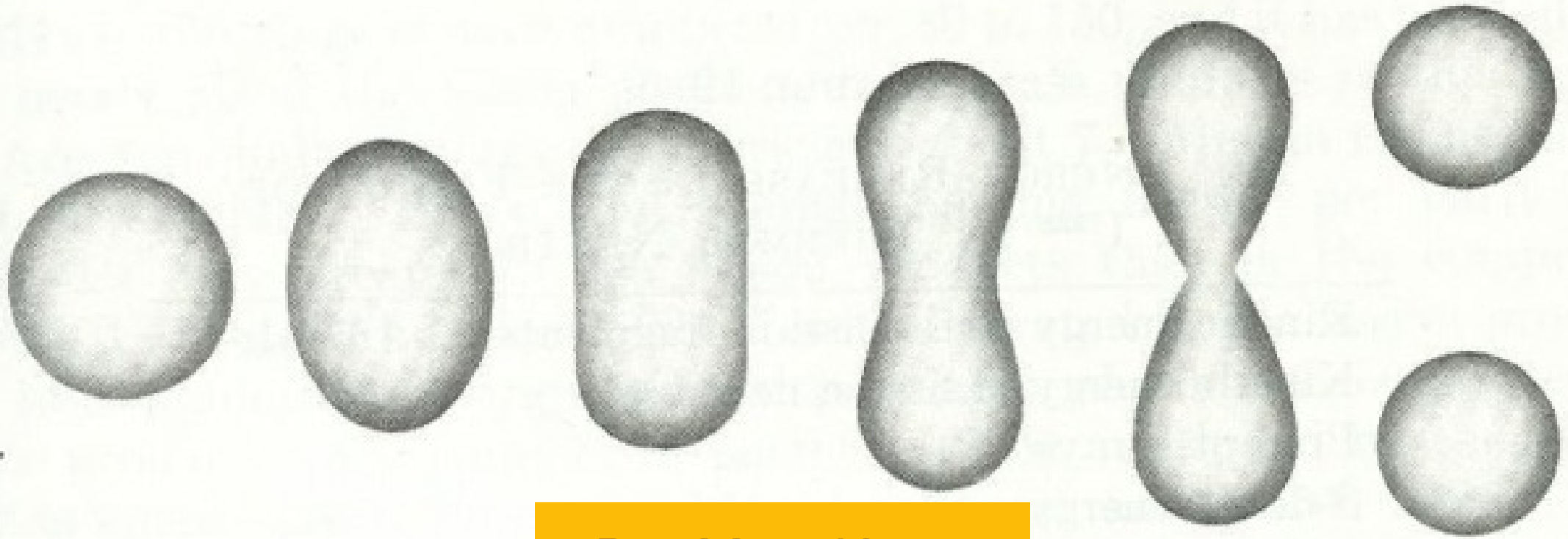


13. Basic Nuclear Properties

Particle and Nuclear Physics



Dr. Matt Kenzie



UNIVERSITY OF
CAMBRIDGE

Welcome back

- Six lectures this term on Nuclear Physics
- Exam on Friday 6th June
- The material in these lectures **IS** examinable!

| date | day | 30 mins | 30 mins | notes | questions |
|----------|-----|-------------------------------|-------------------------|--|--------------|
| 11:30:00 | | | | | |
| 02/05/25 | Fr | 13 – Basic Nuclear Properties | | | 31,32,33 |
| 05/05/25 | Mo | 14 – The Structure of Nuclei | | To excited states | 34 |
| 07/05/25 | We | 14 – The Structure of Nuclei | 15 – Nuclear Decay | Excited states + alpha decay | 35,36, 37,38 |
| 09/05/25 | Fr | 15 – Nuclear Decay | | beta decay | 39,40 |
| 12/05/25 | Mo | 15 – Nuclear Decay | 16 – Fission and Fusion | gamma decay + fission (to neutron induced) | 41 |
| 14/05/25 | We | 16 – Fission and Fusion | | | 42,43,44 |
| | | | | | |
| | | Exam Friday 6th June 9:00 am | | | |

Admin

- Same format as Alex
- [Link to TIS](#)
- **Lecture slides** available as handouts on TIS
- **Lecture examples** on board in lectures
- **Problem sheet** is Part 4, Q31–Q44
- **Song requests** at this [Google Sheet](#)
 - You will need to be logged into google with your Uni account to edit this
 - Please be sensible

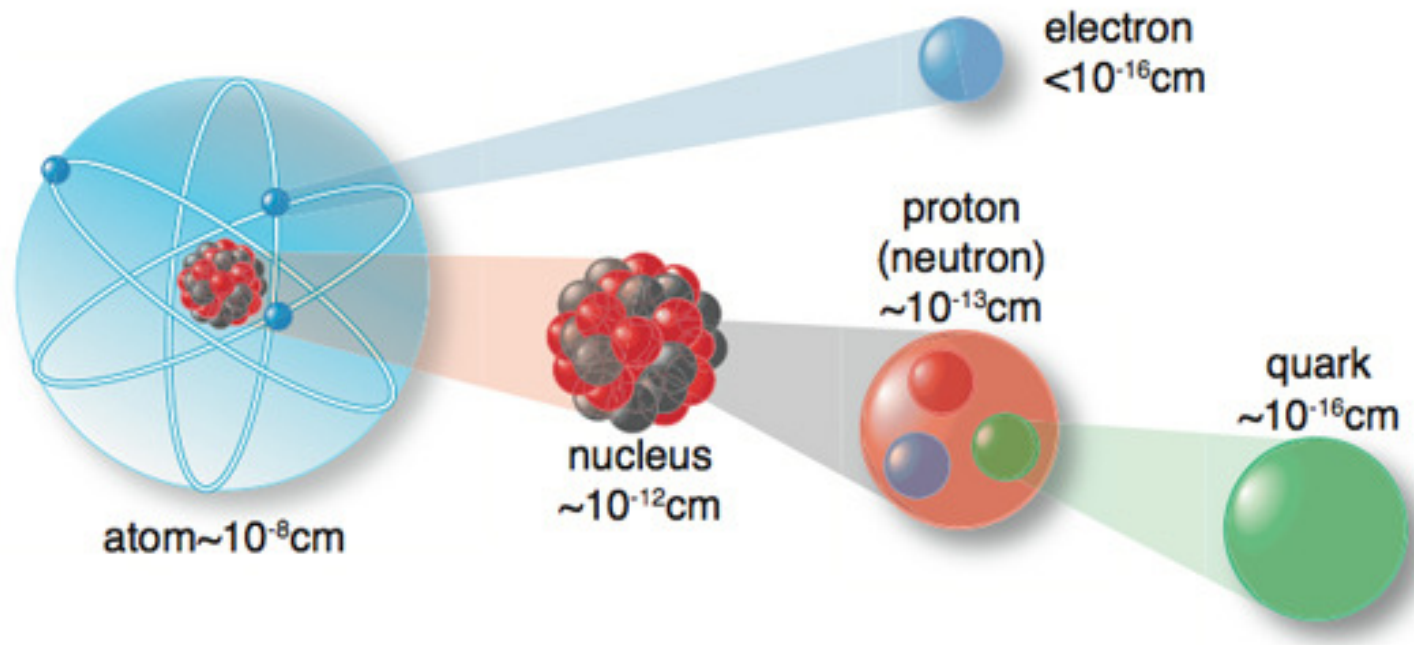
In this section...

- Motivation for study
- The strong nuclear force
- Stable nuclei
- Binding energy & nuclear mass (SEMF)
- Spin & parity
- Nuclear size (scattering, muonic atoms, mirror nuclei)
- Nuclear moments (electric, magnetic)

Introduction

Nuclear processes play a fundamental role in the physical world:

- Origin of the universe
- Creation of chemical elements
- Energy of stars
- Constituents of matter; influence properties of atoms

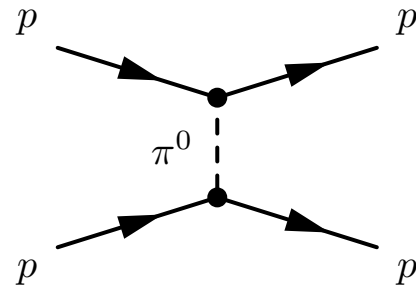


Nuclear processes also have many practical applications:

- Uses of radioactivity in research, health and industry, e.g. NMR, radioactive dating.
- Various tools for the study of materials, e.g. Mössbauer, NMR.
- Nuclear power and weapons.

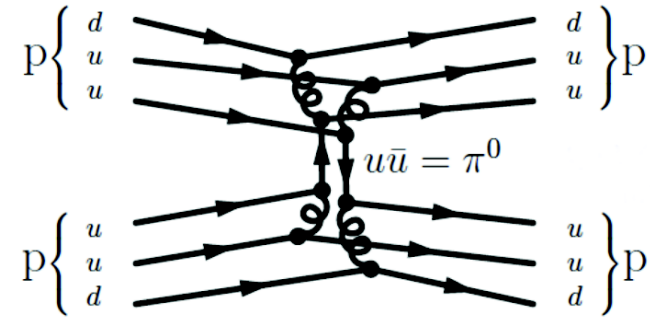
The Nuclear Force

Consider the pp interaction, Range $\sim \hbar/m_\pi c \sim 1\text{fm}$



Hadron level

\equiv



Quark-gluon level

Pion vs. gluon exchange is similar to the Coulomb potential vs. van der Waals' force in QED.

The treatment of the strong nuclear force between nucleons is a **many-body problem** in which

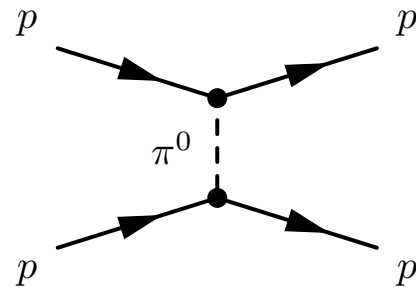
- quarks do not behave as if they were completely independent.
- nor do they behave as if they were completely bound.

The nuclear force is **not yet calculable** in detail at the quark level and can **only** be deduced empirically from nuclear data.

- There is no B^\pm approach to nuclear physics yet

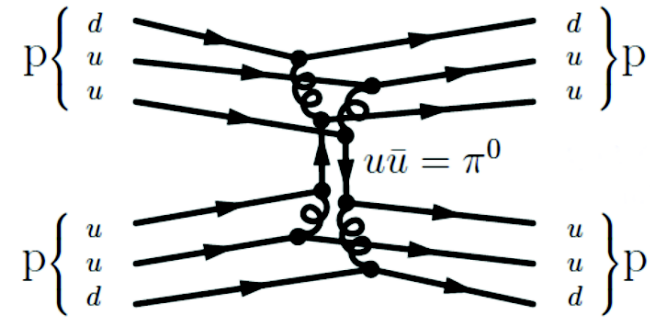
The Nuclear Force

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- nor do they behave as if they were completely bound.

The nuclear force is **not yet calculable** in detail at the quark level and can **only** be deduced empirically from nuclear data.

- There is no B^\pm (bottom-up) approach to nuclear physics yet

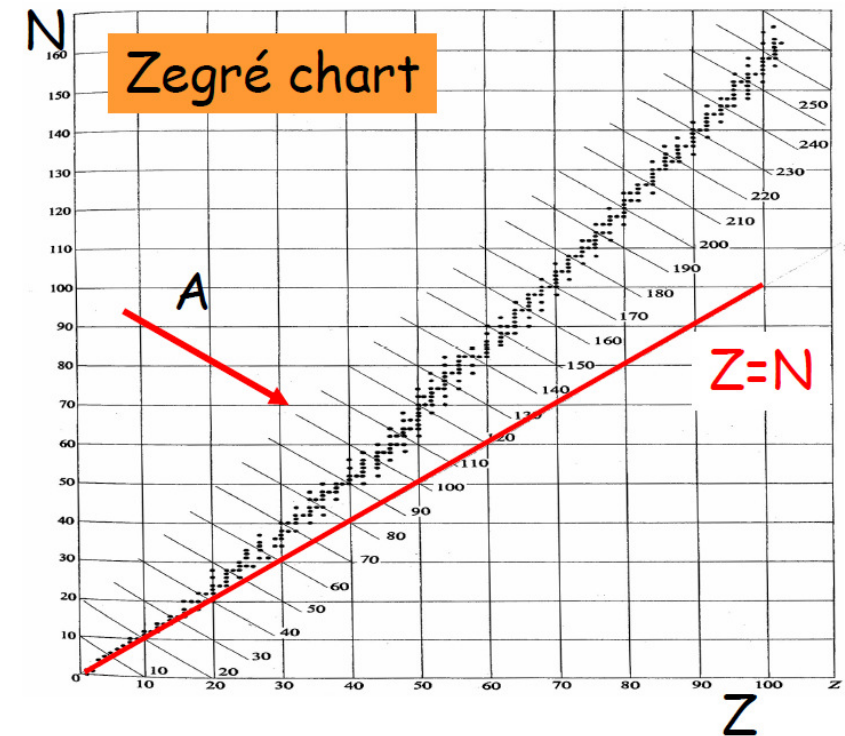
Stable Nuclei

Stable nuclei do not decay by the strong interaction.

They may transform by β and α emission (weak or electromagnetic) with long lifetimes.

Characteristics

- Light nuclei tend to have $N=Z$.
Heavy nuclei have more neutrons, $N > Z$.
Can you think why this is?
- Most have even N and/or Z .
Protons and neutrons tend to form pairs (only 8/284 have odd N and Z).
- Certain values of Z and N exhibit larger numbers of isotopes and isotones.



We would like to be able to explain these characteristics

Binding Energy

Binding Energy is the energy required to split a nucleus into its constituents.

$$\text{Mass of nucleus } m(N, Z) = Zm_p + Nm_n - B$$

Binding energy is **very important**: gives information on

- forces between nucleons
- stability of nucleus
- energy released or required in nuclear decays or reactions

Relies on precise measurement of nuclear masses (mass spectrometry).

Used less in this course, but important nonetheless.

Separation Energy of a nucleon is the energy required to remove a single nucleon from a nucleus.

$$\text{e.g. } n: \quad B\left({}_Z^AX\right) - B\left({}_Z^{A-1}X\right) = m\left({}_Z^{A-1}X\right) + m(n) - m\left({}_Z^AX\right)$$

$$p: \quad B\left({}_Z^AX\right) - B\left({}_{Z-1}^{A-1}X'\right) = m\left({}_{Z-1}^{A-1}X'\right) + m\left({}^1H\right) - m\left({}_Z^AX\right)$$

Binding Energy

Binding Energy per nucleon

Key Observations

Peaks for light nuclei with $A = 4n$. “ α stability”

Broad maximum at $A \sim 60$

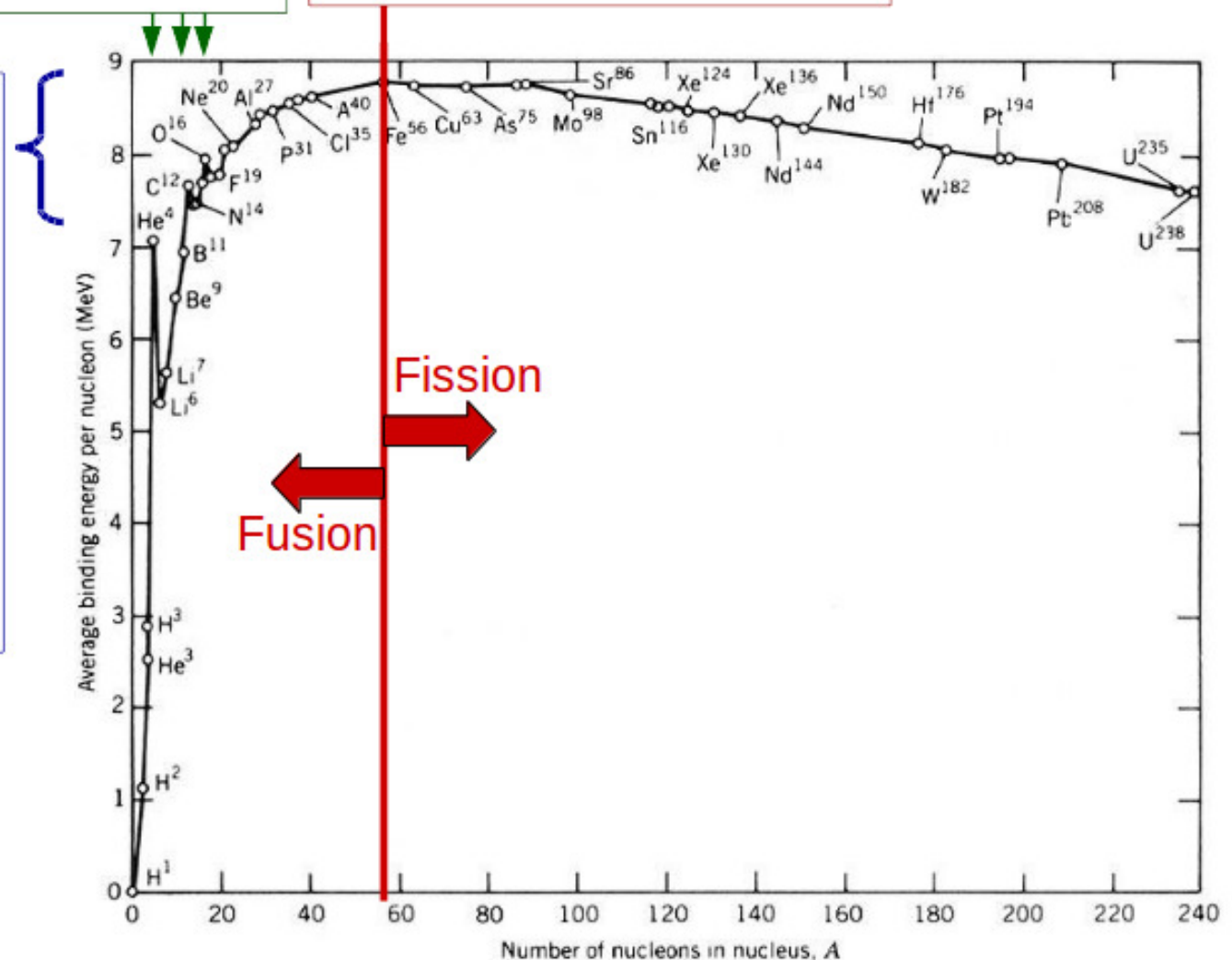
For $A > 20$, $B/A \sim \text{constant}$
(~ 8 MeV per nucleon)

Compare to B of atomic electrons per nucleon < 3 keV

Implies that nucleons are only attracted by nearby nucleons

→ Nuclear force is **short range** and **saturated**

“Saturated” means each nucleus only interacts with a limited number of neighbours; not with all nucleons.



Nuclear mass

The liquid drop model

Atomic mass: $M(A, Z) = Z(m_p + m_e) + (A - Z)m_n - B$

Nuclear mass: $m(A, Z) = Zm_p + (A - Z)m_n - B$

Liquid drop model

Approximate the nucleus as a sphere with a uniform interior density, which drops to zero at the surface.



Liquid Drop

- Short-range intermolecular forces.
- Density independent of drop size.
- Heat required to evaporate fixed mass independent of drop size.

Nucleus

- Nuclear force short range.
- Density independent of nuclear size.
- $B/A \sim \text{constant}$.

Nuclear mass

The liquid drop model

Predicts the binding energy as: $B = a_v A - a_s A^{2/3} - \frac{a_c Z^2}{A^{1/3}}$

Volume term

$$a_v A$$

Strong force between nucleons **increases** B and reduces mass by a constant amount per nucleon.

Nuclear volume $\sim A$

Surface term

$$-a_s A^{2/3}$$

Nucleons on surface are not as strongly bound \Rightarrow **decreases** B .

Surface area $\sim R^2 \sim A^{2/3}$

Coulomb term

$$-\frac{a_c Z^2}{A^{1/3}}$$

Protons repel each other \Rightarrow **decreases** B .

Electrostatic P.E. $\sim Q^2/R \sim Z^2/A^{1/3}$

But there are problems. Does not account for

- $N \sim Z$
- Nucleons tend to pair up; even N , Z favoured

Nuclear mass

The Fermi gas model

Fermi gas model: assume the nucleus is a Fermi gas, in which confined nucleons can only assume certain discrete energies in accordance with the Pauli Exclusion Principle.

Addresses problems with the liquid drop model with additional terms:

$$-a_A \frac{(N - Z)^2}{A}$$

Asymmetry term Nuclei tend to have $N \sim Z$.

Kinetic energy of Z protons and N neutrons is minimised if $N=Z$. The greater the departure from $N=Z$, the smaller the binding energy. Correction scaled down by $1/A$, as levels are more closely spaced as A increases.

$$+\delta(A)$$

Pairing term Nuclei tend to have even Z , even N .

Pairing interaction energetically favours the formation of pairs of like nucleons (pp , nn) with spins $\uparrow\downarrow$ and symmetric spatial wavefunction.

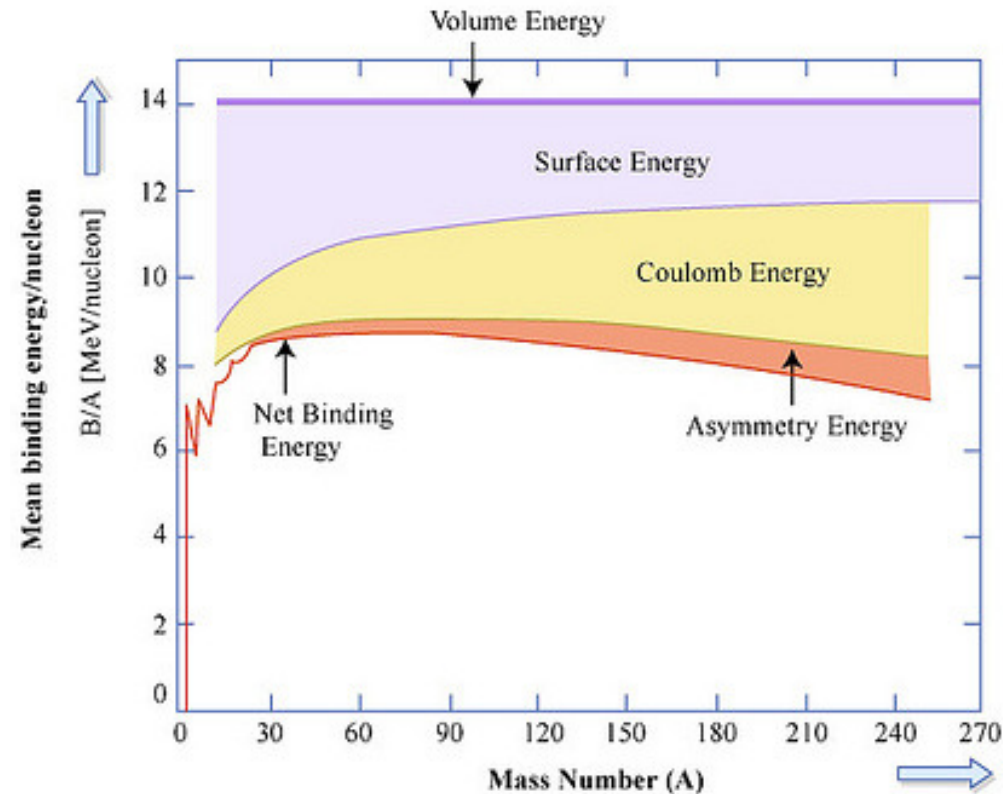
The form is simply empirical.

$$\begin{aligned}\delta(A) &= +a_P A^{-3/4} && N, Z \text{ even-even} \\ &= -a_P A^{-3/4} && N, Z \text{ odd-odd} \\ &= 0 && N, Z \text{ even-odd}\end{aligned}$$

Nuclear mass

The semi-empirical mass formula

Putting all these terms together, we have various contributions to B/A :



Nuclear mass is well described by the **semi-empirical mass formula**

$$m(A, Z) = Zm_p + (A - Z)m_n - B$$

$$B = a_V A - a_S A^{2/3} - \frac{a_C Z^2}{A^{1/3}} - a_A \frac{(N - Z)^2}{A} + \delta(A)$$

with the following coefficients (in MeV) obtained by fitting to data

$$a_V = 15.8, a_S = 18.0, a_C = 0.72, a_A = 23.5, a_P = 33.5$$

Nuclear Spin

The nucleus is an isolated system and so has a well defined **nuclear spin**

Nuclear spin quantum number J

$$|J| = \sqrt{J(J+1)} \quad \hbar = 1$$
$$m_J = -J, -(J-1), \dots, J-1, J.$$

Nuclear spin is the sum of the **individual nucleons** total angular momentum, j_i ,

$$\vec{J} = \sum_i \vec{j}_i, \quad \vec{j}_i = \vec{L}_i + \vec{S}_i$$

$j - j$ coupling always applies because of strong spin-orbit interaction (see later)

where the total angular momentum of a nucleon is the sum of its **intrinsic spin** and **orbital angular momentum**

- intrinsic spin of p or n is $s = 1/2$
- orbital angular momentum of nucleon is integer
 - A even $\rightarrow J$ must be integer
 - A odd $\rightarrow J$ must be $1/2$ integer

All nuclei with even N and even Z have $J = 0$.

- Spin important as it determines **magnetic moments** affects **nuclear decay rates** and is crucial in **NMR**, **MRI** and **spectroscopy**

Nuclear Parity

- All particles are eigenstates of parity $\hat{P}|\Psi\rangle = P|\Psi\rangle$, $P = \pm 1$
- Label nuclear states with the nuclear spin and parity quantum numbers.
Example: 0^+ ($J = 0$, parity even), 2^- ($J = 2$, parity odd)
- The parity of a nucleus is given by the product of the parities of all the neutrons and protons
$$P = \left(\prod_i P_i \right) (-1)^L$$
for ground state nucleus, $L = 0$
- The parity of a single proton or neutron is $P = (+1)(-1)^L$
intrinsic $P = +1$ (3 quarks) nucleon L is important
- For an odd A , the parity is given by the unpaired p or n . (*Nuclear Shell Model*)
- Parity is conserved in nuclear processes (strong interaction).
- Parity of nuclear states can be extracted from experimental measurements, e.g. γ transitions.

Nuclear Size

The **size** of a nucleus may be determined using two sorts of interaction:

Electromagnetic Interaction gives the **charge** distribution of protons inside the nucleus, e.g.

- electron scattering
- muonic atoms
- mirror nuclei

Strong Interaction gives **matter** distribution of protons and neutrons inside the nucleus. Sample nuclear and charge interactions at the same time \Rightarrow more complex, e.g.

- α particle scattering (Rutherford)
- proton and neutron scattering
- Lifetime of α particle emitters (see later)
- π -mesic X-rays.

\Rightarrow Find charge and matter radii EQUAL for all nuclei.

Nuclear Size

Electron scattering

Use electron as a probe to study deviations from a point-like nucleus.

Electromagnetic Interaction

Coulomb potential $V(\vec{r}) = -\frac{Z\alpha}{r}$

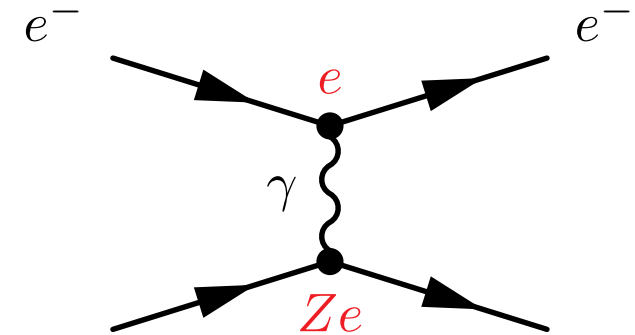
Born Approximation
$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2} \left| \int e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) d^3\vec{r} \right|^2$$

 $\vec{q} = \vec{p}_i - \vec{p}_f$ is the momentum transfer

Rutherford Scattering
$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4E^2 \sin^4 \theta/2}$$

To measure a distance of ~ 1 fm, need large energy (*ultra-relativistic*)

$$E = \frac{1}{\lambda} = 1 \text{ fm}^{-1} \sim 200 \text{ MeV} \quad \hbar c = 197 \text{ MeV}\cdot\text{fm}$$



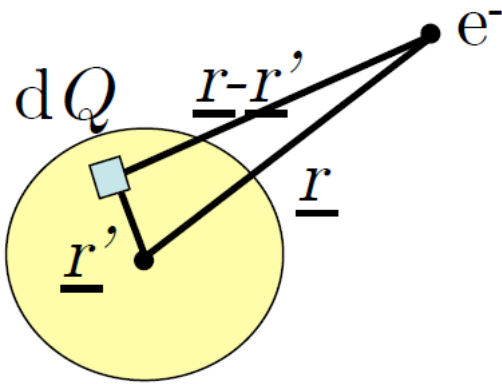
Nucleus, Z protons

Nuclear Size

Scattering from an extended nucleus

But the nucleus is not point-like!

$V(\vec{r})$ depends on the distribution of charge in nucleus.



Potential energy of electron
due to charge dQ

$$dV = -\frac{e dQ}{4\pi |\vec{r} - \vec{r}'|}$$

where $dQ = Ze\rho(\vec{r}') d^3\vec{r}'$

$\rho(\vec{r}')$ is the charge distribution (normalised to 1)

$$V(\vec{r}) = \int -\frac{e^2 Z \rho(\vec{r}')}{4\pi |\vec{r} - \vec{r}'|} = -Z\alpha \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \quad \alpha = \frac{e^2}{4\pi}$$

This is just a convolution of the pure Coulomb potential $Z\alpha/r$ with the normalised charge distribution $\rho(r)$.

Hence we can use the convolution theorem to help evaluate the matrix element which enters into the Born Approximation.

Nuclear Size

Scattering from an extended nucleus

Matrix Element $M_{if} = \int e^{i\vec{q}\vec{r}} V(\vec{r}) d^3\vec{r} = -Z\alpha \int \frac{e^{i\vec{q}\vec{r}}}{r} d^3\vec{r} \int \rho(\vec{r}) e^{i\vec{q}\vec{r}} d^3\vec{r}$

Rutherford scattering $F(q^2)$

Hence, $\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_{\text{point}} |F(q^2)|^2$

where $F(q^2) = \int \rho(\vec{r}) e^{i\vec{q}\vec{r}} d^3\vec{r}$ is called the **Form Factor** and is the fourier transform of the normalised charge distribution.

Spherical symmetry, $\rho = \rho(r)$, a simple calculation (similar to our treatment of the Yukawa potential) shows that

$$F(q^2) = \int_0^\infty \rho(r) \frac{\sin qr}{qr} 4\pi r^2 dr \quad ; \quad \rho(r) = \frac{1}{2\pi^2} \int_0^\infty F(q^2) \frac{\sin qr}{qr} q^2 dq$$

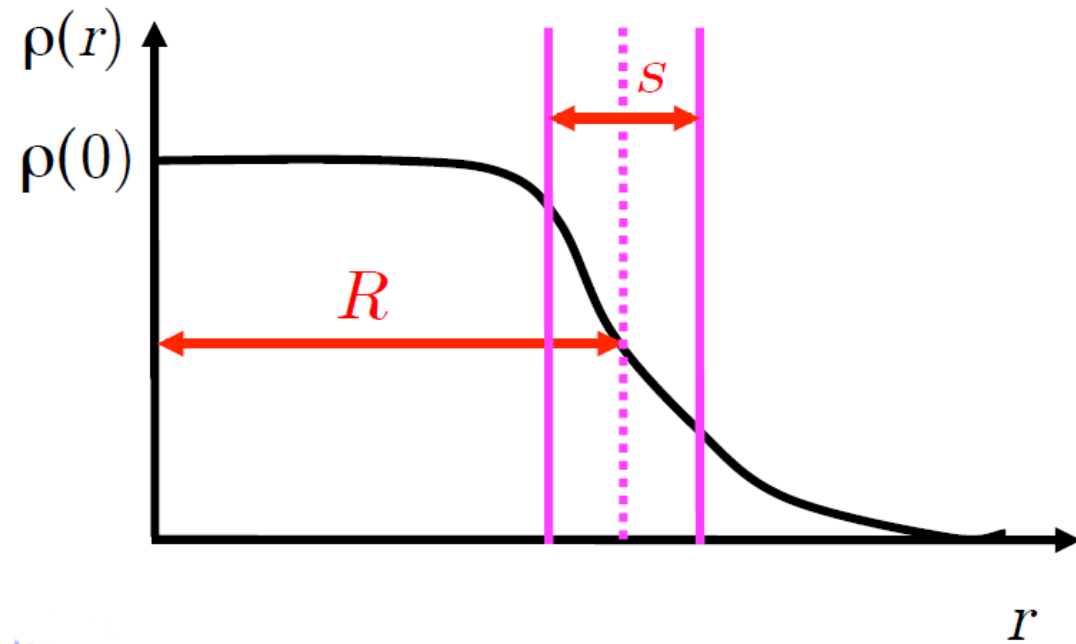
So if we measure cross-section, we can infer $F(q^2)$ and get the charge distribution by Fourier transformation.

Nuclear Size

Modelling charge distribution

Use nuclear diffraction to measure scattering, and find the charge distribution inside a nucleus is well described by the **Fermi parametrisation**.

$$\rho(r) = \frac{\rho(0)}{1 + e^{(r-R)/s}}$$



Fit this to data to determine parameters R and s .

- R is the **radius** at which $\rho(r) = \rho(0)/2$

Find R increases with A : $R = r_0 A^{1/3}$ $r_0 \sim 1.2 \text{ fm}$.

- s is the **surface width** or **skin thickness** over which $\rho(r)$ falls from 90% \rightarrow 10%.

Find s is approximately the same for all nuclei ($s \sim 2.5 \text{ fm}$); governed by the range of the strong nuclear interaction

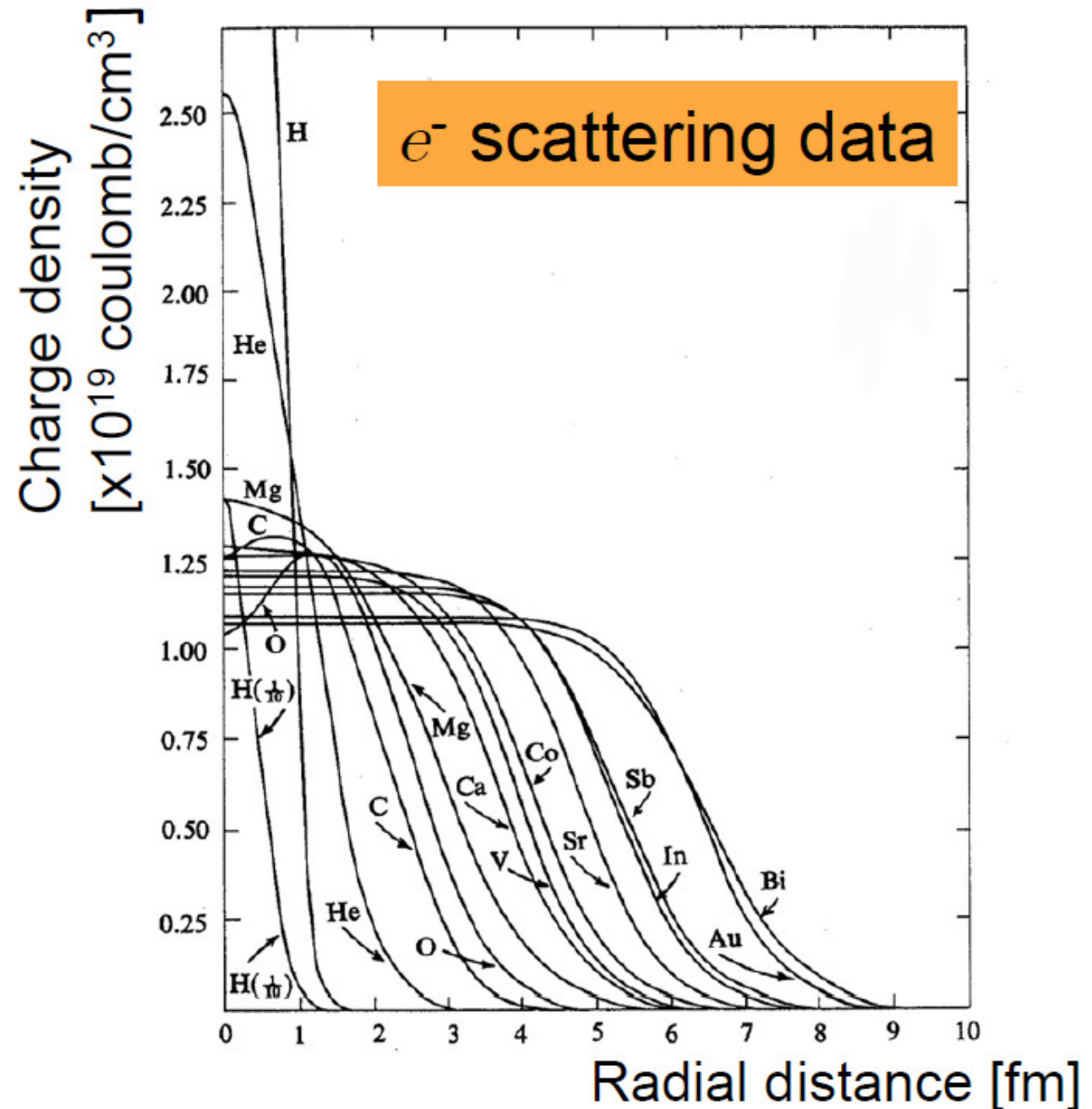
Nuclear Size

Modelling charge distribution

Fits to e^- scattering data show the Fermi parametrisation models nuclear charge distributions well.

Shows that all nuclei have roughly the same density in their interior.

Radius $\sim R_0 A^{1/3}$ with $R_0 \sim 1.2 \text{ fm} \Rightarrow$ consistent with short-range saturated forces.



Nuclear Size *Muonic Atoms*

Muons can be brought to rest in matter and trapped in orbit → probe EM interactions with nucleus.

The large muon mass affects its orbit, $m_\mu \sim 207 m_e$

Bohr radius, $r \propto 1/Zm$

Hydrogen atom with electrons: $r = a_0 \sim 53,000$ fm
with muons: $r \sim 285$ fm

Lead ($Z = 82$) with muons: $r \sim 3$ fm **Inside nucleus!**

Energy levels, $E \propto Z^2 m$

Rapid transitions to lower energy levels $\sim 10^{-9}$ s

Factor of 2 effect seen from nuclear size in muonic lead

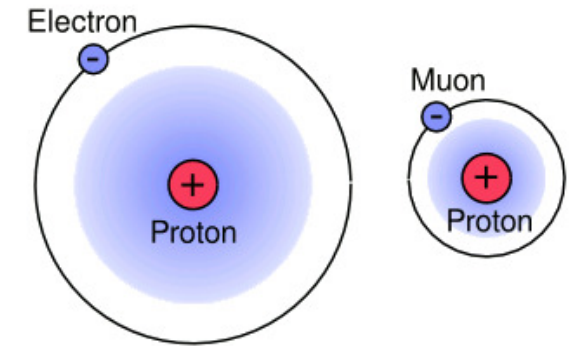
Transition energy ($2P_{3/2} \rightarrow 1S_{1/2}$) : 16.41 MeV (Bohr theory) vs 6.02 MeV (measured)

Muon lifetime, $\tau_\mu \sim 2\mu\text{s}$

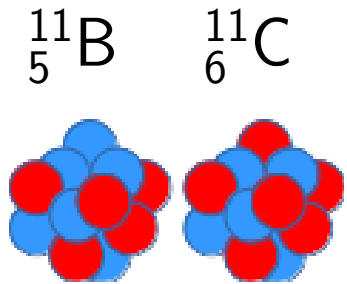
Decays via $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ – Plenty of time spent in 1s state.

$Z_{\text{effective}}$ and E are changed relative to electrons.

Measure X-ray energies → **nuclear radius**.



Nuclear Size *Mirror Nuclei*



Different nuclear masses from p - n difference and the different Coulomb terms.

$$m(A, Z) = Zm_p + (A - Z)m_n - \left[a_V A - a_S A^{2/3} - \frac{a_C Z^2}{A^{1/3}} - a_A \frac{(N - Z)^2}{A} + \delta(A) \right]$$

For the *atomic* mass difference, don't forget the electrons!

$$M(A, Z + 1) - M(A, Z) = \Delta E_c + m_p + m_e - m_n$$

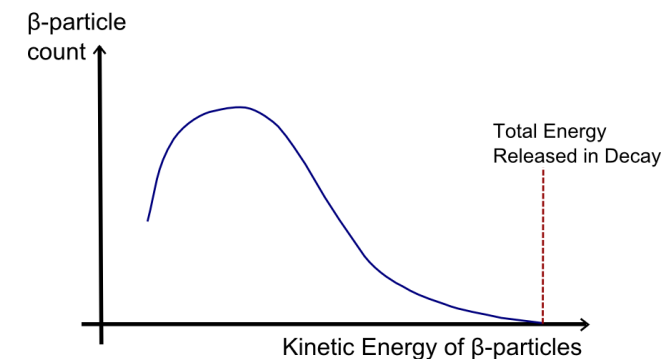
$$\text{where } \Delta E_c = \frac{3A\alpha}{5R} \quad (\text{see Question 33})$$

Probe the atomic mass difference between two mirror nuclei by observing β^+ decay spectra (3-body decay).



$$M(A, Z + 1) - M(A, Z) = 2m_e + E_{\text{max}} \quad m_\nu \sim 0$$

where E_{max} is the maximum kinetic energy of the positron.

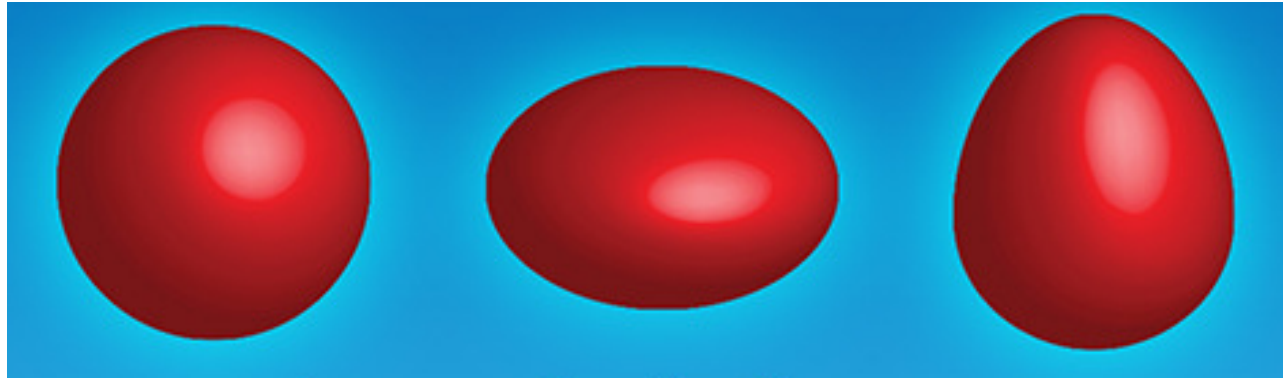


Relate mass difference to ΔE_c
and extract the nuclear radius

$$R = \frac{3A\alpha}{5} \left[\frac{1}{E_{\text{max}} - m_p + m_n + m_e} \right]$$

Nuclear Shape

The shape of nuclei can be inferred from measuring their **electromagnetic moments**.



Nuclear moments give information about the way magnetic moment and charge is distributed throughout the nucleus.

The two most important moments are:

Electric Quadrupole Moment Q

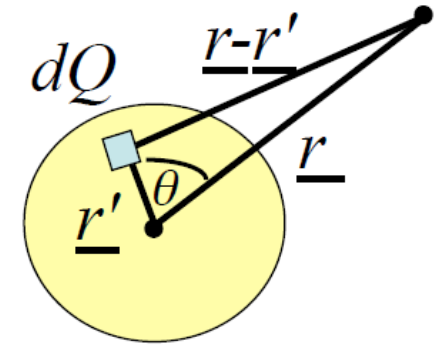
Magnetic Dipole Moment μ

Nuclear Shape *Electric Moments*

Electric moments depend on the **charge distribution** inside the nucleus.

Parameterise the nuclear shape using a multipole expansion of the external electric field or potential

$$V(r) = \frac{1}{4\pi} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$



where $\rho(\vec{r}') d^3\vec{r}' = Ze$ and $r(r')$ = distance to observer (charge element) from origin.

$$\begin{aligned} |\vec{r} - \vec{r}'| &= [r^2 + r'^2 - 2rr' \cos \theta]^{1/2} \Rightarrow |\vec{r} - \vec{r}'|^{-1} = r^{-1} \left[1 + \frac{r'^2}{r^2} - 2\frac{r'}{r} \cos \theta \right]^{-1/2} \\ |\vec{r} - \vec{r}'|^{-1} &= r^{-1} \left[1 - \frac{1}{2} \left(\frac{r'^2}{r^2} - 2\frac{r'}{r} \cos \theta \right) + \frac{3}{8} \left(\frac{r'^2}{r^2} - 2\frac{r'}{r} \cos \theta \right)^2 + \dots \right] \\ &\sim r^{-1} \left[1 + \frac{r'}{r} \cos \theta + \frac{1}{2} \frac{r'^2}{r^2} (3 \cos^2 \theta - 1) + \dots \right] \end{aligned}$$

$r' \ll r \Rightarrow$ expansion in powers of $r'r$; or equivalently Legendre polynomials

$$V(r) = \frac{1}{4\pi r} \left[Ze + \frac{1}{r} \int r' \cos \theta \rho(r') d^3\vec{r}' + \frac{1}{2r^2} \int r'^2 (3 \cos^2 \theta - 1) \rho(r') d^3\vec{r}' + \dots \right]$$

Nuclear Shape

Electric Moments

Let r define z -axis, $z = r' \cos \theta$

$$V(r) = \frac{1}{4\pi r} \left[Ze + \frac{1}{r} \int z \rho(r') d^3 \vec{r}' + \frac{1}{2r^2} \int (3z^2 - r'^2) \rho(r') d^3 \vec{r}' + \dots \right]$$

Quantum limit: $\rho(r') = Ze \cdot |\psi(\vec{r}')|^2$

The electric moments are the coefficients of each successive power of $1/r$

E0 moment $\int Ze \cdot \psi^* \psi d^3 \vec{r}' = Ze$ *charge*

No shape information

E1 moment $\int \psi^* z \psi d^3 \vec{r}'$ *electric dipole*

Always zero since ψ have definite parity

$$|\psi(\vec{r})|^2 = |\psi(-\vec{r})|^2$$

E2 moment $\int \frac{1}{e} \psi^* (3z^2 - r'^2) \psi d^3 \vec{r}'$ *electric quadrupole*

First interesting moment!

Nuclear Shape *Electric Moments*

Electric Quadrupole Moment

$$Q = \frac{1}{e} \int (3z^2 - r^2) \rho(\vec{r}) d^3\vec{r}$$

Units: m^2 or barns (though sometimes the factor of e is left in)

If spherical symmetry, $\bar{z}^2 = \frac{1}{3}\bar{r}^2 \Rightarrow Q = 0$

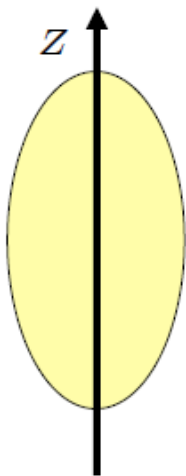
- $Q = 0$ Spherical nucleus.
- Large Q Highly deformed nucleus.

All $J = 0$ nuclei have $Q = 0$.
e.g. Na

Two cases:

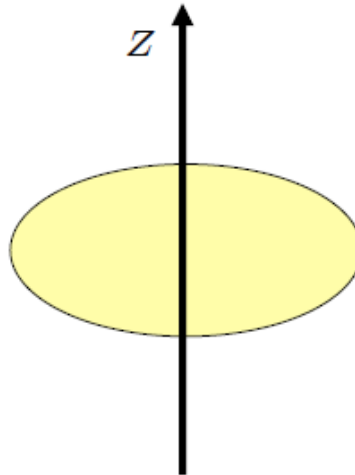
Prolate spheroid

$Q > 0$



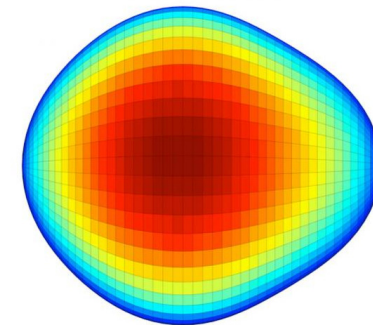
Oblate spheroid

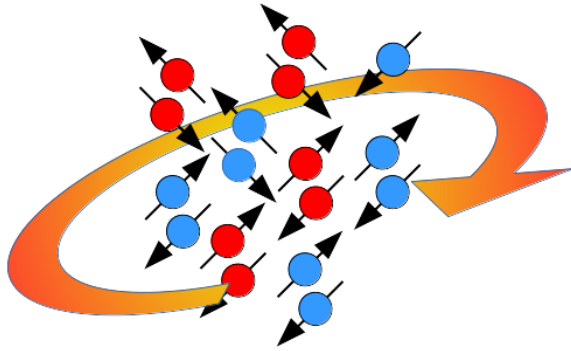
$Q < 0$



Aside: Radium-224 is pear-shaped!
Non-zero quadrupole and octupole moments.

(ISOLDE, CERN, 2013)





- Nuclear magnetic dipole moments arise from
- intrinsic spin magnetic dipole moments of the protons and neutrons
 - circulating currents (motion of the protons)

The **nuclear magnetic dipole moment** can be written as

$$\vec{\mu} = \frac{\mu_N}{\hbar} \sum_i \left[g_L \vec{L} + g_s \vec{S} \right]$$

summed over all p, n

where $\mu_N = e\hbar/2m_p$ is the Nuclear Magneton.

or $\mu = g_J \mu_N J$ where J total nuclear spin quantum number
 g_J nuclear g -factor (analogous to Landé g -factor in atoms)

g_J may be predicted using the Nuclear Shell Model (see later), and measured using magnetic resonance (see Advanced Quantum course).

All even-even nuclei have $\mu = 0$ since $J = 0$

Summary

- Nuclear binding energy – short range saturated forces
- Semi-empirical Mass Formula – based on liquid drop model + simple inclusion of quantum effects

$$m(A, Z) = Zm_p + (A - Z)m_n - B$$

$$B = a_V A - a_S A^{2/3} - \frac{a_C Z^2}{A^{1/3}} - a_A \frac{(N - Z)^2}{A} + \delta(A)$$

- Nuclear size from electron scattering, muonic atoms, and mirror nuclei.
Constant density; radius $\propto A^{1/3}$
- Nuclear spin, parity, electric and magnetic moments.

Problem Sheet: q.31-33

Up next...

Section 14: The Structure of Nuclei