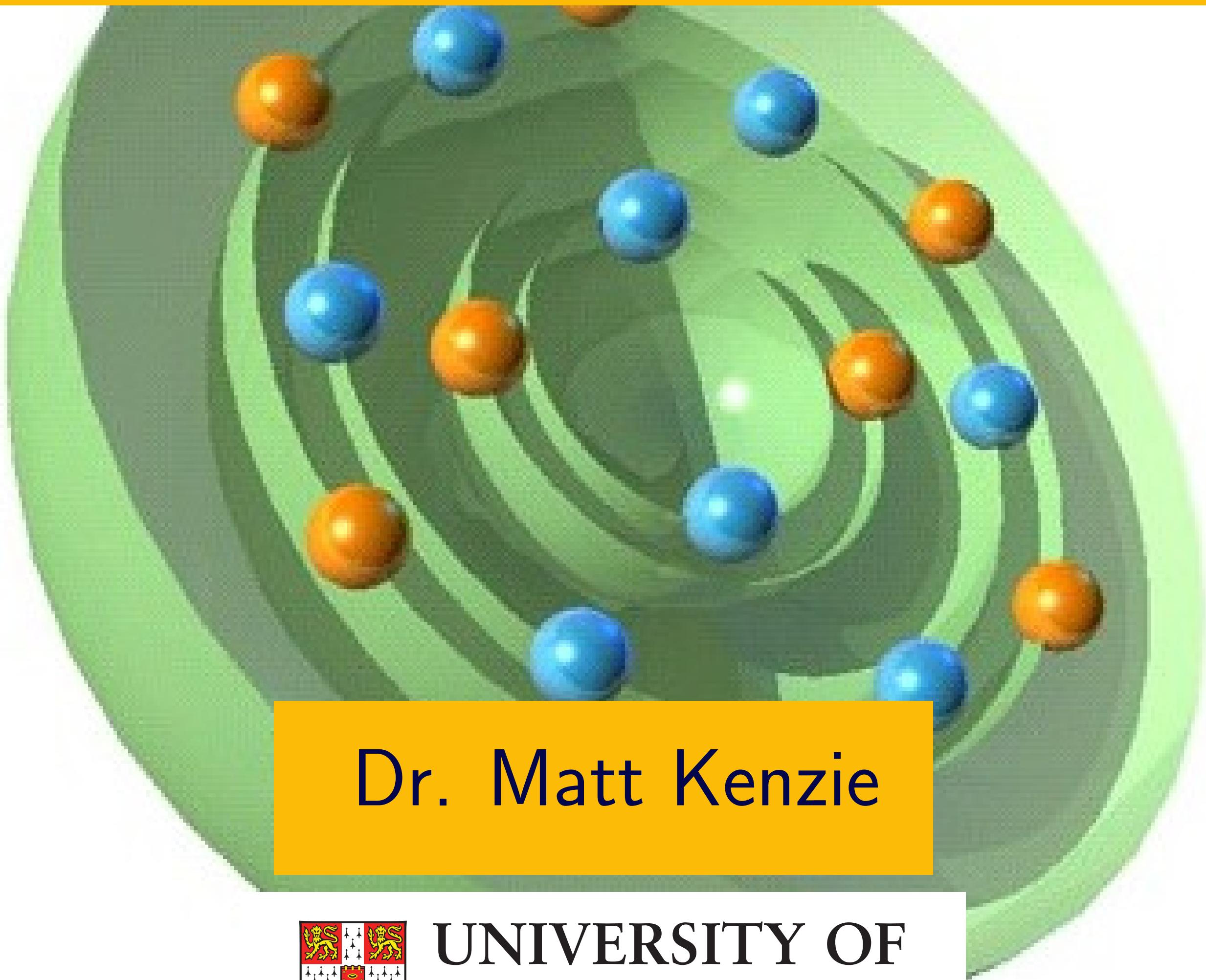
14. Structure of Nuclei

Particle and Nuclear Physics



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In this section...

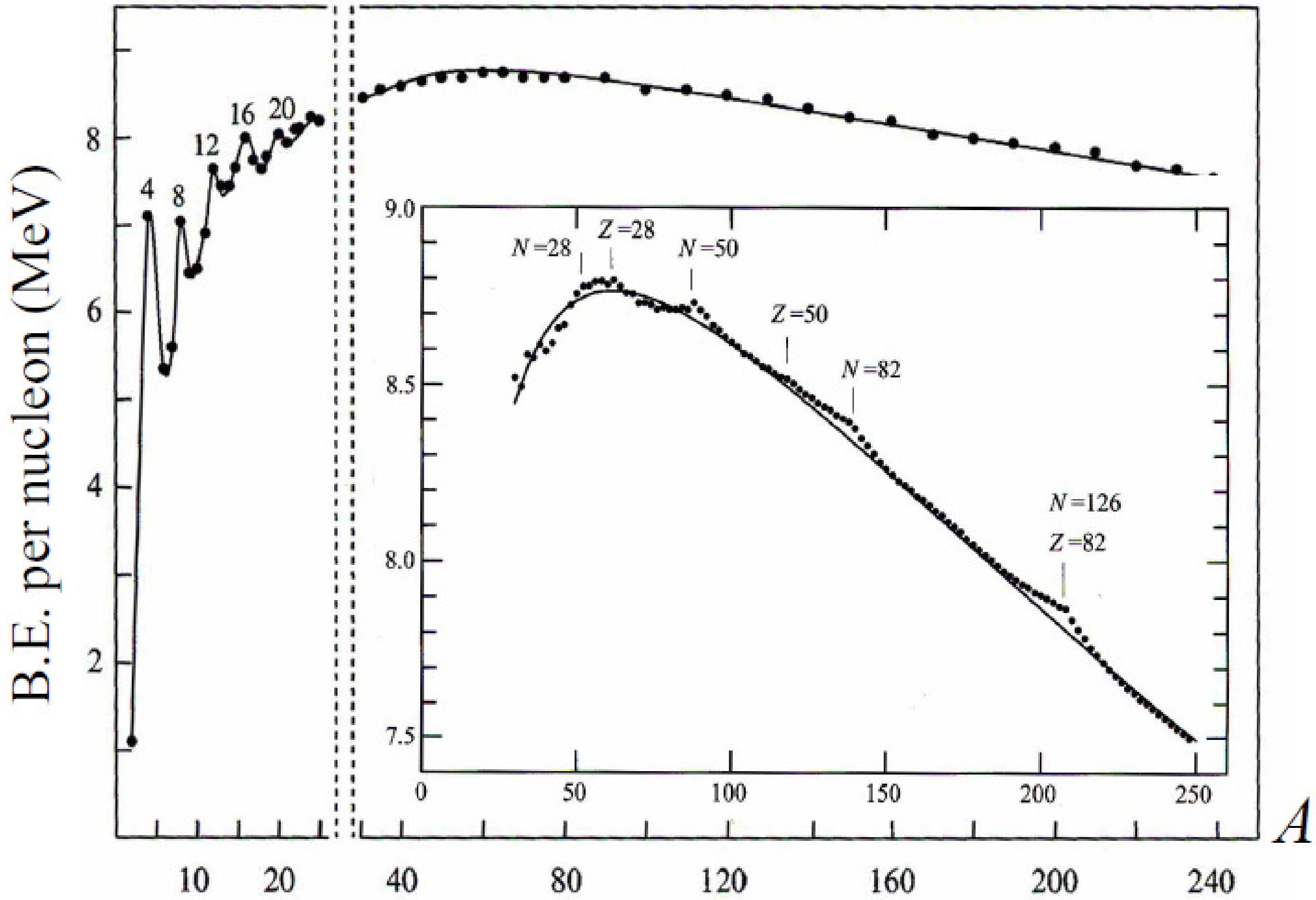
- Magic Numbers
- The Nuclear Shell Model
- Excited States

Magic Numbers

Magic Numbers = 2, 8, 20, 28, 50, 82, 126...

Nuclei with a magic number of Z and/or N are particularly stable,

e.g. Binding energy per nucleon is large for magic numbers



Doubly magic nuclei are especially stable.

Magic Numbers

Other notable behaviour includes

- Greater abundance of isotopes and isotones for magic numbers
  e.g. $Z = 20$ has 6 stable isotopes (average $= 2$)
  $Z = 50$ has 10 stable isotopes (average $= 4$)
- Odd A nuclei have small quadrupole moments when magic
- First excited states for magic nuclei higher than neighbours
- Large energy release in α, β decay when the daughter nucleus is magic
- Spontaneous neutron emitters have $N = magic + 1$
- Nuclear radius shows only small change with Z, N at magic numbers.

etc... etc...

Magic Numbers

Analogy with atomic behaviour as electron shells fill.

Atomic case - reminder

- Electrons move independently in central potential $V(r) \sim 1/r$ (Coulomb field of nucleus).
- Shells filled progressively according to Pauli exclusion principle.
- Chemical properties of an atom defined by valence (unpaired) electrons.
- Energy levels can be obtained (to first order) by solving Schrödinger equation for central potential.

$$E_n = \frac{1}{n^2} \qquad n = \text{ principle quantum number}$$

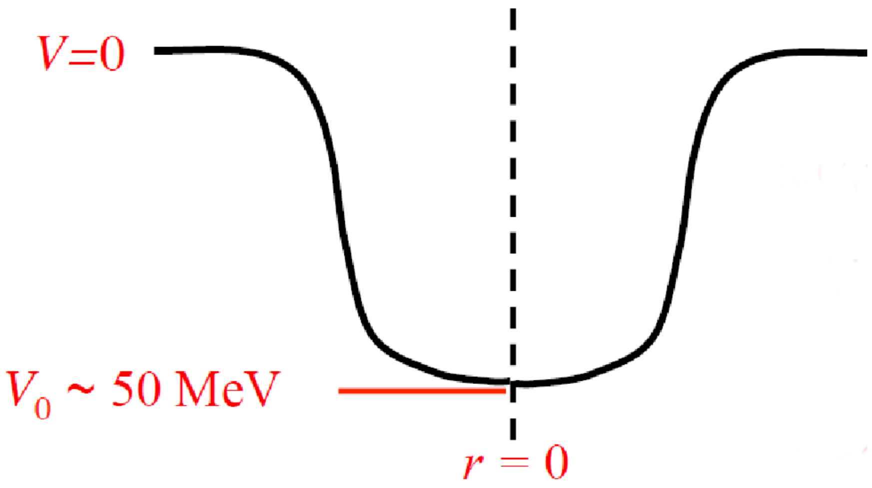
- Shell closure gives noble gas atoms.

Are magic nuclei analogous to the noble gas atoms?

Magic Numbers

Nuclear case (Fermi gas model)

Nucleons move in a net nuclear potential that represents the *average effect* of interactions with the other nucleons in the nucleus.



Nuclear Potential

$$V(r) \sim \frac{-V_0}{\left(1 + e^{(r-R)/s}\right)}$$

"Saxon-Woods potential", i.e. a Fermi function, like the nuclear charge distribution

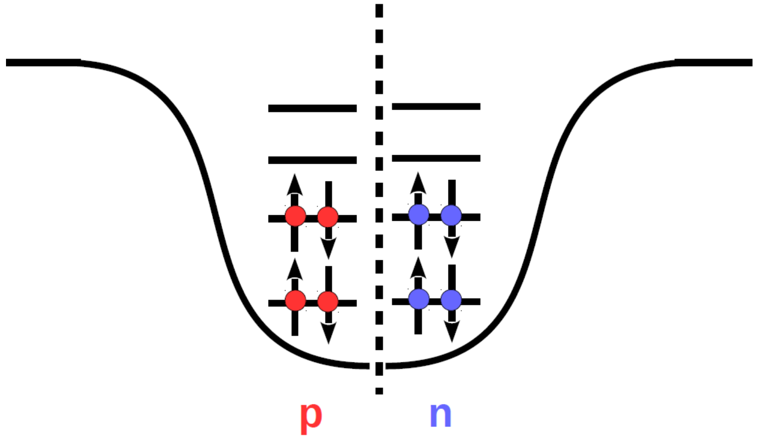
- Nuclear force short range + saturated ⇒ near centre $V(r)$ ∼constant.
- Near surface: density and no. of neighbours decreases ⇒ $V(r)$ decreases
- For protons, $V(r)$ is modified by the Coulomb interaction

Magic Numbers

In the ground state, nucleons occupy energy levels of the nuclear potential so as to minimise the total energy without violating the Pauli principle.

The exclusion principle operates independently for protons and neutrons.

Tendency for Z=N
to give the minimum E



Postulate: nucleons move in well-defined orbits with discrete energies.

Objection: nucleons are of similar size to nucleus ∴ expect many collisions. How can there be well-defined orbits?

Pauli principle: if energy is transferred in a collision then nucleons must move up/down to new states. However, all nearby states are occupied ∴ no collision. i.e. almost all nucleons in a nucleus move freely within nucleus if it is in its ground state.

The Nuclear Shell Model

- Treat each nucleon independently and solve Schrödingers equation for nuclear potential to obtain nucleon energy levels.
- Consider spherically symmetric central potential e.g. Saxon-Woods potential
$$V(r) \sim \frac{-V_0}{\left(1 + \mathrm{e}^{(r-R)/s}\right)}$$
- Solution of the form $\psi(\vec{r}) = R_{nL}(r) Y_L^m(\theta, \phi)$
- Obtain 2 equations separately for radial and angular coordinates.

Radial Equation: $\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L(L+1)}{r^2} + 2M(E - V(r))\right] R_{nL}(r) = 0$

Allowed states specified by n, L, m:

n radial quantum number (n.b. different to atomic notation)

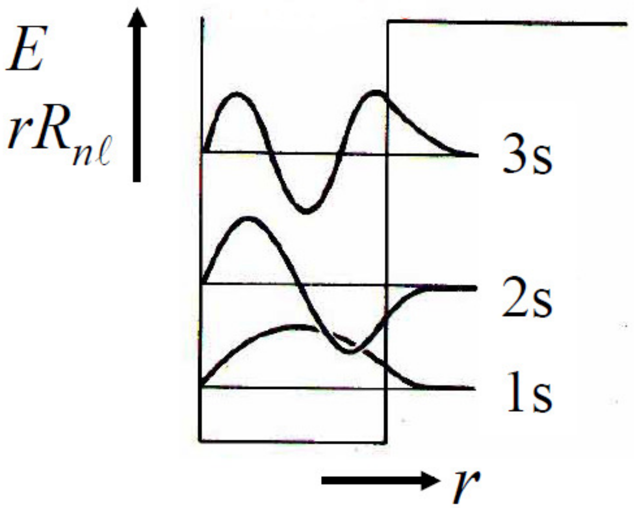
L orbital a.m. quantum no. n.b. any L for given n (c.f. Atomic $L < n$)

m magnetic quantum number $m = -L ... + L$

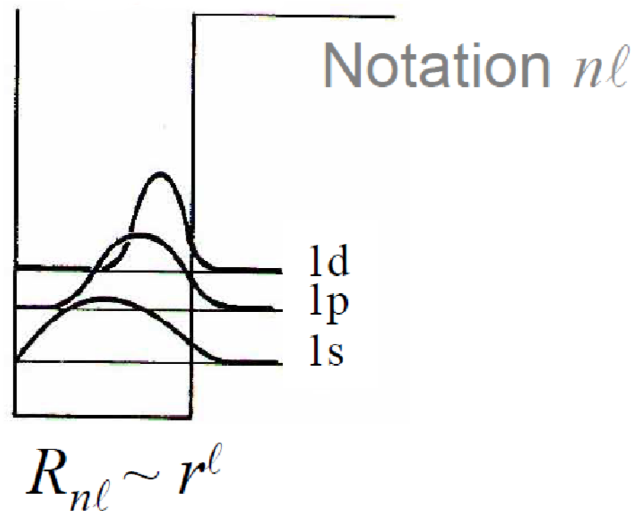
The Nuclear Shell Model

Energy levels increase with n and L (similar to atomic case)

Fix L, increase n



Fix n, increase L



As n increases:
rR_{nL} has more nodes, greater curvature and E increases.

As L increases:
rR_{nL} has greater curvature and E increases.

Fill shells for both p and n:

$$\text{Degeneracy} = (2s+1)(2L+1) = 2(2L+1) \quad (s = 1/2)$$

But, this central potential alone cannot reproduce the observed magic numbers. Need to include **spin-orbit interaction**.

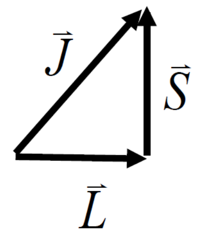
Spin-orbit interaction

Mayer and Jensen (1949) included (strong) spin-orbit potential to explain magic numbers.

$$V(r) = V_{\text{central}}(r) + V_{\text{so}}(r)\hat{\vec{L}}.\hat{\vec{S}}$$

n.b. V_{so} is negative

Spin-orbit interaction splits L levels into their different j values

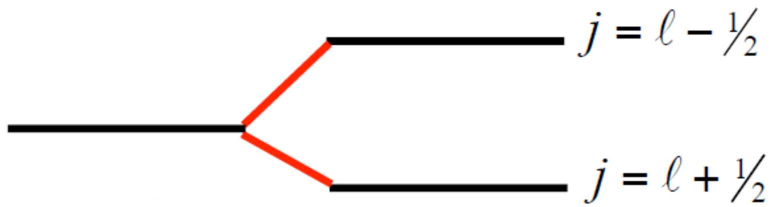


$$\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}; \qquad \hat{\vec{J}}^2 = \hat{\vec{L}}^2 + \hat{\vec{S}}^2 + 2\hat{\vec{L}}.\hat{\vec{S}}; \qquad \hat{\vec{L}}.\hat{\vec{S}} = \frac{1}{2}\left[\hat{\vec{J}}^2 - \hat{\vec{L}}^2 - \hat{\vec{S}}^2\right]$$

$$\hat{\vec{L}}.\hat{\vec{S}}|\psi\rangle = \frac{1}{2}\left[j(j+1) - L(L+1) - s(s+1)\right]|\psi\rangle$$

For a single nucleon with $s = \frac{1}{2}$,

- $j = L - \frac{1}{2}$: $\quad \hat{\vec{L}}.\hat{\vec{S}}|\psi\rangle = -\frac{1}{2}(L+1)|\psi\rangle \quad V = V_{\text{central}} - \frac{1}{2}(L+1)V_{\text{so}}$
- $j = L + \frac{1}{2}$: $\quad \hat{\vec{L}}.\hat{\vec{S}}|\psi\rangle = \frac{1}{2}L|\psi\rangle \quad V = V_{\text{central}} + \frac{1}{2}LV_{\text{so}}$

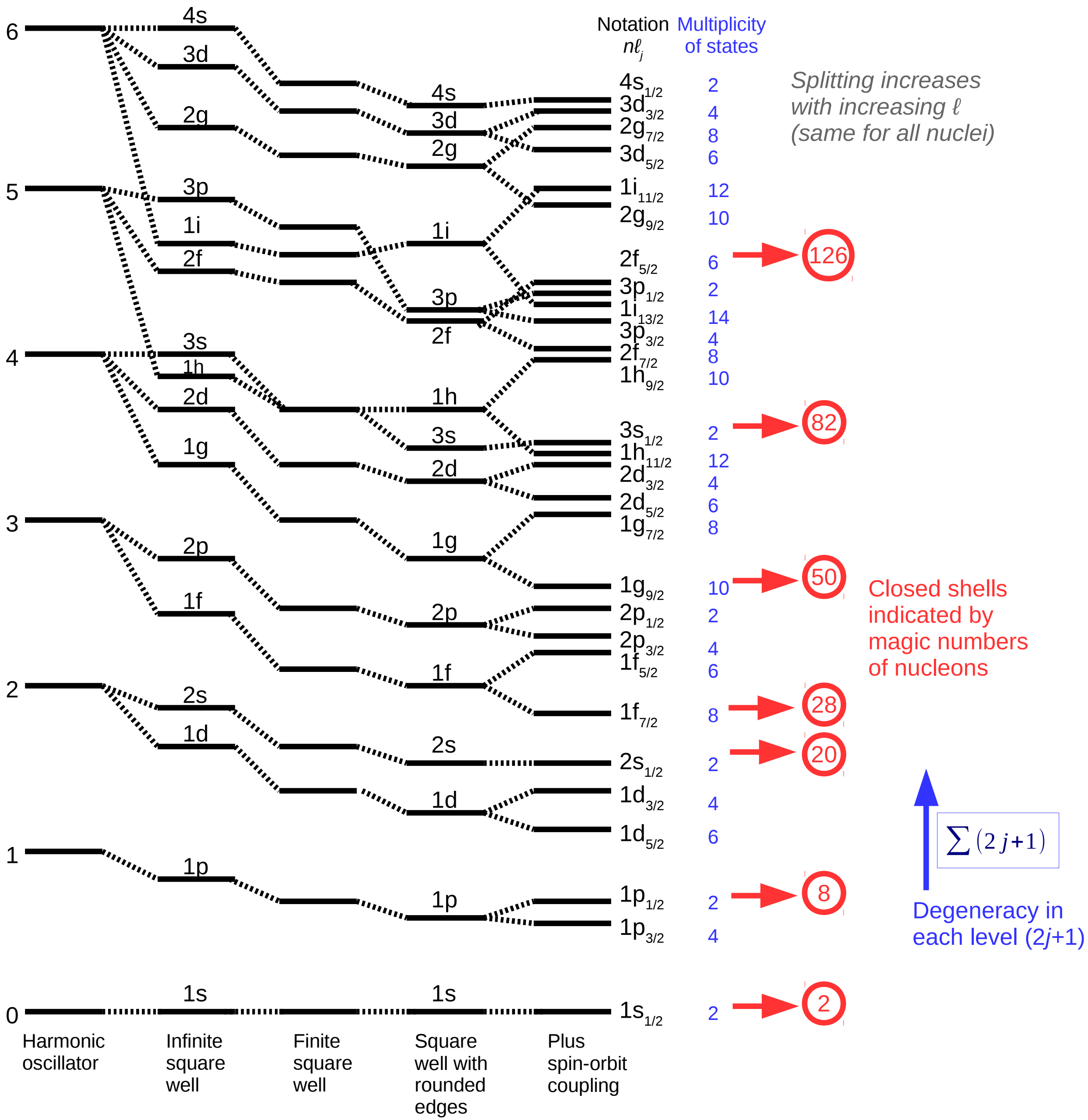


V_{central} with $\hat{\vec{L}}.\hat{\vec{S}}$

$$\Delta E = \frac{1}{2}(2L+1)V_{\text{so}}$$

n.b. larger j lies lower

Nuclear Shell Model *Energy Levels*



Nuclear Shell Model Predictions

1. Magic Numbers.
   The Shell Model successfully predicts the origin of the magic numbers. It was constructed to achieve this.
2. Spin & Parity.
3. Magnetic Dipole Moments.

Nuclear Shell Model *Spin and Parity*

The Nuclear Shell Model predicts the spin & parity of ground state nuclei.

Case 1: Near closed shells

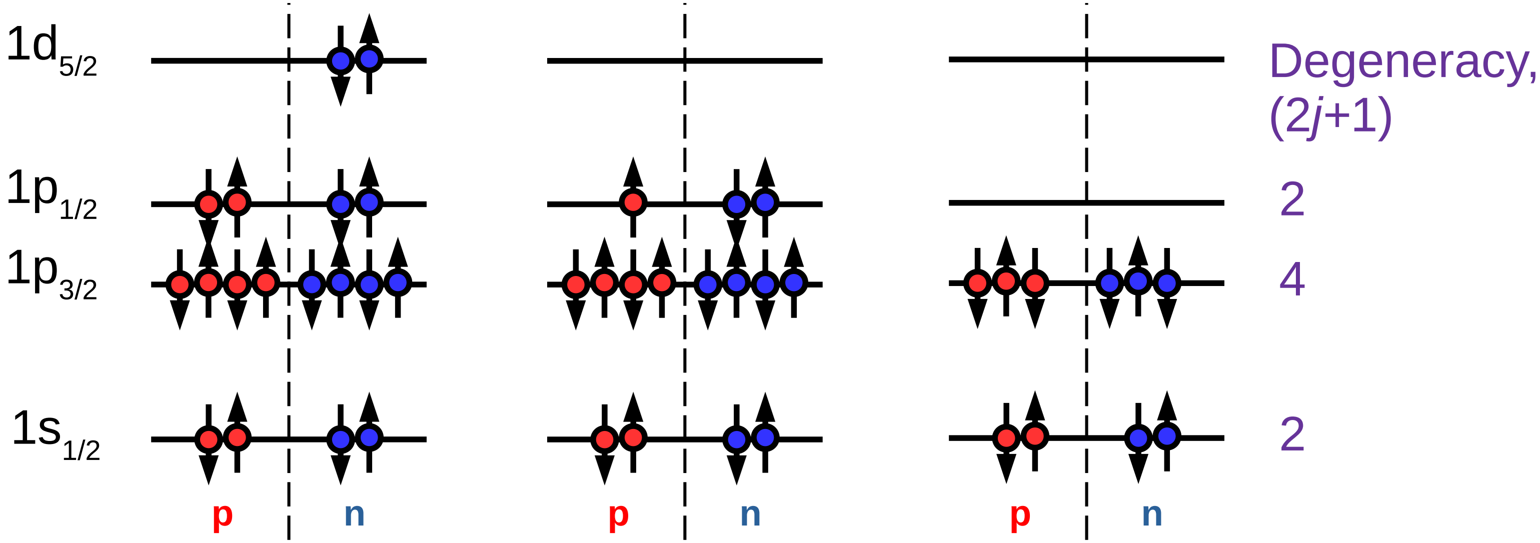
Even-Even Nuclei : $J^P = 0^+$

Even-Odd Nuclei : J^P given by unpaired nucleon or hole; $P = (-1)^L$

Odd-Odd Nuclei : Find J values of unpaired p and n, then apply jj coupling

i.e. $|j_p - j_n| \leq J \leq j_p + j_n$, Parity $= (-1)^{Lp}(-1)^{Ln}$

e.g.



$^{18}_{8}O$: $J^P = 0^+$ (obs)

$^{15}_{7}N$: $J^P = 1/2^-$ (obs)

$^{10}_{5}B$: $j_p = 3/2^-$, $j_n = 3/2^-$

$J^P = 0^+, 1^+, 2^+, 3^+$ ($J^P = 3^+$ observed)

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i.e. $|j_p - j_n| \leq J \leq j_p + j_n$, Parity $= (-1)^{L_p}(-1)^{L_n}$

There are however cases where this simple prescription fails.

The pairing interaction between identical nucleons is not described by a spherically symmetric potential nor by the spin-orbit interaction.
Lowest energy spin state of pair: $\uparrow\downarrow$ with (j, m) and $(j, -m)$. Total $J = 0$.
Need antisymmetric $\psi_{\text{total}} = \psi_{\text{spin}}\psi_{\text{spatial}}$: ψ_{spin} antisymmetric, thus ψ_{spatial} is symmetric.
This maximises the overlap of their wavefunctions, increasing the binding energy (attractive force). The pairing energy increases with increasing L of nucleons.

Example: $^{207}_{82}$**Pb** naively expect odd neutron in $2f_{5/2}$ subshell.
But, pairing interaction means it is energetically favourable for the $2f_{5/2}$ neutron and a neutron from nearby $3p_{1/2}$ to pair and leave hole in $3p_{1/2}$. $\Rightarrow J^P = 1/2^-$ (observed)

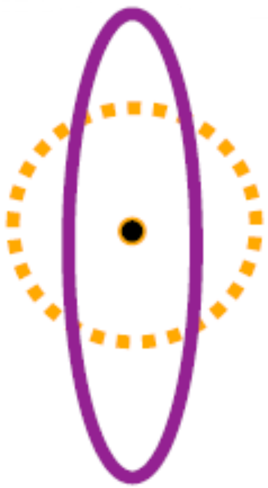
Nuclear Shell Model *Spin and Parity*

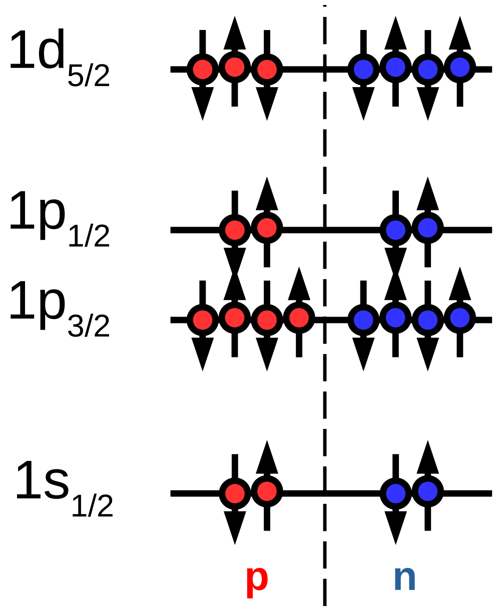
The Nuclear Shell Model predicts the spin & parity of ground state nuclei.

Case 2: Away from closed shells

More than one nucleon can contribute and electric quadrupole moment Q is often large
$\Rightarrow V(r)$ no longer spherically symmetric.

Example: $^{23}_{11}$**Na** Q is observed to be large, i.e. non-spherical.
Three protons in $1d_{5/2}$; if two were paired up, we expect $J^P = 5/2^+$.





In fact, all three protons must contribute
$\Rightarrow$ can get $J^P = 3/2^+$
(observed)

Nuclear Shell Model *Magnetic Dipole Moments*

The Nuclear Shell Model predicts the magnetic dipole moments of ground state nuclei.

Even-even nuclei : $J = 0 \Rightarrow \mu = 0$

Odd A nuclei: μ corresponds to the unpaired nucleon or hole

For a single nucleon $\vec{\mu} = \frac{\mu_N}{\hbar}(g_L \vec{L} + g_s \vec{s})$ with $p: \quad g_L = 1, \quad g_s = +5.586,$

$n: \quad g_L = 0, \quad g_s = -3.826,$

where $\mu_N = \frac{e\hbar}{2m_p}$ is the Nuclear Magneton.

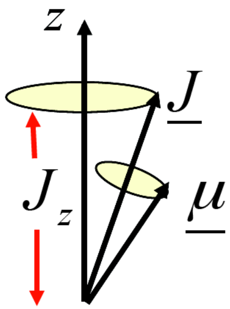
$\vec{\mu}$ is not parallel to $\vec{j}$ (since $\vec{j} = \vec{L} + \vec{s}$).

However, the *angle* between $\vec{\mu}$ and $\vec{j}$ is constant, because

$$\cos\theta \sim \vec{\mu}.\vec{j} \sim g_L \vec{L}.\vec{j} + g_s \vec{s}.\vec{j} = \frac{1}{2}\left[g_L(L^2 + j^2 - s^2) + g_s(s^2 + j^2 - L^2)\right]$$

and j^2, L^2 and s^2 are all constants of motion.

Hence, we can calculate the nuclear magnetic moment (projection of $\vec{\mu}$ along the z-axis)



$$\mu_z = \frac{\vec{\mu}.\vec{J}}{|\vec{J}|} \times \frac{J_z}{|\vec{J}|}$$

project $\vec{\mu}$ onto $\vec{J}$ then $\vec{J}$ onto $\vec{z}$

c.f. derivation of Landé g-factor in Quantum course

$$\therefore \mu_z = \mu_N \frac{m_J}{2j(j+1)} \left(g_L \left[L(L+1) + j(j+1) - s(s+1)\right] + g_s \left[s(s+1) + j(j+1) - L(L+1)\right] \right)$$

The Nuclear Shell Model predicts the magnetic dipole moments of ground state nuclei.

Even-even nuclei : $J = 0 \Rightarrow \mu = 0$
Odd A nuclei: μ corresponds to the unpaired nucleon or hole

Thus $\boldsymbol{\mu = g_J \mu_N J}$ for $m_J = J$ and

$$g_J = \frac{1}{2j(j+1)} \left(g_L \left[L(L+1) + j(j+1) - s(s+1) \right] + g_s \left[s(s+1) + j(j+1) - L(L+1) \right] \right)$$

For a single nucleon ($s = 1/2$), there are two possibilities ($j = L + 1/2$ or $L - 1/2$)

$$g_J = g_L \pm \frac{g_s - g_L}{2L+1} \qquad j = L \pm 1/2$$

Odd p: $g_L = 1$ $g_s = +5.586$
Odd n: $g_L = 0$ $g_s = -3.826$

called the **"Schmidt Limits"**.

Nuclear Shell Model *Magnetic Dipole Moments*

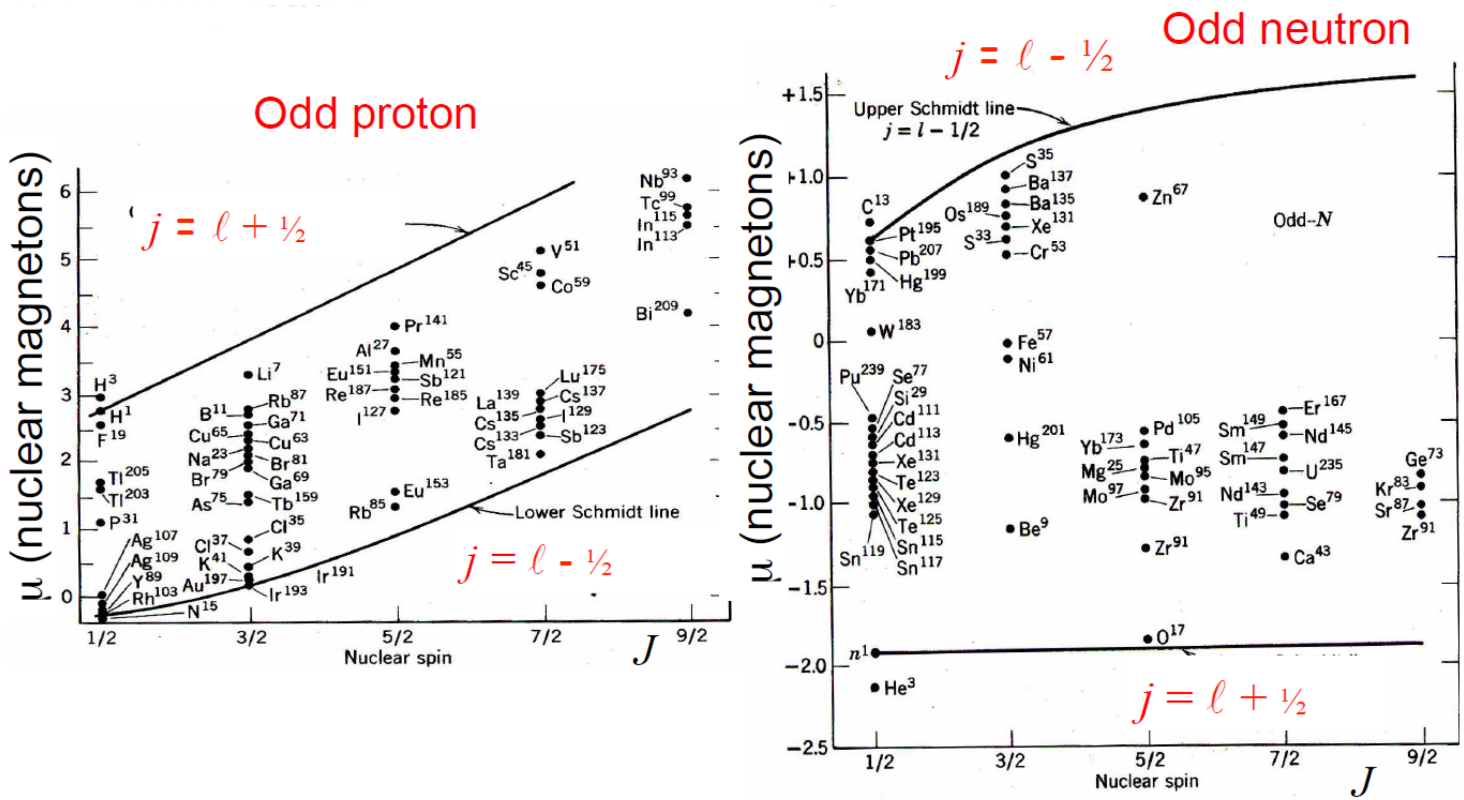
The Nuclear Shell Model predicts the magnetic dipole moments of ground state nuclei.

Even-even nuclei : $J = 0 \Rightarrow \mu = 0$

Odd A nuclei: μ corresponds to the unpaired nucleon or hole

Schmidt Limits compared to data: The Nuclear Shell Model predicts the broad trend of the magnetic moments. But not good in detail, except for closed shell ± 1 nucleon or so.

$\Rightarrow$ wavefunctions must be more complicated than our simple model.



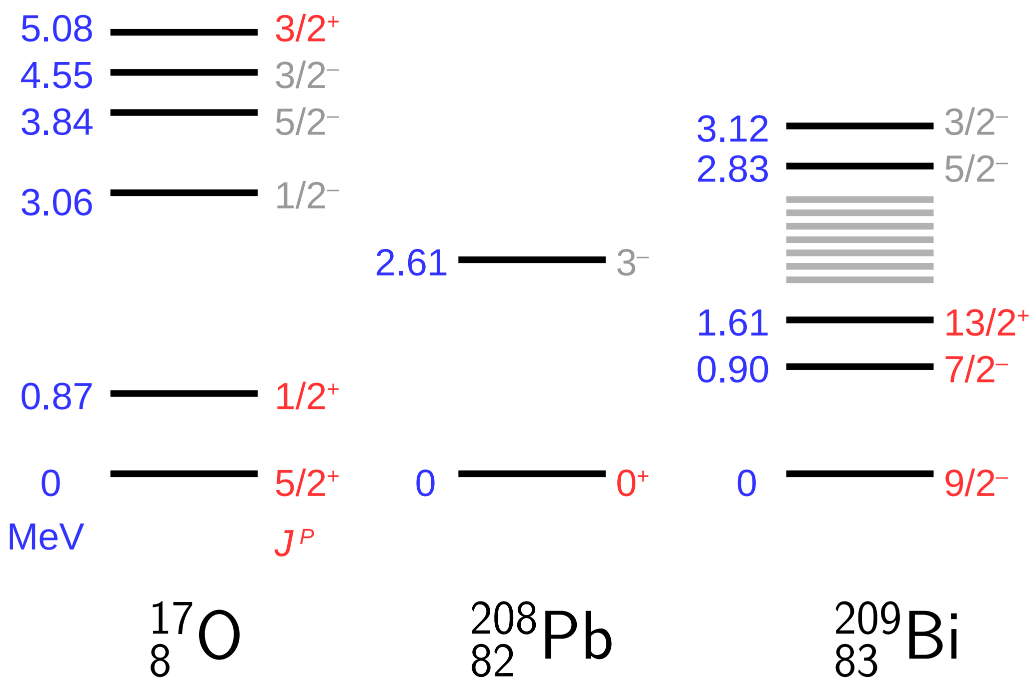
Excited States of Nuclei

In nuclear spectra, we can identify three kinds of excitations:

- Single nucleon excited states
- Vibrational excited states
- Rotational excited states

Single nucleon excited states may, to some extent, be predicted from the simple Shell Model. Most likely to be successful for lowest-lying excitations of odd *A* nuclei near closed shells.

e.g.



Excited States of Nuclei

Vibrational and **rotational** motion of nuclei involve the collective motion of the nucleons in the nucleus.

Collective motion can be incorporated into the shell model by replacing the static symmetrical potential with a potential that undergoes deformations in shape.

$\Rightarrow$ Collective vibrational and rotational models.

Here we will only consider even Z, even N nuclei

Ground state : $J^P = 0^+$

Lowest excited state (nearly always): $J^P = 2^+$

Tend to divide into two categories:

A	$E(2^+)$	Type
30–150	~ 1 MeV	Vibrational
150–190 (rare earth) $>$220 (actinides)	~ 0.1 MeV	Rotational

Nuclear Vibrations

Vibrational excited states occur when a nucleus oscillates about a spherical equilibrium shape (low energy surface vibrations, near closed shells). Form of the excitations can be represented by a multipole expansion (just like underlying nuclear shapes).

Monopole

Incorporated into the average radius

Dipole

Involves a net displacement of centre of mass ⇒ cannot result from action of nuclear forces (can be induced by applied e/m field i.e. a photon)

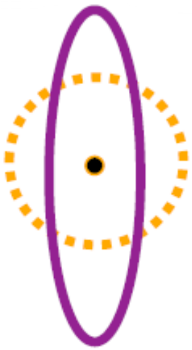
Quadrupole

Octupole

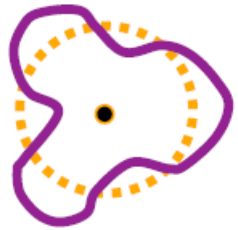
Quadrupole oscillations are the lowest order nuclear vibrational mode.

Similar to SHM – the quanta of vibrational energy are called phonons.

Nuclear Vibrations



A **quadrupole phonon** carries 2 units of angular momentum and has even parity $\Rightarrow J^P = 2^+$



An **octupole phonon** carries 3 units of angular momentum and has odd parity $\Rightarrow J^P = 3^-$

Phonons are bosons and must satisfy Bose-Einstein statistics (overall symmetric wavefunction under the interchange of two phonons).

e.g. for quadrupole phonons:

Even-even ground state $\quad 0^+ \xrightarrow{\text{1 phonon}} 2^+$

$$\xrightarrow{\text{2 phonons}} 0^+,\ 2^+,\ 4^+$$

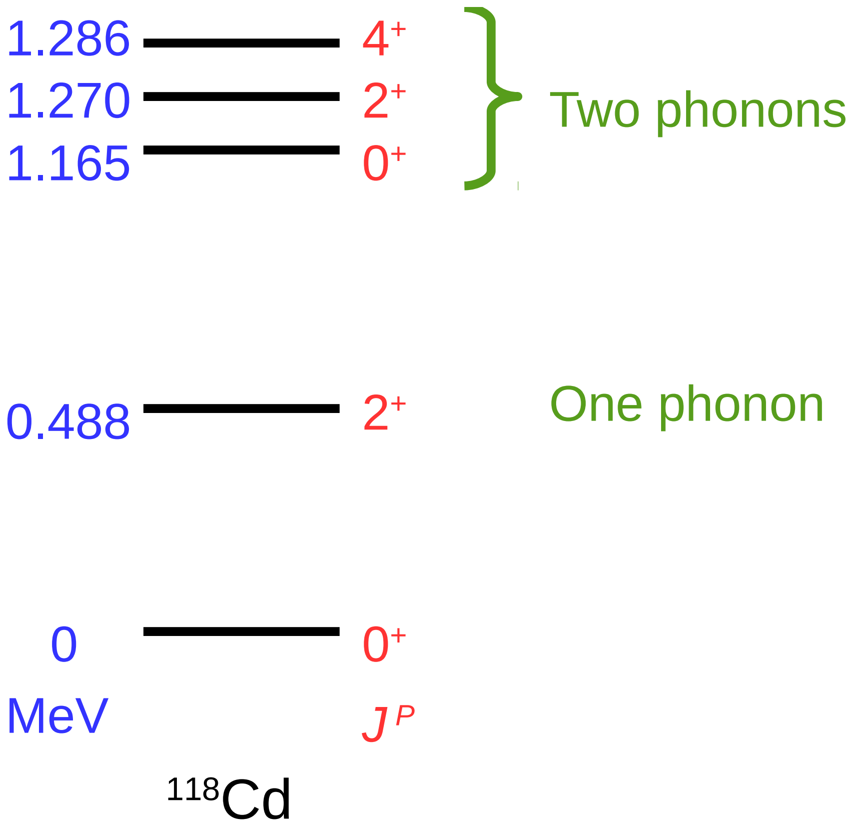
(in practice not degenerate)

Energies of vibrational excitations are not predicted, but we can predict the ratios

$$\frac{\text{Second excited (2 phonons; } 0^+,\ 2^+,\ 4^+)}{\text{First excited (1 phonon; } 2^+)} \sim 2$$

Nuclear Vibrations

Example of vibrational excitations:



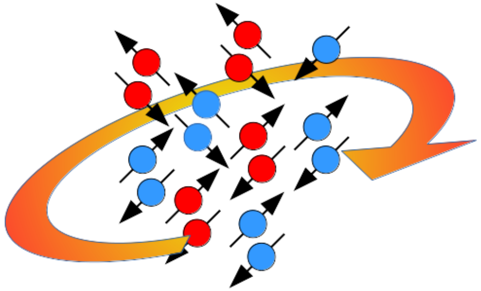
$$\text{Predict} \quad \frac{\text{2nd excited}}{\text{1st excited}} \sim 2$$

$$\text{Observe} \quad \frac{\text{2nd excited}}{\text{1st excited}} \sim 2.4$$

Octupole states ($J^P = 3^-$) are often seen near the triplet of two-phonon quadrupole states.

Vibrational states decay rapidly by γ emission (see later).

Nuclear Rotations



Collective rotational motion can only be observed in nuclei with non-spherical equilibrium shapes (i.e. far from closed shells, large Q).

Rotating deformed nucleus: nucleons in rapid internal motion in the nuclear potential + entire nucleus rotating slowly. Slow to maintain a stable equilibrium shape and not to affect the nuclear structure.

Nuclear mirror symmetry restricts the sequence of rotational states to even values of angular momentum.

Even-even ground state $0^+ \rightarrow 2^+, 4^+, 6^+$

... (total angular momentum = nuclear a.m. + rotational a.m.)

Energy of a rotating nucleus

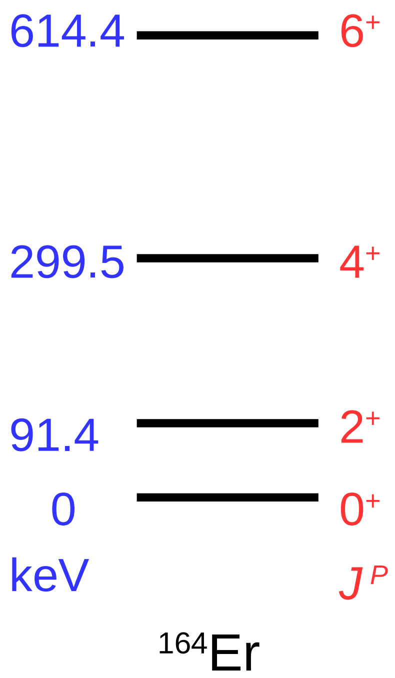
$$E = \frac{\hbar^2}{2I_{\mathrm{eff}}} J(J+1)$$

where I_{eff} is the effective moment of inertia.

Nuclear Rotations

Energies of rotational excitations are not predicted, but we can predict the ratios

e.g.



Predict $\dfrac{E(4^+)}{E(2^+)} = \dfrac{4(4+1)}{2(2+1)} = 3.33$

Observe $\dfrac{E(4^+)}{E(2^+)} = \dfrac{299.5}{91.4} = 3.28$

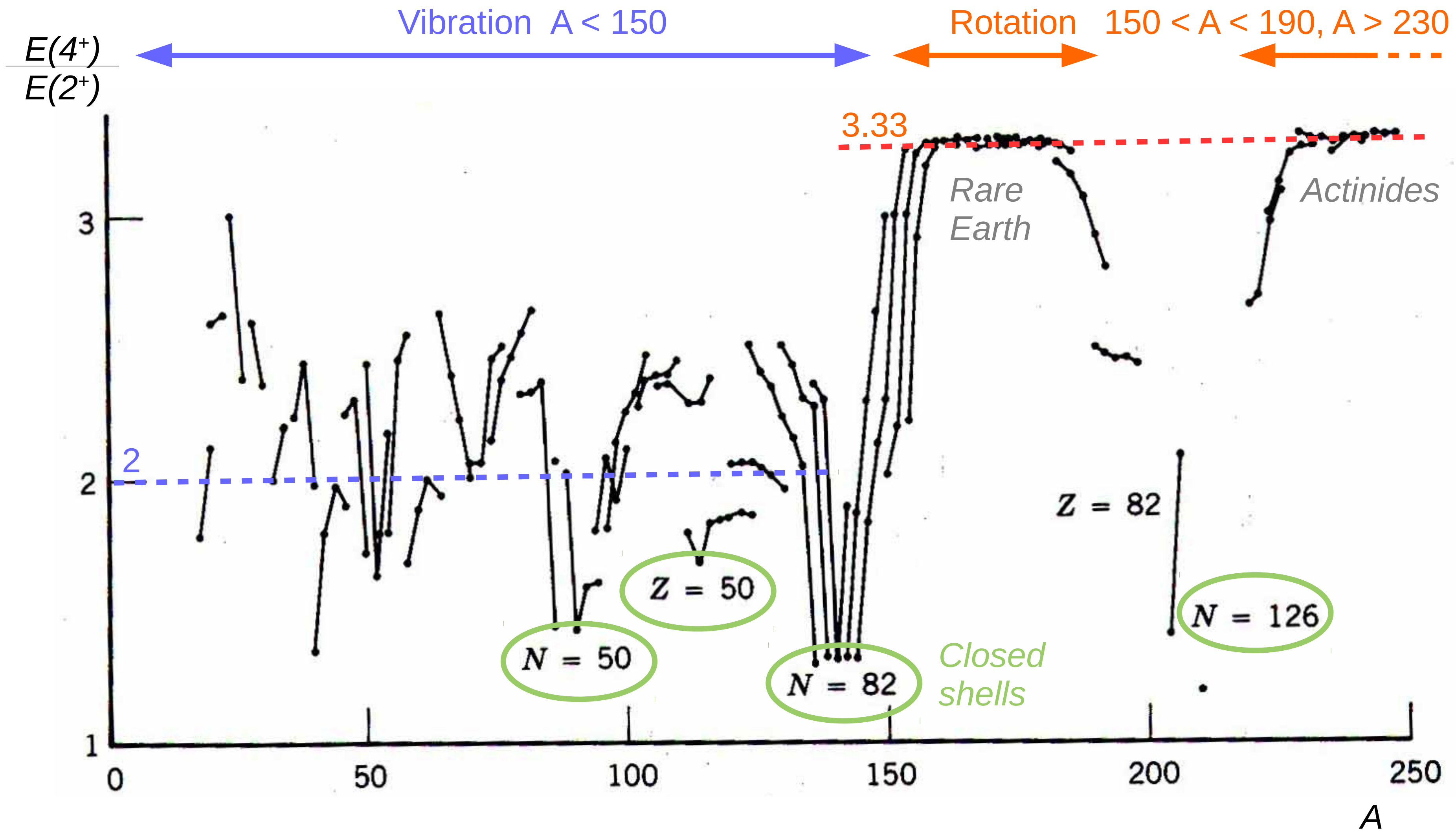
Deduce I_{eff} from the absolute energies; it is found that $I_{\text{rigid}} > I_{\text{eff}} > I_{\text{fluid}}$

$\rightarrow$ the nucleus does not rotate like a rigid body. Only some of its nucleons are in collective motion (presumably the outer ones).

Rotational behaviour is intermediate between the nucleus being tightly bonded and weakly bonded i.e. the strong force is not long range.

Nuclear Vibrations and Rotations

For even-even ground state nuclei, the ratio of excitation energies $\frac{E(4^+)}{E(2^+)}$ is a diagnostic of the type of excitation.



Summary

The Nuclear Shell Model is successful in predicting

- Origin of magic numbers
- Spins and parities of ground states
- Trend in magnetic moments
- Some excited states near closed shells, small excitations in odd A nuclei

In general, it is not good far from closed shells and for non-spherically symmetric potentials.

The collective properties of nuclei can be incorporated into the Nuclear Shell Model by replacing the spherically symmetric potential by a deformed potential.
Improved description for

- Even A excited states
- Electric quadrupole and magnetic dipole moments.

Many more sophisticated models exist (see Cont. Physics 1994 vol. 35 No. 5 329 http://www.tandfonline.com/doi/pdf/10.1080/00107519408222099)

Problem Sheet: q.34-36

Up next... Section 15: Nuclear Decays