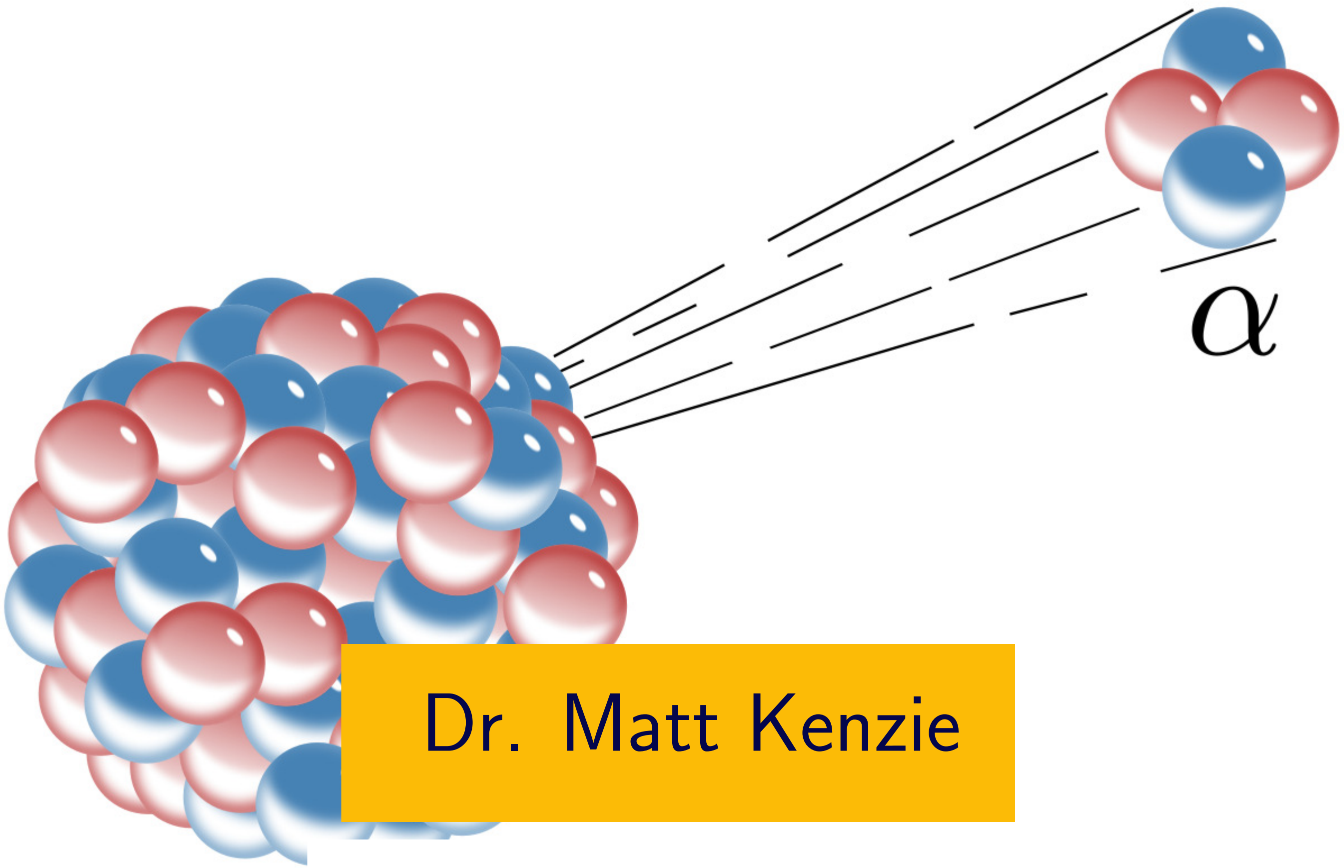
15. Nuclear Decay

Particle and Nuclear Physics



Dr. Matt Kenzie

UNIVERSITY OF CAMBRIDGE

In this section...

- Radioactive decays
- Radioactive dating
- α decay
- β decay
- γ decay

Radioactivity

Natural radioactivity: three main types α, β, γ, and in a few cases, spontaneous fission.

α decay $\quad {}^4_2\text{He}$ nucleus emitted.

$${}^A_Z\text{X} \rightarrow {}^{A-4}_{Z-2}\text{Y} + {}^4_2\text{He} \qquad \text{Occurs for } A \geq 210$$

For decay to occur, energy must be released $Q > 0$

$$Q = m_{\text{X}} - m_{\text{Y}} - m_{\text{He}} = B_{\text{Y}} + B_{\text{He}} - B_{\text{X}}$$

β decay $\quad$ emission of electron e^- or positron e^+

$$n \rightarrow p + e^- + \bar{\nu}_e \qquad {}^A_Z\text{X} \rightarrow {}^{A}_{Z+1}\text{Y} + e^- + \bar{\nu}_e \qquad \beta^- \text{ decay}$$

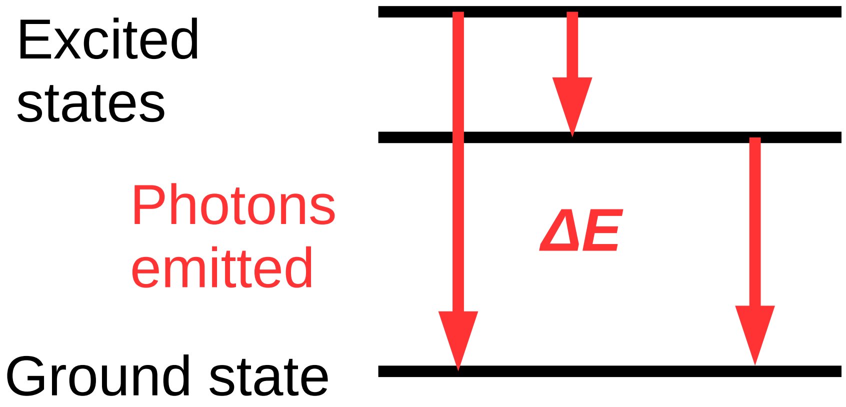
$$p \rightarrow n + e^+ + \nu_e \qquad {}^A_Z\text{X} \rightarrow {}^{A}_{Z-1}\text{Y} + e^+ + \nu_e \qquad \beta^+ \text{ decay}$$

$$p + e^- \rightarrow n + \nu_e \qquad {}^A_Z\text{X} + e^- \rightarrow {}^{A}_{Z-1}\text{Y} + \nu_e \qquad \text{Electron capture}$$

n.b. of these processes, only $n \rightarrow pe\nu$ can occur outside a nucleus.

Radioactivity

γ decay Nuclei in excited states can decay by emission of a photon γ. Often follows α or β decay.



	ΔE	λ	
Atom	~ 10 eV	$\sim 10^{-7}$ m	optical
	~ 10 keV	$\sim 10^{-10}$ m	X-ray
Nucleus	$\sim$ MeV	$\sim 10^{-12}$ m	γ-ray

A variant of γ decay is Internal Conversion:

- an excited nucleus loses energy by emitting a virtual photon,
- the photon is absorbed by an atomic e^-, which is then ejected
- n.b. not β decay, as nucleus composition is unchanged (e^- not from nucleus)

Natural Radioactivity

The half-life, $\tau_{1/2}$, is the time over which 50% of the nuclei decay

$$\tau_{1/2} = \frac{\ln 2}{\lambda} = 0.693\tau$$

λ Transition rate

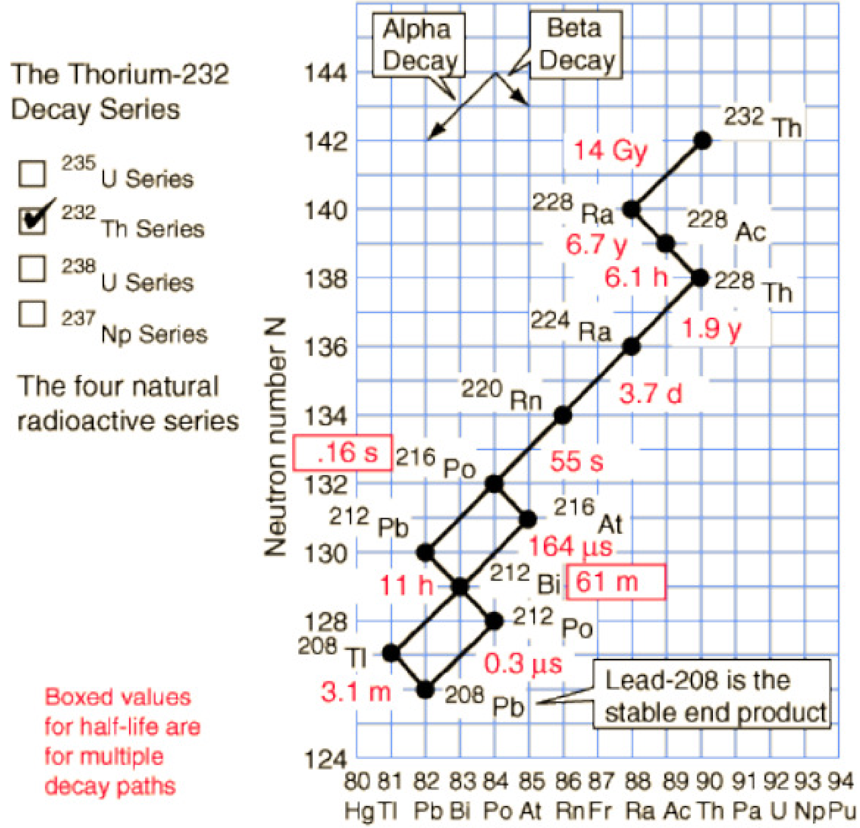
τ Average lifetime

Some $\tau_{1/2}$ values may be long compared to the age of the Earth.

Series Name	Type	Final Nucleus (stable)	Longest-lived Nucleus	$\tau_{1/2}$ (years)
Thorium	4n	^{208}Pb	^{232}Th	1.41×10^{10}
Neptunium	4n+1	^{209}Bi	^{237}Np	2.14×10^{6}
Uranium	4n+2	^{206}Pb	^{238}U	4.47×10^{9}
Actinium	4n+3	^{207}Pb	^{235}U	7.04×10^{8}

n is an integer

4n series



Radioactive Dating *Geological Dating*

Can use β^- decay to age the Earth, $\underset{N_1}{^{87}\text{Rb}} \rightarrow \underset{N_2}{^{87}\text{Sr}}$ $(\tau_{1/2} = 4.8 \times 10^{10}$ years)

^{87}Sr is stable $\rightarrow \lambda_2 = 0$

So in this case, we have (using expressions from Chapter 2)

$$N_2(t) \; = \; N_1(0)\left[1 - e^{-\lambda_1 t}\right] + N_2(0) \quad = N_1(t)\left[e^{\lambda_1 t} - 1\right] + N_2(0)$$

Assume we know λ_1, and can measure $N_1(t)$ and $N_2(t)$ e.g. chemically.
But we don't know $N_2(0)$.

Solution is to normalise to another (stable) isotope – ^{86}Sr – for which number is $N_0(t) = N_0(0)$.

$$\frac{N_2(t)}{N_0} = \frac{N_1(t)}{N_0}\left[e^{\lambda_1 t} - 1\right] + \frac{N_2(0)}{N_0}$$

Method: plot $N_2(t)/N_0$ vs $N_1(t)/N_0$ for lots of minerals.
Gradient gives $\left[e^{\lambda_1 t} - 1\right]$ and hence t.
Intercept $= N_2(0)/N_0$, which should be the same for all minerals (determined by chemistry of formation).

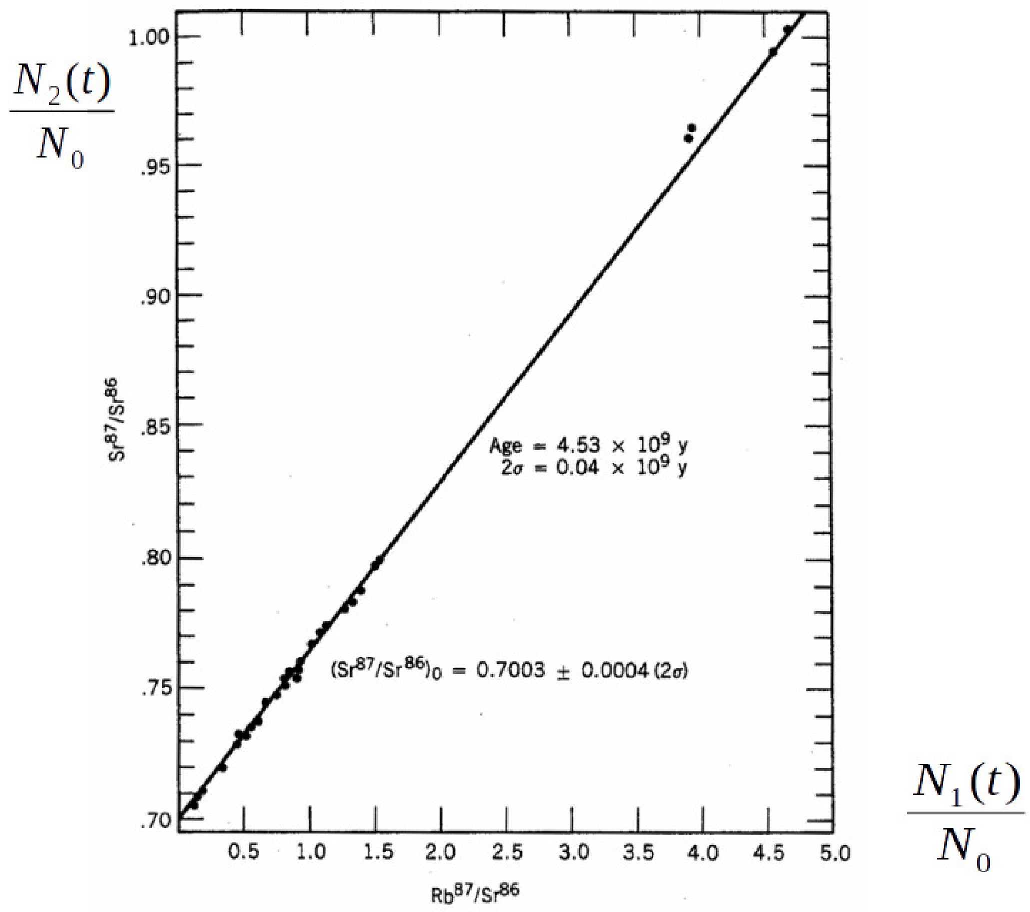
$$\frac{N_2(t)}{N_0} = \frac{N_1(t)}{N_0}\left[e^{\lambda_1 t} - 1\right] + \frac{N_2(0)}{N_0}$$

Method: plot $N_2(t)/N_0$ vs $N_1(t)/N_0$ for lots of minerals.
Gradient gives $\left[e^{\lambda_1 t} - 1\right]$ and hence t.
Intercept $= N_2(0)/N_0$, which should be the same for all minerals (determined by chemistry of formation).

Using minerals from the Earth, Moon and meteorites.

Intercept gives $N_2(0)/N_0 = 0.70$

Slope gives the age of the Earth $= 4.5 \times 10^9$ yrs



For recent organic matter, use ^{14}C dating

Continuously formed in the upper atmosphere at approx. constant rate.
$^{14}N + n \rightarrow {}^{14}C + p$

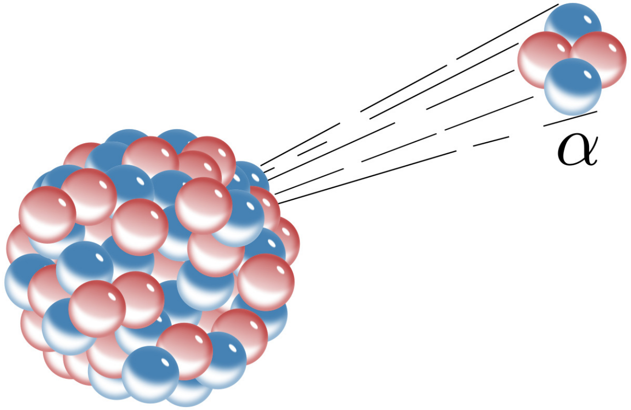
Atmospheric carbon continuously exchanged with living organisms.
Equilibrium: 1 atom of ^{14}C to every 10^{12} atoms of other carbon isotopes
(98.9% ^{12}C, 1.1% ^{13}C)

Undergoes β^- decay
$^{14}C \rightarrow {}^{14}N + e^- + \bar{\nu}_e$ $\tau_{1/2} = 5730$ yrs

No more ^{14}C intake for dead organisms.
Fresh organic material
~11 decays/minute/gram of carbon.

Measure the **specific activity** of material to obtain age, i.e. number of decays per second per unit mass

Complications for the future!
Burning of fossil fuels increases ^{12}C in atmosphere,
Nuclear bomb testing (adds ^{14}C to atmosphere)



- α decay is due to the emission of a ${}^4_2$He nucleus.
- ${}^4_2$He is doubly magic and very tightly bound.
- α decay is energetically favourable for almost all with A$\geq$190 and for many A$\geq$150.

Why α rather than any other nucleus?

Consider energy release (Q) in various possible decays of ^{232}U

	n	p	^{2}H	^{3}H	^{3}He	^{4}He	^{5}He	^{6}Li	^{7}Li
Q/MeV	-7.26	-6.12	-10.70	-10.24	-9.92	+5.41	-2.59	-3.79	-1.94

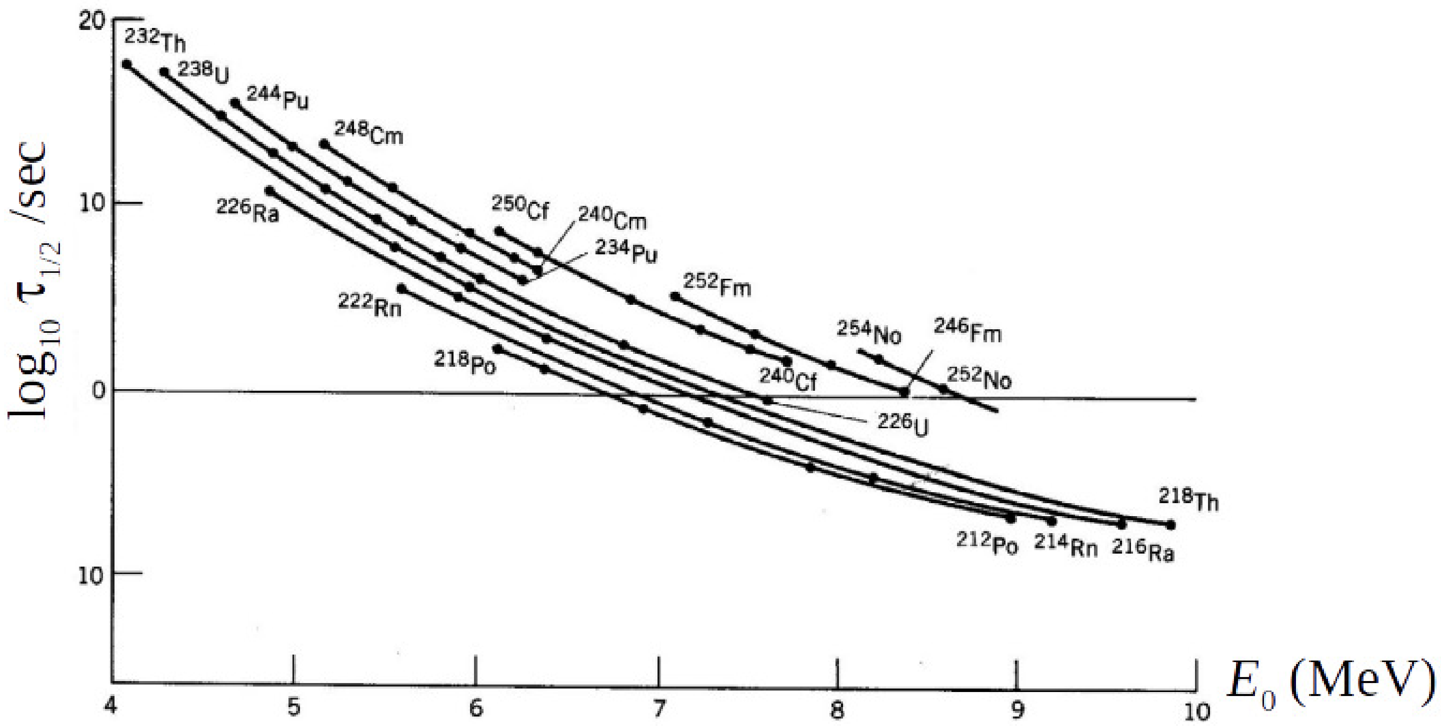
α is easy to form inside a nucleus $2p\uparrow\downarrow + 2n\uparrow\downarrow$

(though the extent to which α particles really exist inside a nucleus is still debatable)

(Geiger and Nuttall 1911)

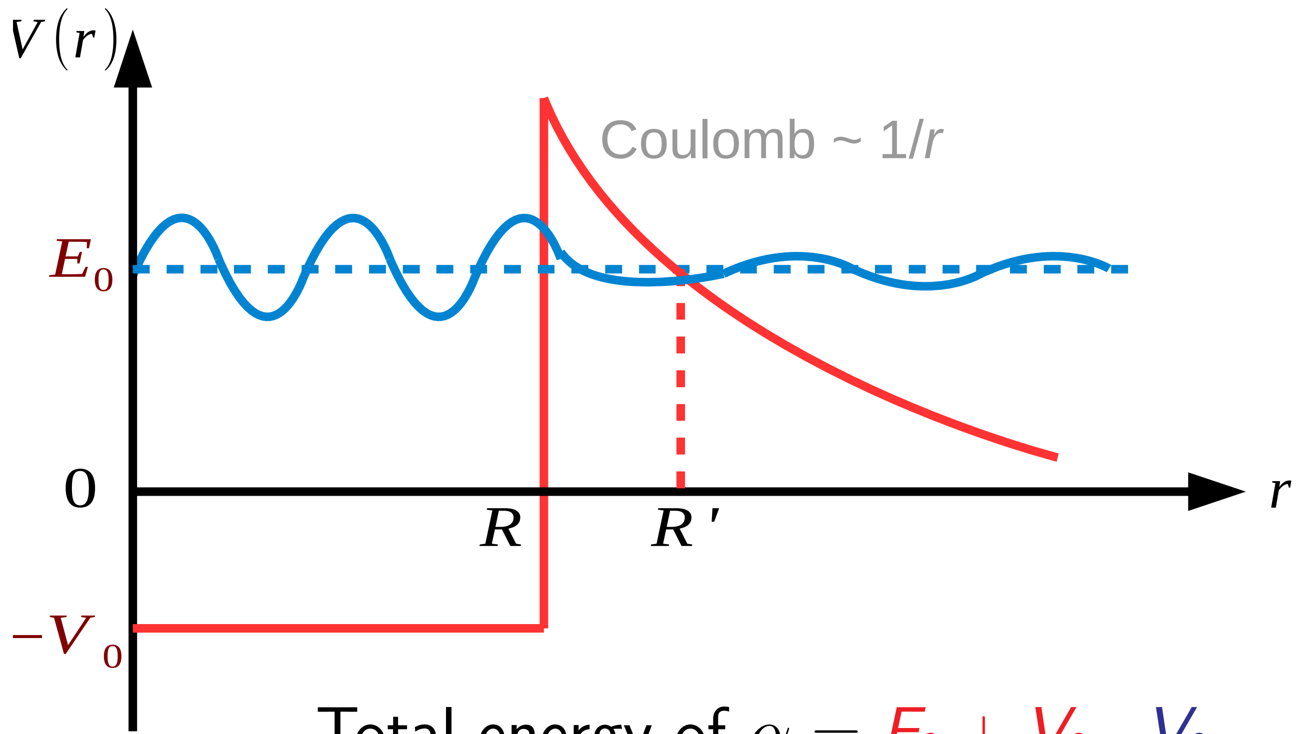
A very striking feature of α decay is the strong dependence of lifetime on E_0

Example ^{232}Th $E_0 = 4.08\ \mathrm{MeV}$ $\tau_{1/2} = 1.4 \times 10^{10}$ yrs
^{218}Th $E_0 = 9.85\ \mathrm{MeV}$ $\tau_{1/2} = 1.0 \times 10^{-7}$ s
A factor of $\sim$2.5 in E_0 $\Rightarrow$ factor 10^{24} in $\tau_{1/2}$!



e.g. even N, even Z nuclei for a given Z see smooth trend ($\tau_{1/2}$ increases as Z does)

The nuclear potential for the α particle due to the daughter nucleus includes a Coulomb barrier which inhibits the decay.



Total energy of $\alpha = \underbrace{E_0 + V_0}_{\text{K.E.}} \underbrace{- V_0}_{\text{P.E.}}$

Classically, α particle cannot enter or escape from nucleus.
Quantum mechanically, α particle can penetrate the Coulomb barrier

$\Rightarrow$ Quantum Mechanical Tunnelling

α Decay *Simple Theory* *(Gamow, Gurney, Condon 1928)*

Assume α exists inside the nucleus and hits the barrier.

$$\alpha \text{ decay rate}, \quad \lambda = f\,P$$

f = escape trial frequency, P = probability of tunnelling through barrier

$$\text{semi} - \text{classically}, \quad f \sim v/2R$$

v= velocity of a particle inside nucleus, given by: $v^2 = (2E_\alpha/m_\alpha)$
and R = radius of nucleus

Typical values: $V_0 \sim 35$ MeV, $E_0 \sim 5$ MeV $\Rightarrow E_\alpha = 40$ MeV inside nucleus

$$f \sim \frac{v}{2R} = \frac{1}{2R}\sqrt{\frac{2E_\alpha}{m_\alpha}} \sim 10^{22}\,\mathrm{s}^{-1}$$

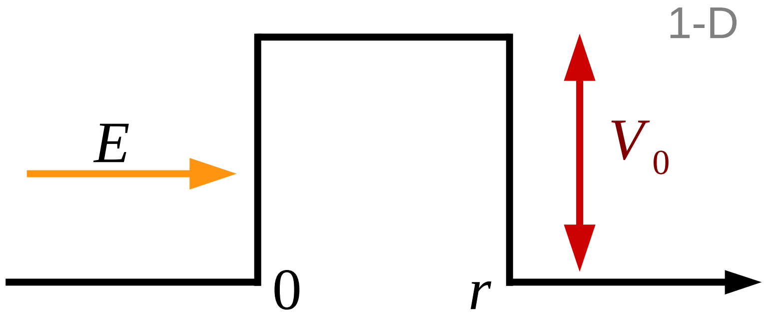
$m_\alpha = 3.7$ GeV
$R \sim 2.1$ fm

Obtain tunnelling probability, P, by solving Schrödinger equation in three regions and using boundary conditions.

Transmission probability (1D square barrier):

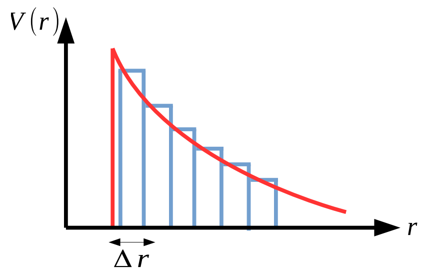
$$P = \left[1 + \frac{V_0^2}{4(V_0 - E)E}\sinh^2 ka\right]^{-1}$$

$$\frac{\hbar^2 k^2}{2m} = V_0 - E \qquad m = \text{reduced mass}$$



For $ka \gg 1$, P is dominated by the exp. decay within barrier $\Rightarrow P \sim \mathrm{e}^{-2ka}$.

Coulomb potential, $V \propto 1/r$, and thus k varies with r.
Divide into rectangular pieces and multiply together exponentials, i.e. sum exponents.



Probability to tunnel through Coulomb barrier

$$P = \prod_i \mathrm{e}^{-2k_i \Delta R} = \mathrm{e}^{-2G} \qquad k = \frac{[2m_\alpha(V(r) - E_0)]^{1/2}}{\hbar}$$

The Gamow Factor $$G = \int_R^{R'} \frac{[2m_\alpha(V(r) - E_0)]^{1/2}}{\hbar}\,\mathrm{d}r = \int_R^{R'} k(r)\,\mathrm{d}r$$

α Decay *Simple Theory* *(Gamow, Gurney, Condon 1928)*

For $r > R$, $V(r) = \dfrac{Z_\alpha Z' e^2}{4\pi\epsilon_0 r} = \dfrac{B}{r}$ $\quad Z' = Z - Z_\alpha \quad (Z_\alpha = 2)$

α-particle escapes at $r = R'$, $V(R') = E_0 \Rightarrow R' = B/E_0$

$$\therefore G = \int_R^{R'} \left(\frac{2m_\alpha}{\hbar^2}\right)^{1/2} \left[\frac{B}{r} - E_0\right]^{1/2} \mathrm{d}r = \left(\frac{2m_\alpha B}{\hbar^2}\right)^{1/2} \int_R^{R'} \left[\frac{1}{r} - \frac{1}{R'}\right]^{1/2} \mathrm{d}r$$

See Appendix H

$$G = \left(\frac{2m_\alpha}{E_0}\right)^{1/2} \frac{B}{\hbar} \left[\cos^{-1}\left(\frac{R}{R'}\right)^{1/2} - \left\{\left(1 - \frac{R}{R'}\right)\left(\frac{R}{R'}\right)\right\}^{1/2}\right]$$

To perform integration, substitute $r = R' \cos^2\theta$

In most practical cases $R \ll R'$, so term in [...] $\sim \pi/2$

$$G \sim \left(\frac{2m_\alpha}{E_0}\right)^{1/2} \frac{B}{\hbar}\frac{\pi}{2} \qquad B = \frac{Z_\alpha Z' e^2}{4\pi\epsilon_0}$$

e.g. typical values: $Z = 90$, $E_0 \sim 6\ \mathrm{MeV} \Rightarrow R' \sim 40$ fm $\gg R$

$$G \sim Z' \left(\frac{3.9\ \mathrm{MeV}}{E_0}\right)^{1/2}$$

α Decay *Simple Theory* *(Gamow, Gurney, Condon 1928)*

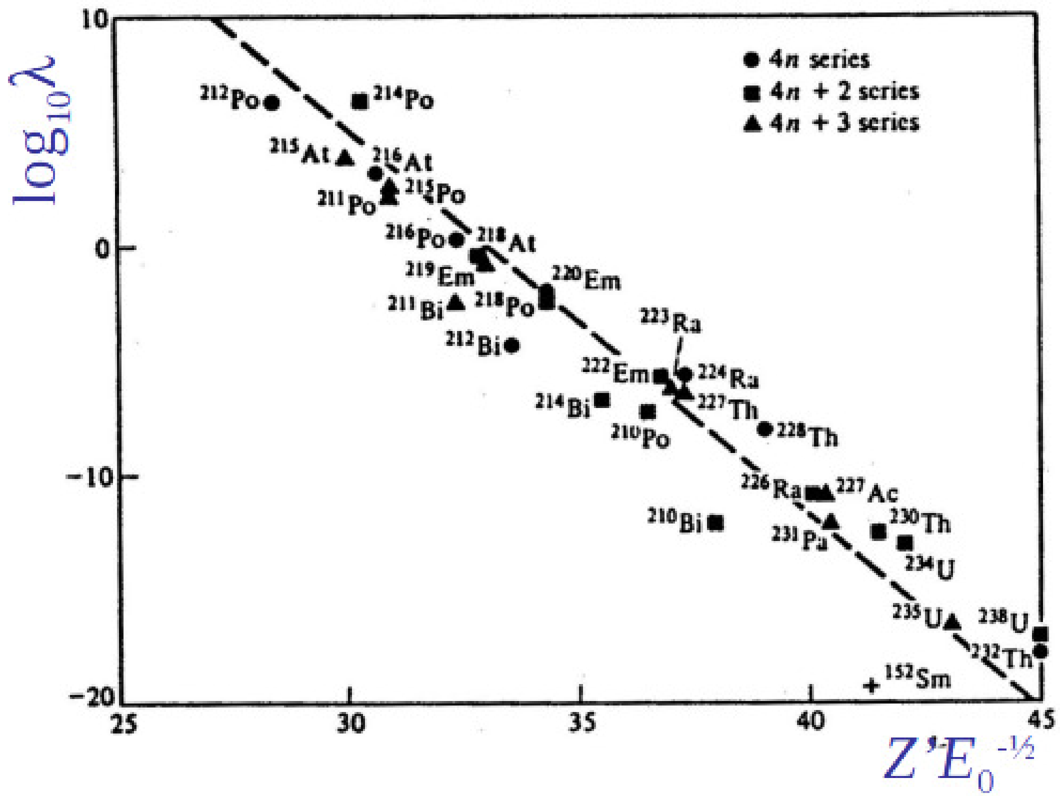
Lifetime $\tau = \dfrac{1}{\lambda} = \dfrac{1}{fP} \sim \dfrac{2R}{v} \mathrm{e}^{2G}$

$$\Rightarrow \quad \ln \tau \sim 2G + \ln \frac{2R}{v}$$

$$\ln \lambda \sim -\frac{Z'}{E_0^{1/2}} + \text{constant}$$

Geiger-Nuttall Law

Not perfect, but provides an explanation of the dominant trend of the data



Simple tunnelling model accounts for

- strong dependence of $\tau_{1/2}$ on E_0
- $\tau_{1/2}$ increases with Z
- disfavoured decay to heavier fragments e.g. ^{12}C

$$G \propto m^{1/2} \quad \text{and} \quad G \propto \text{ charge of fragment}$$

Deficiencies/complications with simple tunnelling model:

- Assumed existence of a single α particle in nucleus and have taken no account of probability of formation.
- Assumed "semi-classical" approach to estimate escape trial frequency, $f \sim v/2R$, and make absolute prediction of decay rate.
- If α is emitted with some angular momentum, L, the radial wave equation must include a centrifugal barrier term in Schrödinger equation

$$V' = \frac{L(L+1)\hbar^2}{2\mu r^2}$$

$L =$ relative a.m. of α and daughter nucleus

$\mu =$ reduced mass

which raises the barrier and suppresses emission of α in in high L states.

α Decay *Selection rules*

Nuclear Shell Model: α has $J^P = 0^+$

Angular momentum

e.g. $X \to Y + \alpha$

Conserve J: $J_X = J_Y \oplus J_\alpha = J_Y \oplus L_\alpha$

L_α can take values from $J_X + J_Y$ to $|J_X - J_Y|$

Parity

Parity is conserved in α decay (strong force).

Orbital wavefunction has $P = (-1)^L$

X, Y same parity $\Rightarrow L_\alpha$ must be even

X, Y opposite parity $\Rightarrow L_\alpha$ must be odd

e.g. if X, Y are both even-even nuclei in their ground states, shell model predicts both have $J^P = 0^+ \;\Rightarrow\; L_\alpha = 0$.

More generally, if X has $J^P = 0^+$, the states of Y which can be formed in α decay are $J^P = 0^+, 1^-, 2^+, 3^-, 4^+$, ...

β Decay

β^-	$n \to p + e^- + \bar{\nu}_e$	${}^A_Z\mathrm{X} \to {}^{A}_{Z+1}\mathrm{Y} + e^- + \bar{\nu}_e$
β^+	$p \to n + e^+ + \nu_e$	${}^A_Z\mathrm{X} \to {}^{A}_{Z-1}\mathrm{Y} + e^+ + \nu_e$
electron capture	$p + e^- \to n + \nu_e$	${}^A_Z\mathrm{X} + e^- \to {}^{A}_{Z-1}\mathrm{Y} + \nu_e$

- β decay is a weak interaction mediated by the W boson.
- Parity is violated in β decay.
- Responsible for Fermi postulating the existence of the neutrino.
- Kinematics: Decay is possible if energy release $E_0 > 0$

	Nuclear Masses	*Atomic Masses*
β^-	$E_0 = m_X - m_Y - m_e - m_\nu$	$E_0 = M_X - M_Y - m_\nu$
β^+	$E_0 = m_X - m_Y - m_e - m_\nu$	$E_0 = M_X - M_Y - 2m_e - m_\nu$
e.c.	$E_0 = m_X - m_Y + m_e - m_\nu$	$E_0 = M_X - M_Y - m_\nu$
	(and note that $m_\nu \sim 0$)	using $M(A,Z) = m(A,Z) + Zm_e$

n.b. electron capture may be possible even if β^+ not allowed

β Decay *Nuclear stability against β decay*

Consider nuclear mass as a function of N and Z

$$m(A,Z) = Zm_p + (A-Z)m_n - a_V A + a_S A^{2/3} + \frac{a_C Z^2}{A^{1/3}} + a_A \frac{(N-Z)^2}{A} - \delta(A)$$

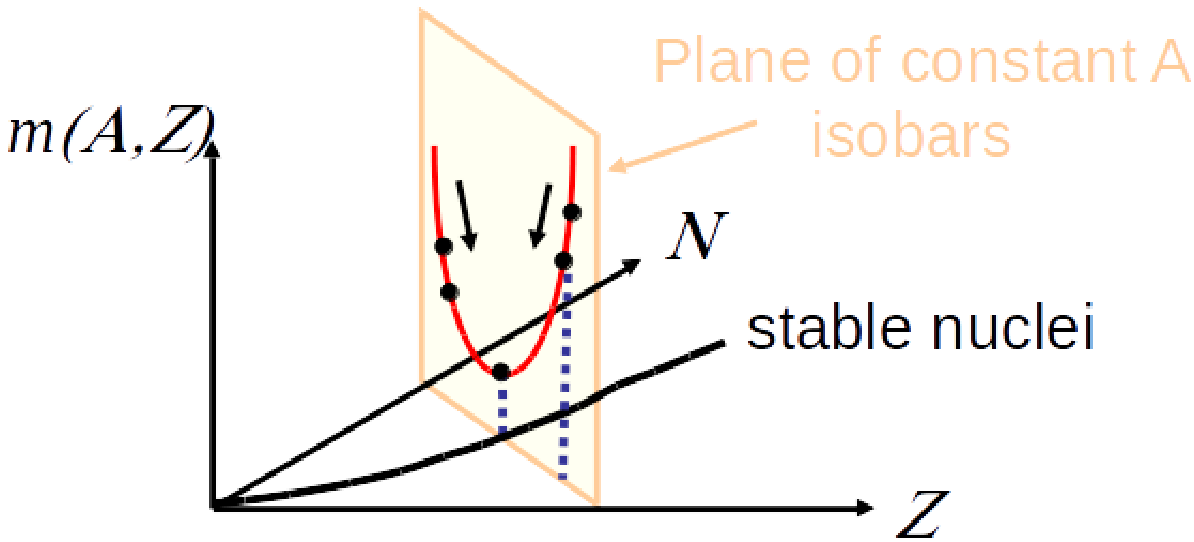
using SEMF

For β decay, A is constant,

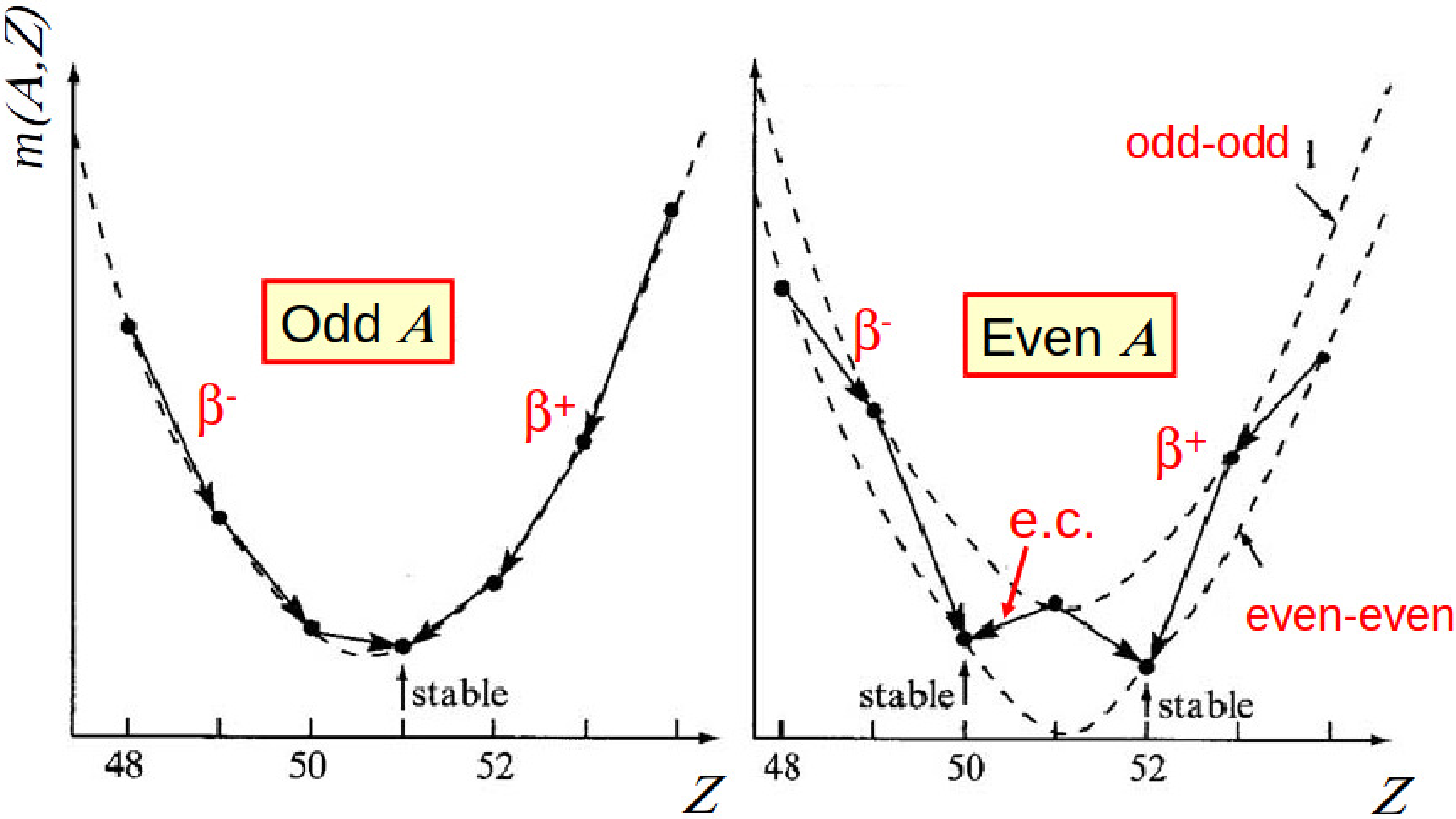
but Z changes by ± 1 and $m(A,Z)$ is quadratic in Z

Most stable nuclide when

$$\left[\frac{\partial m(A,Z)}{\partial Z}\right]_A = 0$$



β Decay *Typical situation at constant A*



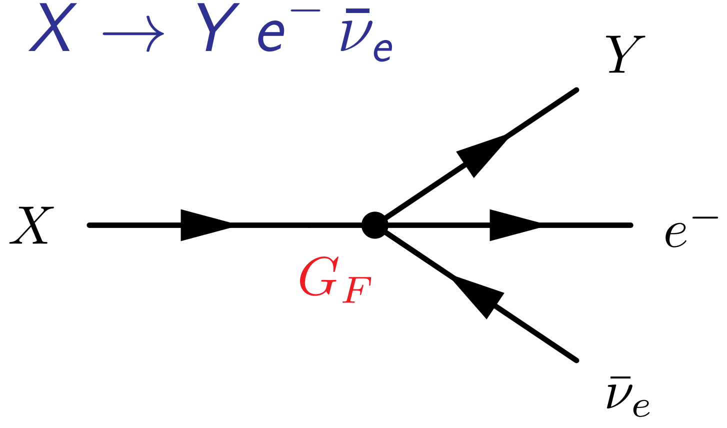
Usually only one isotope table against β-decay; occasionally two.

Typically two even-even nuclides are stable against β-decay; almost no odd-odd ones (pairing term).

Fermi Theory of β-decay

In nuclear decay, weak interaction taken to be a 4-fermion contact interaction:

$X \rightarrow Y\, e^-\, \bar{\nu}_e$



No "propagator" – absorb the effect of the exchanged W boson into an effective coupling strength given by the Fermi constant $G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$.

Use Fermi's Golden Rule to get the transition rate $\quad \Gamma = 2\pi |M_{\mathrm{fi}}|^2 \rho(E_{\mathrm{f}})$

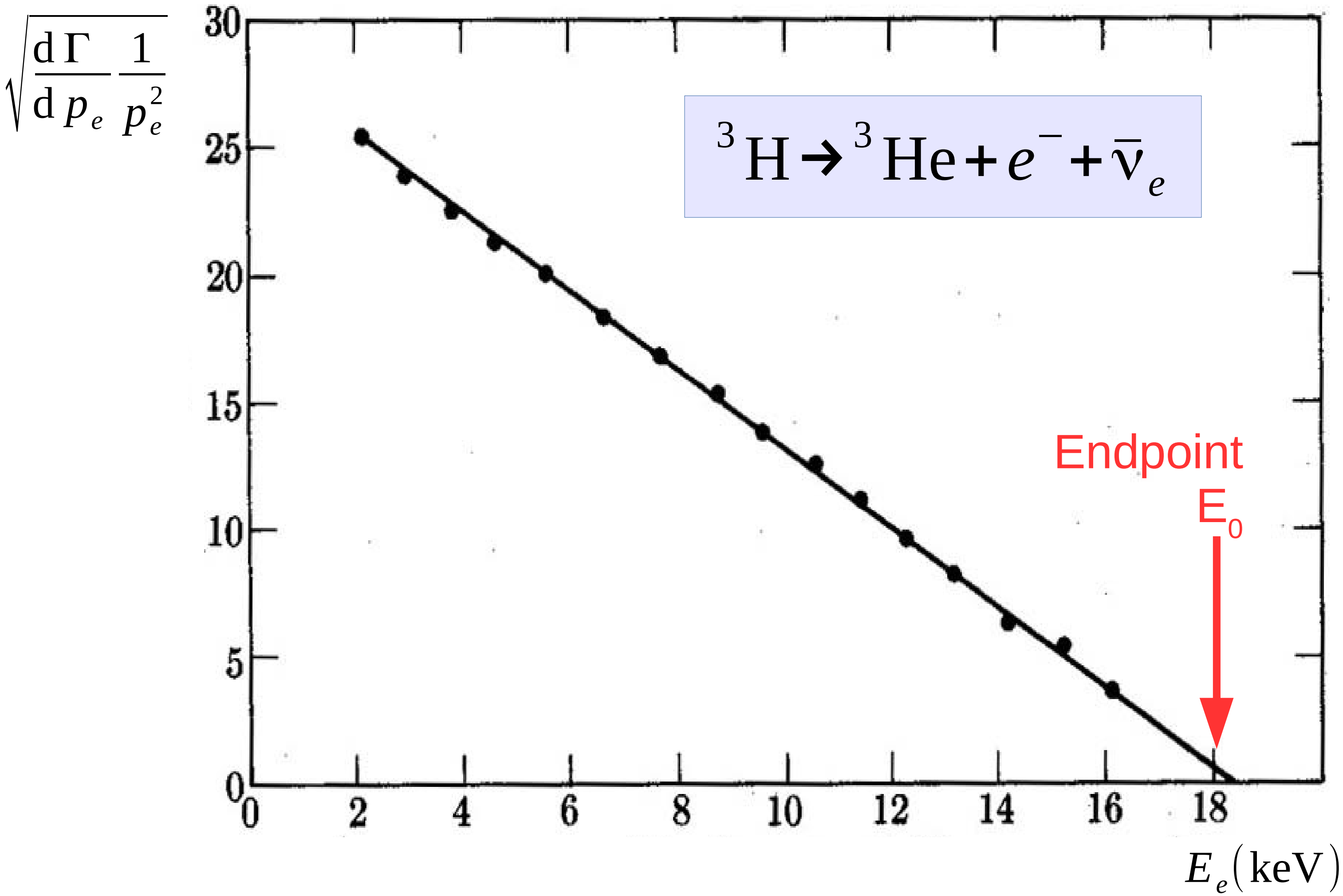
where M_{fi} is the matrix element and $\rho(E_{\mathrm{f}}) = \frac{\mathrm{d}N}{\mathrm{d}E_{\mathrm{f}}}$ is the density of final states.

$$\Gamma = \frac{G_F^2\, |M_{\mathrm{nuclear}}|^2}{2\pi^3} \int_0^{E_0} (E_0 - E_e)^2 E_e^2 \,\mathrm{d}E_e$$

Total decay rate given by Sargent's Rule, $\Gamma \propto E_0^5$

Fermi Theory of β-decay

β decay spectrum described by $$\sqrt{\frac{\mathrm{d}\Gamma}{\mathrm{d}p_e}\frac{1}{p_e^2}} \propto (E_0 - E_e)$$ Kurie Plot



Fermi Theory of β-decay

BUT, the momentum of the electron is modified by the Coulomb interaction as it moves away from the nucleus (different for e^- and e^+).

$\Rightarrow$ Multiply spectrum by Fermi function $F(Z_Y, E_e)$

$$\Gamma = \frac{G_F^2 \left|M_{\text{nuclear}}\right|^2}{2\pi^3} \int_0^{E_0} (E_0 - E_e)^2 E_e^2 \, F(Z_Y, E_e) \, \mathrm{d}E_e$$

All the information about the nuclear wavefunctions is contained in the matrix element. Values for the complicated Fermi Integral are tabulated.

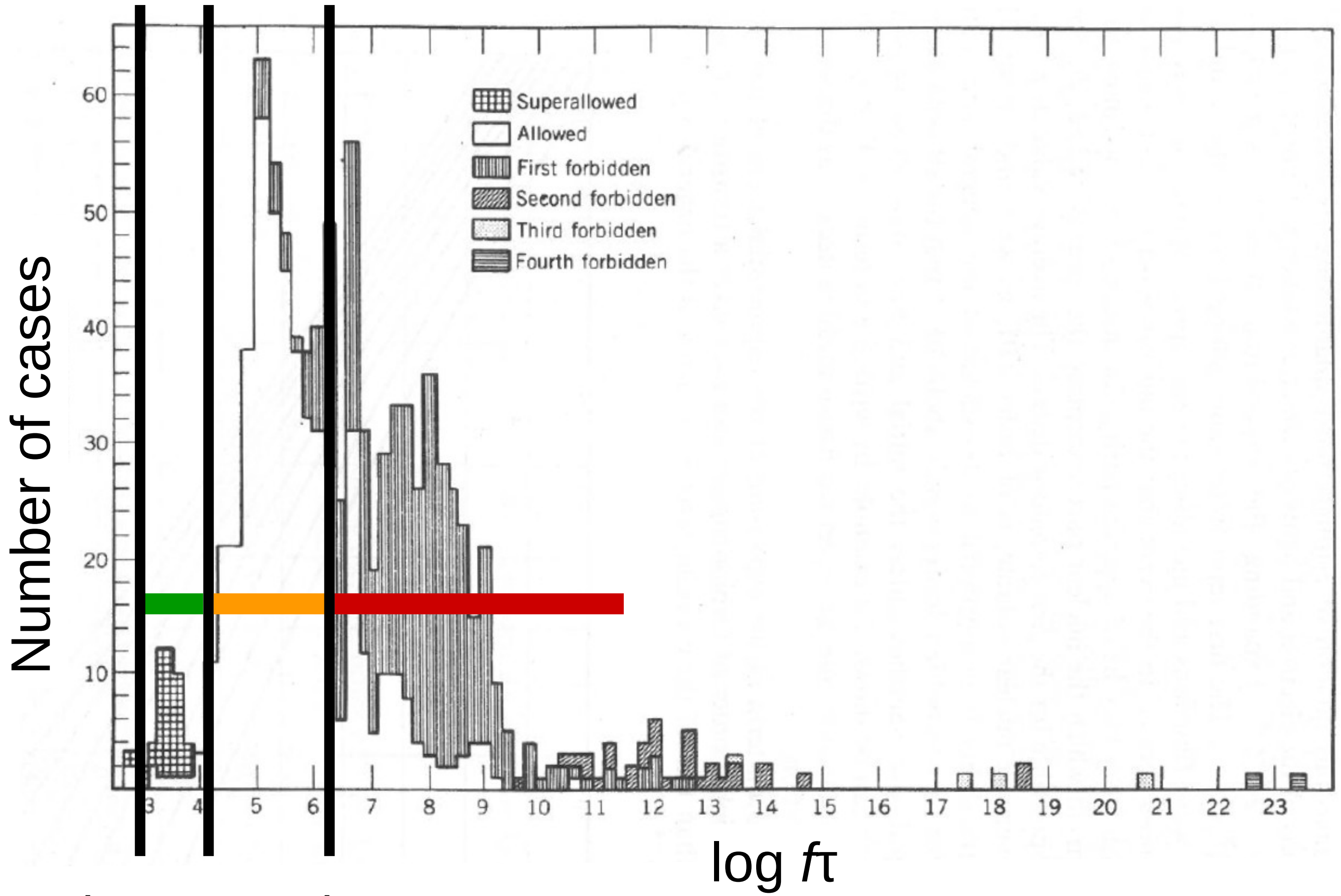
$$f(Z_Y, E_0) = \frac{1}{m_e^5} \int_0^{E_0} (E_0 - E_e)^2 E_e^2 \, F(Z_Y, E_e) \, \mathrm{d}E_e$$

Mean lifetime $\tau = 1/\Gamma$, half-life $\tau_{1/2} = \frac{\ln 2}{\Gamma}$

$$f\tau_{1/2} = \ln 2 \frac{2\pi^3}{m_e^5 G_F^2 \left|M_{\text{nuclear}}\right|^2}$$

Comparative half-life

this is rather useful because it depends only on the nuclear matrix element



In rough terms, decays with

$$
\begin{aligned}
\log f\tau_{1/2} \quad &\sim 3-4 && \text{known as } \textbf{super-allowed} \\
&\sim 4-7 && \text{known as } \textbf{allowed} \\
&\geq 6 && \text{known as } \textbf{forbidden} \text{ (i.e. suppressed, small } M_{\text{if}})
\end{aligned}
$$

Fermi theory
$$M_{\rm fi} = G_F \int \psi_p^* \, e^{-i(\vec{p}_e+\vec{p}_\nu).\vec{r}} \, \psi_n \, d^3\vec{r}$$

e, ν wavefunctions

Allowed Transitions $\log_{10} f\tau_{1/2} \sim 4-7$

Angular momentum of $e\nu$ pair relative to nucleus, $L=0$.

Equivalent to: $e^{-i(\vec{p}_e+\vec{p}_\nu).\vec{r}} \sim 1$

Superallowed Transitions $\log_{10} f\tau_{1/2} \sim 3-4$

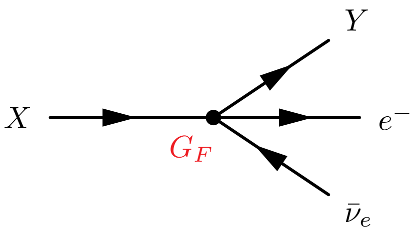
subset of Allowed transitions: often **mirror nuclei** in which p and n have approximately the same wavefunction
$$M_{\rm nuclear} \sim \int \psi_p^* \psi_n \, d^3\vec{r} \sim 1$$

e, ν both have spin $1/2 \Rightarrow$ Total spin of $e\nu$ system can be $S_{e\nu} = 0$ or 1.

There are two types of allowed/superallowed transitions depending on the relative spin states of the emitted e and ν...

Fermi Theory of β-decay *Selection Rules*

For allowed/superallowed transitions, $L_{e\nu} = 0$



$$X \rightarrow Y + e + \nu$$

$$J_X = J_Y \oplus S_{e\nu} \oplus L_{e\nu}$$

e.g. $n \rightarrow p e^- \bar{\nu}_e$
4 spin states of $e\nu$
(3 G-T, 1 Fermi)

$S_{e\nu} = 0$ Fermi transitions

$$n\uparrow \ \rightarrow \ p\uparrow \ + \frac{1}{\sqrt{2}}\left[\left(e^-\uparrow \bar{\nu}_e\downarrow\right) - \left(e^-\downarrow \bar{\nu}_e\uparrow\right)\right] \qquad \Delta J = 0$$

$S_{e\nu} = 0, m_s = 0$ $\qquad J_X = J_Y$

$S_{e\nu} = 1$ Gamow-Teller transitions

$$n\uparrow \ \rightarrow \ p\uparrow \ + \frac{1}{\sqrt{2}}\left[\left(e^-\uparrow \bar{\nu}_e\downarrow\right) + \left(e^-\downarrow \bar{\nu}_e\uparrow\right)\right] \qquad \Delta J = 0$$

$0 \rightarrow 0$ forbidden

$S_{e\nu} = 1, m_s = 0$ $\qquad J_X = J_Y$

$$n\uparrow \ \rightarrow \ p\downarrow \ + \ e^-\uparrow \ + \ \bar{\nu}_e\uparrow \qquad \Delta J = \pm 1$$

$S_{e\nu} = 1, m_s = \pm 1$ $\qquad J_X = J_Y \pm 1$

No change in angular momentum of the $e\nu$ pair relative to the nucleus, $L_{e\nu} = 0$

$\Rightarrow$ Parity of nucleus unchanged

Forbidden Transitions $\log_{10} f\tau_{1/2} \geq 6$

Angular momentum of $e\nu$ pair relative to nucleus, $L_{e\nu} > 0$.

$$e^{-i(\vec{p}_e+\vec{p}_\nu).\vec{r}} = 1 - i(\vec{p}_e + \vec{p}_\nu).\vec{r} + \frac{1}{2}[(\vec{p}_e + \vec{p}_\nu).\vec{r}]^2 - \ldots$$

$L =$	0	1	2
$P = (-1)^L =$	even	odd	even
	Allowed	1^{st} forbidden	2^{nd} forbidden

Transition probabilities for $L > 0$ are small $\Rightarrow$ forbidden transitions (really means "suppressed").

Forbidden transitions are only competitive if an allowed transition cannot occur (selection rules). Then the lowest permitted order of "forbiddeness" will dominate.

In general, n^{th} forbidden $\Rightarrow$ $e\nu$ system carries orbital angular momentum $L = n$, and $S_{e\nu} = 0$ (Fermi) or 1 (G-T). Parity change if L is odd.

Examples

$$^{34}\mathrm{Cl}(0^+) \rightarrow {}^{34}\mathrm{S}(0^+)$$

$$^{14}\mathrm{C}(0^+) \rightarrow {}^{14}\mathrm{N}(1^+)$$

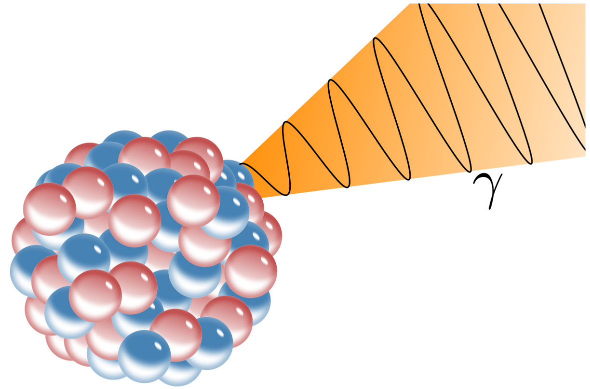
$$n(1/2^+) \rightarrow p(1/2^+)$$

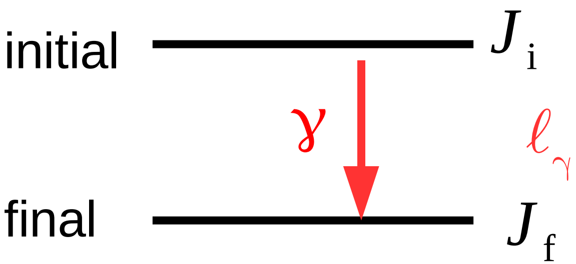
$$^{39}\mathrm{Ar}(7/2^-) \rightarrow {}^{39}\mathrm{K}(3/2^+)$$

$$^{87}\mathrm{Rb}(3/2^-) \rightarrow {}^{87}\mathrm{Sr}(9/2^+)$$

γ Decay

Emission of γ-rays (EM radiation) occurs when a nucleus is created in an excited state (e.g. following α, β decay or collision).





The photon carries away net angular momentum L_γ when a proton in the nucleus makes a transition from its initial a.m. state J_i to its final a.m. state J_f.

$$\vec{J_i} = \vec{L_\gamma} \oplus \vec{J_f} \quad \text{and} \quad |\vec{J_i} - \vec{J_f}| \le L_\gamma \le |\vec{J_i} + \vec{J_f}|$$

The photon carries $J^P = 1^-$ $\Rightarrow L_\gamma \geq 1$.

$\Rightarrow$ Single γ emission is forbidden for a transition between two $J = 0$ states.

($0 \rightarrow 0$ transitions can only occur via internal conversion (emitting an electron) or via the emission of more than one γ.)

γ Decay

Radiative transitions in nuclei are generally the same as for atoms, except

Atom $E_\gamma \sim \text{ eV}$; $\lambda \sim 10^8\,\text{fm} \sim 10^3 \times r_\text{atom}$; $\Gamma \sim 10^9\,\text{s}^{-1}$

Only dipole transitions are important.

Nuclei $E_\gamma \sim \text{ MeV}$; $\lambda \sim 10^2\,\text{fm} \sim 25 \times r_\text{nucl}$; $\Gamma \sim 10^{16}\,\text{s}^{-1}$

Collective motion of many protons lead to higher transition rates.

$\Rightarrow$ Higher order transitions are also important.

Two types of transitions:

Electric (E) transitions arise from an oscillating charge which causes an oscillation in the external electric field.

Magnetic (M) transitions arise from a varying current or magnetic moment which sets up a varying magnetic field.

Obtain transition probabilities using Fermi's Golden Rule

$$\Gamma = 2\pi |M_\text{if}|^2 \rho(E_\text{f})$$

γ Decay *Electric Dipole Transitions (E1) $L = 1$*

Insert dipole matrix element into FGR $$\Gamma_{\mathrm{i}\to\mathrm{f}} = \frac{\omega^3}{3\pi\epsilon_0 c^3\hbar} \, | \, \langle\psi_\mathrm{f}|e\vec{r}|\psi_\mathrm{i}\rangle \, |^2$$

see Adv. Quantum Physics; after averaging over initial and summing over final states

Order of magnitude estimate of this rate,

$$| \, \langle\psi_\mathrm{f}|e\vec{r}|\psi_\mathrm{i}\rangle \, |^2 \sim |eR|^2 \quad \Rightarrow \quad \Gamma \sim \frac{4}{3}\alpha E_\gamma^3 R^2$$

R = radius of nucleus,
$\alpha = \frac{e^2}{4\pi\epsilon_0 c\hbar}$, $E_\gamma = \hbar\omega$, $\hbar = c = 1$.

e.g. $E_\gamma = 1\ \mathrm{MeV}$, $R = 5$ fm ($\hbar c = 197\ \mathrm{MeVfm}$, $\hbar = 6.6 \times 10^{-22}\ \mathrm{MeVs}$)

$$\Gamma(E1) = 0.24\ \mathrm{MeV}^3\mathrm{fm}^2 = \frac{0.24}{(197)^2 \times 6.6 \times 10^{-22}}\mathrm{s}^{-1} = 10^{16}\,\mathrm{s}^{-1}$$ (c.f. atoms $\Gamma \sim 10^9\mathrm{s}^{-1}$)

As nuclear wavefunctions have definite parity, the matrix element can only be non-zero if the initial and final states have opposite parity.

$$e\vec{r} \xrightarrow{\hat{P}} -e\vec{r} \qquad \text{ODD}$$

E1 transition $\Rightarrow$ parity change of nucleus

γ Decay *Magnetic Dipole Transitions (M1) $L = 1$*

Magnetic dipole matrix element $\quad | \langle \psi_\mathrm{f} | \mu \vec{\sigma} | \psi_\mathrm{i} \rangle |^2$

μ = magnetic moment, $\vec{\sigma}$ = Pauli spin matrices

Typically $\quad \langle \mu\sigma \rangle \sim \dfrac{e\hbar}{2m_p} = \mu_N \quad$ Nuclear magneton

For a proton $\quad \dfrac{\hbar}{m_p} \sim 0.2\mathrm{fm} \sim \dfrac{R}{25} \quad$ for $R = 5$ fm

Compare to E1 transition rate $\quad \dfrac{\Gamma(M1)}{\Gamma(E1)} = \left(\dfrac{e\hbar}{2m_p}\right)^2 \dfrac{1}{(eR)^2} = 10^{-3}$

Magnetic moment transforms the same way as angular momentum

$$e\vec{r} \times \vec{p} \xrightarrow{\hat{P}} e(-\vec{r}) \times (-\vec{p}) = e\vec{r} \times \vec{p} \qquad \text{EVEN}$$

M1 transition $\Rightarrow$ no parity change of nucleus

If the initial and final nuclear states differ by more than 1 unit of angular momentum

$\Rightarrow$ higher multipole radiation

The perturbing Hamiltonian is a function of electric and magnetic fields and hence of the vector potential $\langle \psi_\mathrm{f} | H'(\vec{A}) | \psi_\mathrm{i} \rangle$

$\vec{A}$ for a photon is taken to have the form of a plane wave

$$\vec{A}\mathrm{e}^{\mathrm{i}\vec{p}.\vec{r}} = 1 \qquad -\mathrm{i}\vec{p}.\vec{r} \qquad +\frac{1}{2}(\vec{p}.\vec{r})^2 + \qquad \ldots\frac{(-\mathrm{i}\vec{p}.\vec{r})^n}{n!}$$

	Dipole	Quadrupole	Octupole
$L =$	1	2	3
	E1,M1	E2,M2	E3,M3

Each successive term in the expansion of $\vec{A}$ is reduced from the previous one by a factor of roughly $\vec{p}.\vec{r}$.

e.g. Compare E1 to E2 for $p \sim 1\ \mathrm{MeV}$, $R \sim 5\mathrm{fm}$

$\Rightarrow$ $pR \sim 5\ \mathrm{MeVfm} \sim 0.025$, $|pR|^2 \sim 10^{-3}$

$$\frac{\Gamma(E2)}{\Gamma(E1)} \sim 10^{-3} \sim \frac{\Gamma(M1)}{\Gamma(E1)}$$

The matrix element for E2 transitions $\sim r^2$ i.e. even under a parity transformation.

In general, EL transitions Parity $= (-1)^L$
ML transitions Parity $= (-1)^{L+1}$

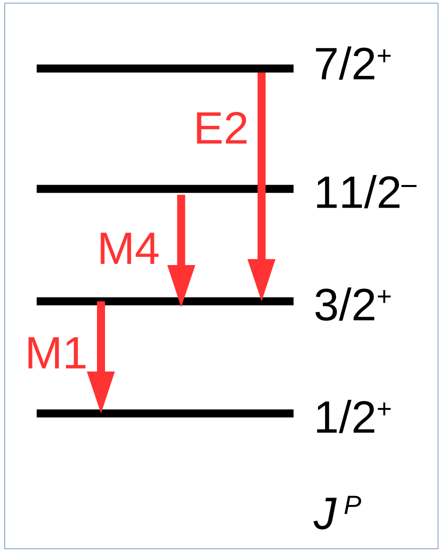
Rate	1	10^{-3}	10^{-6}	10^{-9} ...
	E1	E2	E3	E4 ...
		M1	M2	M3 ...
Parity change	✓	✗	✓	✗
J^P of γ E:	1^-	2^+	3^-	4^+
M:		1^+	2^-	3^+

In general, a decay will proceed dominantly by the lowest order (i.e. fastest) process permitted by angular momentum and parity.

e.g. if a process has $\Delta J = 2$, no parity change, it will go by the E2, even though M3, E4 are also allowed.

γ Decay *Transitions*

e.g. $^{117}_{50}$Sn



$3/2^+ \rightarrow 1/2^+$ M1 (E2 also allowed)

$11/2^- \rightarrow 3/2^+$ M4
More likely than $11/2^- \rightarrow 1/2^+$ (E5)

$7/2^+ \rightarrow 3/2^+$ E2
M2, M3 less likely: $7/2^+ \rightarrow 11/2^-$, $7/2^+ \rightarrow 1/2^+$

Information about the nature of transitions (based on rates and angular distributions) is very useful in inferring the J^P values of states.

Please note: this discussion of rates is fairly naïve. More complete formulae can be found in textbooks.

Also collective effects may be important if

- many nucleons participate in transitions,
- nucleus has a large electric quadrupole moment, Q, $\rightarrow$ rotational excited states enhance E2 transitions.

Summary

- Radioactive decays and dating.
- α-decay
  Strong dependence on E, Z
  Tunnelling model (Gamow) – Geiger-Nuttall law $\ln \tau_{1/2} \sim \frac{Z'}{E_0^{1/2}} + \text{const.}$
- β-decay
  β^+, β^-, electron capture; energetics, stability
  Fermi theory – 4-fermion interaction plus 3-body phase space.
  $$\Gamma = \frac{G_F^2 \left|M_{\text{nuclear}}\right|^2}{2\pi^3} \int_0^{E_0} (E_0 - E_e)^2 p_e^2 \, \mathrm{d}p_e$$
  Electron energy spectrum; Kurie plot.
  Comparative half-lives.
  Selection rules; Fermi, Gamow-Teller; allowed, forbidden.
- γ-decay
  Dipole, quadrupole; electric, magnetic transitions.
  Selection rules.

Problem Sheet: q.37-41

Up next... Section 16: Fission and Fusion