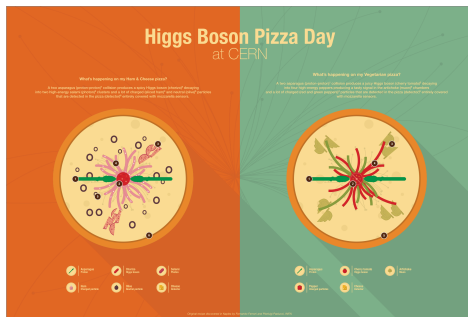


Part III Physics

Particle Physics

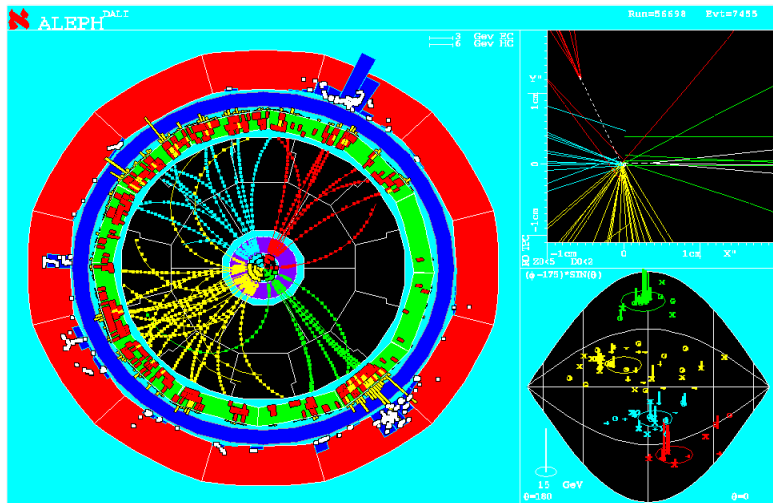
Prof. Alex Mitov



Sub-divisions (Handouts)

- 1 H1: Introduction
- 2 H2: The Dirac Equation
- 3 H3: Interaction by Particle Exchange and QED
- 4 H4: Electron-Positron Annihilation
- 5 H5: Electron-Proton Elastic Scattering
- 6 H6: Deep Inelastic Scattering
- 7 H7: Symmetries and the Quark Model
- 8 H8: Quantum Chromodynamics
- 9 H9: The Weak Interaction and $V - A$
- 10 H10: Leptonic Weak Interactions and Neutrino Deep Inelastic Scattering
- 11 H11: Neutrino Oscillations
- 12 H12: The CKM Matrix and CP Violation
- 13 H13: Electroweak Unification and the W and Z Bosons
- 14 H14: Precision Tests of the Standard Model
- 15 References
- 16 Appendices

Prof. Alex Mitov, 2026



H1: Introduction

Preliminaries

Web-page / Classes / Feedback / Correction

<https://www.precision.hep.phy.cam.ac.uk/people/mitov/teaching-pp>

- Course web-page above contains examples sheet, old exam papers and solutions, errata, lecturer's contact details, and (when ready) arrangements for examples classes ,etc.
- Please send corrections on course and/or feedback to me: adm74@cam.ac.uk

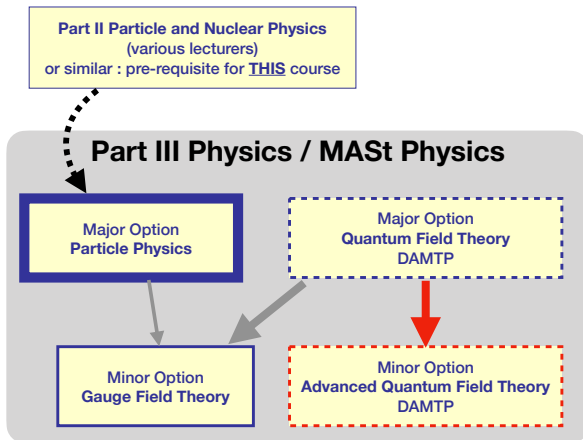
Format

- For historical reasons, the fourteen sections of the course are called 'handouts'. (List on page 2.)
- Some handouts contain additional theoretical background in non-examinable appendices.

Books

- **"Modern Particle Physics"**, Mark Thomson (Cambridge) **BASED ON THIS COURSE!**
- **"Particle Physics"**, Martin and Shaw (Wiley): fairly basic but good.
- **"Introductory High Energy Physics"**, Perkins (Cambridge): slightly below level of the course but well written.
- **"Introduction to Elementary Physics"**, Griffiths (Wiley): about right level but doesn't cover the more recent material.
- **"Quarks and Leptons"**, Halzen & Martin (Wiley): good graduate level textbook (slightly above level of this course).

Cambridge Particle Physics Courses



Aims of this course

The course is intended as an overview-style course.

It aims to provide:

- a context for the other more rigorous courses (QFT, AQFT, Gauge Field Theory),
- examples of the experiments and types of experimental evidence which have led to our current understanding of The Standard Model, and
- 'just enough' of the theory to understand how/why the experiments constrain theory.

Level of rigour

Since the QFT, AQFT and Gauge Field Theory courses are either not yet lectured or are lectured in parallel, it is necessary for many results in this course to be presented without proof, or with only plausibility arguments, or with outline theoretical motivations. That will be dissatisfying for some taking the course – but are a necessary evil if this course is to complement those other courses.

Non-examinable material

What material is deemed to be **non-examinable** ?

- any slide content explicitly labelled as '**non-examinable**', or
- any slide content explicitly labelled as '**not examinable**', or
- any material located in an appendix of the course, whether or not it carries either of the labels above, unless it is explicitly labelled '**examinable**'.

What does it mean if material is deemed to be **non-examinable** ?

no student is expected or required to read, learn or revise any such material for Tripos; the exam should not *require* prior knowledge of such material.

Terms and conditions, small print, etc.

(i) It is not the case that a Tripos question could never have a domain overlapping with or including material deemed to be **non-examinable**. Such overlaps, in the rare cases in which they appear, simply indicate that an examiner has judged that material in the overlap can be reasonably deduced from material which was deemed to be examinable. Therefore, a more specific (though wordier) name for the material could be 'material-which-does-not-need-to-be-learned-or-revised'. (ii) In the event that material has been mis-labelled, a correction would be issued to the class (either by email or verbally in lectures) before the end of Michaelmas Term and then recorded on the course website. (iii) If in doubt about the status of any material, ask the lecturer for clarifications before the end of Michaelmas Term. (iv) Occasionally material from non-examinable sections will be discussed in lectures if the lecturer judges that it could be of benefit to some members of the audience. The discussion of such material in lectures does not change its status unless an official announcement to that effect is given.

Lecture Zero

The course proper begins on Wednesday!

Before then, here are a few things which fit nowhere else:

- Units.
- Assumed knowledge about Dirac δ -Functions.
- Standard Model - review.
- Special Relativity - things you should be familiar with.
- Why Mandelstam variables matter.

Units in Particle Physics

S.I. Units measure:

mass in *kg*, **length** in *m*, **time** in *s*, **charge** in *C*.

In principle, particle physicists using 'natural' units measure:

mass in GeV/c^2 , **length** in $\hbar c/\text{GeV}$, **time** in $\hbar/\text{GeV}$, **charge** in $(\epsilon_0 \hbar c)^{\frac{1}{2}}$.

NB: You could change *GeV* to *MeV*, *TeV* or any other eV-based energy unit without upsetting anyone at CERN.

Heaviside-Lorentz convention:

$c = \hbar = \epsilon_0 = 1$ (and $\mu_0 = 1$ too since $c = (\epsilon_0 \mu_0)^{-\frac{1}{2}}$)

In practice, particle physicists using natural units measure:

mass in *GeV*, **length** in $1/\text{GeV}$, **time** in $1/\text{GeV}$, and **charge** is *dimensionless* on account of using that Heaviside-Lorentz convention!

$1 = \hbar c = 197 \text{ MeV fm}$: (the one unit-based fact memorized in CERN)

Lazy particle physicists correct their omissions of $\hbar$ and c factors with the following tricks: They get the “GeV”s on the bottom/top correctly they remembering that:

- they like **high** energies and **high** momenta: so these are GeV
- they like **small** lengths and **small** times: so these are GeV^{-1}

To help them recover c and $\hbar$ factors omitted from energies they (mostly) use the following *aides-mémoire*:

- (for **mass**): $E \sim mc^2$, so mass is $\propto \text{GeV}/c^2$,
- (for **momentum**): $E \sim (mc)(c) \sim pc$, so momentum is $\propto \text{GeV}/c$,
- (for **time**): $\Delta E \Delta t \sim \hbar$, so time is $\propto \hbar/\text{GeV}$,
- (for **length**): $1 = \hbar c \approx 197 \text{ MeV} \cdot \text{fm}$, so length is $\propto \hbar c/\text{GeV}$,

and to get proper S.I. units like Joules and meters:

- (energies to Joules): $\text{eV} \approx 1.60 \times 10^{-19} \text{ J}$,
- (lengths to meters): $1 = \hbar c \approx 197 \text{ MeV} \cdot \text{fm}$.

Easy to remember: the previous lecturer's height in centimeters ...

Assumed knowledge concerning Dirac δ -functions of one real variable:

It is assumed you know this:

$$\int_X g(x) \delta(u(x)) dx = \int g(x(u)) \delta(u) \left| \frac{dx}{du} \right| du = \sum_{x \in X \text{ s.t. } u(x)=0} \frac{g(x)}{\left| \frac{du}{dx} \right|}, \quad (1)$$

If the above is not familiar to you, convince yourself of it using other sources.

Examples of the above:

$$\int_{-\infty}^{\infty} \sin x \delta(x - a) dx = \sin a, \quad (\text{trivial example})$$

$$\int_{-\infty}^{\infty} e^x \delta(x^2 - a^2) dx = \sum_{x=\pm a} \frac{e^x}{|2x|} = \frac{e^a}{|2(a)|} + \frac{e^{-a}}{|2(-a)|} = \frac{1}{2|a|} (e^a + e^{-a}) = \frac{1}{|a|} \cosh a.$$

The second example is presumably non-trivial, as many past students of the course have mistakenly produced other answers for it like $e^a + e^{-a}$ or $2 \cosh a$.

The generalisation to two or more variables ...

Not examinable

This is not examinable, but the generalisations of the result (1) to Dirac δ -functions of more than one variable would be as follows:

Two variables:

$$\begin{aligned} \int_X g(x, y) \delta(u(x, y)) \delta(v(x, y)) dx dy &= \int g(x(u, v), y(u, v)) \delta(u) \delta(v) \left\| \frac{\partial(x, y)}{\partial(u, v)} \right\| du dv \\ &= \sum_{(x, y) \in X \text{ s.t. } u(x, y) = v(x, y) = 0} \frac{g(x, y)}{\left\| \frac{\partial(u, v)}{\partial(x, y)} \right\|}. \end{aligned}$$

Many variables:

$$\int_X g(\vec{x}) \delta^n(\vec{u}(\vec{x})) d^n x = \int g(\vec{x}(\vec{u})) \delta^n(\vec{u}) \left\| \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} \right\| d^n u = \sum_{\vec{x} \in X \text{ s.t. } \vec{u}(\vec{x}) = 0} \frac{g(\vec{x})}{\left\| \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)} \right\|}.$$

Not examinable

Reminder: Matter in the Standard Model

The Standard Model's point-like spin-1/2 fermions:

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	-1	0.0005	d	$-1/3$	~ 0.3
	ν_1	0	≈ 0	u	$+2/3$	~ 0.3
Second Generation	μ^-	-1	0.106	s	$-1/3$	~ 0.5
	ν_2	0	≈ 0	c	$+2/3$	~ 1.5
Third Generation	τ^-	-1	1.77	b	$-1/3$	~ 4.5
	ν_3	0	≈ 0	t	$+2/3$	173

Open Questions (not addressed in this course!) :

- Why are there three generations, rather than some other number?
- Why are there generations at all?
- Why are the neutrinos so light compared to the others? (e.g. ν_1 has a non-zero mass below ~ 3 eV)
- Are the neutrinos Majorana?
- Why are there no particles that could explain Dark Matter?

Reminder: Forces in the Standard Model

Spin-1 Gauge Bosons mediate the four forces of the SM:

Force	Boson(s)	J^P	m/GeV	Charge "g"	coupling constant "α"	Notes
EM (QED)	Photon γ	1^-	0	e	α	Handout 3
Weak	$W^\pm/Z$	1^-	80/91	g_W, g_Z	α_W, α_Z	Handout 13
Strong (QCD)	8 Gluons g	1^-	0	g_s	α_s	Handout 8
Gravity	Graviton?	$2^+ ?$	0 ?	?	?	Not in SM!

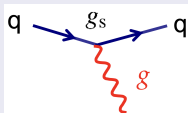
Charges g vs Coupling Constants α

- Coupling constants α are always dimensionless whereas charges g might have strange units.
- $g = \sqrt{4\pi\alpha}$ is true for all forces, but only in natural units with Heaviside Lorentz convention.
- $g = \sqrt{4\pi\alpha\epsilon_0\hbar c} = e$ for QED without Heaviside Lorentz convention.
- Since always dimensionless, the coupling constant α sets the intrinsic strength of each force and does not depend on choice of units.

Reminder: Standard Model Vertices

Although much more detail will be provided later, it will be assumed that you have some partial knowledge of the following properties of the SM interactions:

Strong

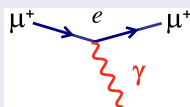


Only quarks

Never changes
flavour

$$\alpha_S \sim 1$$

Electromagnetic

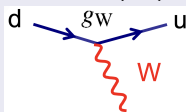


All charged
fermions

Never changes
flavour

$$\alpha \simeq 1/137$$

Weak Charged Current (CC)

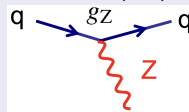


All fermions

Always changes
flavour

$$\alpha_W \sim 1/40$$

Weak Neutral Current (NC)



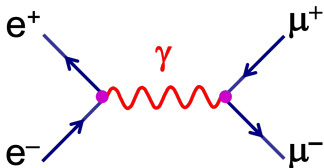
All fermions

Never changes
flavour

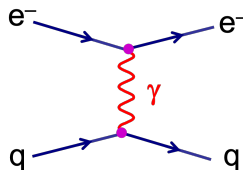
$$\alpha_Z \sim 1/40$$

Feynman Diagram conventions

electron-positron annihilation example:



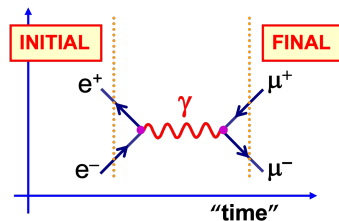
electron-quark scattering example:



Usually in this course: 'time' runs from left to right, but only in the sense that:

- LHS of diagram is initial state
- RHS of diagram is final state
- Middle is how process 'might' have happened

Anti-particles are indicated by arrows in the negative 'time' direction.



Much more detail appears in Handout 3 and onwards!

Special Relativity and 4-Vector Notation

- Will use 4-vector notation with t as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\} \text{ (contravariant)}$$

$$p_\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\} \text{ (covariant)}$$

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, usually deal with relativistic particles. Require **all** calculations to be **Lorentz Invariant**. **L.I.** quantities formed from **4-vector scalar products**, e.g.

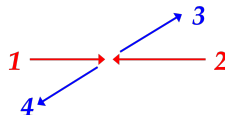
$$p^\mu p_\mu = E^2 - p_x^2 - p_y^2 - p_z^2 = E^2 - p^2 = m^2 \quad \text{(Invariant Mass)}$$

$$x^\mu p_\mu = Et - p_x x - p_y y - p_z z = Et - \vec{p} \cdot \vec{r} \quad \text{(Phase)}$$

- A few words on NOTATION: Four vectors are written as either: p^μ or p . Four vector scalar products as either: $p^\mu p_\mu$ or $p \cdot r$. Three vectors are written as: $\vec{p}$. Quantities evaluated in the centre of mass frame are written: $\vec{p}^*$ or p^* etc.

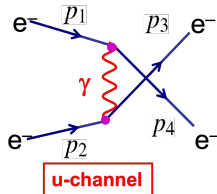
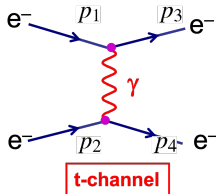
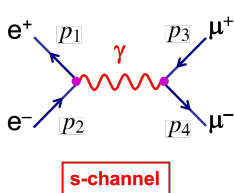
The Mandelstam Variables: s , t , and u

Scattering processes like $1 + 2 \rightarrow 3 + 4$ are often characterised by three useful real **Lorentz Invariant** quantities: s , t , and u . These are built from the squares of particular sums or differences of the Lorentz four-vectors p_1 , p_2 , p_3 and p_4 :



$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

Simple Feynman diagrams for two-to-two processes are named s -, t - or u -channel according to the squared four-momentum of the exchanged particle:



Why care about s , t and u ?

Only two of s , t and u are independent if the masses of the four external particles are known.

Proof is in **example sheet question 2** which will ask you to show that

$$s + t + u \equiv m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Without loss of generality, take s and t to be the pair of independent variables.

s contains information about the **initial state**, and

t contains information about the **final state**.

Therefore, t encodes information about the interaction that occurred.

Hardly surprising then that energy in centre-of-mass frame is $\sqrt{s}$

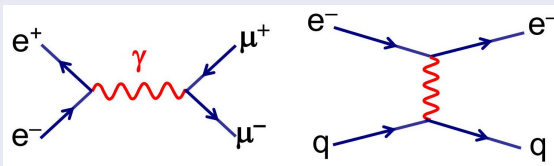
s is **Lorentz invariant** so takes same value in any frame. In the c.o.m. frame $p_1^* = (E_1, \vec{p}^*)$ and $p_2^* = (E_2, -\vec{p}^*)$ so $s = (E_1 + E_2, \vec{0})^2 = (\text{centre-of-mass energy})^2$.

The first five lectures

Aiming towards proper calculations of decay rates and scattering probabilities

Will concentrate on:

- $e^+ e^- \rightarrow \mu^+ \mu^-$
- $e^- q \rightarrow e^- q$ (to probe proton structure)



We will need:

- relativistic calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

- relativistic treatment of spin-half particles: (see Dirac Equation, etc, Handout 2)
- relativistic calculation of interaction Matrix Element

Master formula for computing decay rates

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1(p_1) - E_2(p_2)) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}. \quad (2)$$

- It follows from Fermi's Golden Rule:

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- $\rho(E_f)$: the phase space (density of final states); in the formula above, the δ -functions and the $\frac{d^3\vec{p}}{(2\pi)^3 2E}$ factors.
- M_{fi} : the transition amplitude, normalised to $2E$ particles per unit volume:
 $M_{fi} = (2E_i 2E_1 2E_2)^{1/2} T_{fi}$. It is Lorentz invariant.
- The formula is written for a two-body final state; each additional final-state particle brings one more factor $\frac{d^3\vec{p}}{(2\pi)^3 2E}$.

Full details can be found in Appendix I (page 489)

Decay Rate Calculations

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1(p_1) - E_2(p_2)) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}.$$

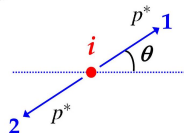
The integral is Lorentz invariant (i.e. frame independent). Therefore, it can be evaluated in any frame we choose. The C.o.M. frame is most convenient.

- In the C.o.M. frame $E_i = m_i$ and $\vec{p}_i = 0 \Rightarrow$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1(p_1) - E_2(p_2)) \delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3\vec{p}_1}{2E_1(p_1)} \frac{d^3\vec{p}_2}{2E_2(p_2)}.$$

- Integrating over $\vec{p}_2$ using the δ -function:

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1(p_1) - E_2(\vec{p}_1)) \frac{d^3\vec{p}_1}{4E_1(p_1)E_2(\vec{p}_1)}$$



(note $E_2(p_1) = \sqrt{m_2^2 + |\vec{p}_1|^2}$ since the δ -function imposes $\vec{p}_2 = -\vec{p}_1$) ... and so writing $|\vec{p}_1|$ as p_1 :

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 dp_1 d\Omega}{E_1(p_1)E_2(\vec{p}_1)}. \quad (3)$$

- The last expression for Γ_{fi} can be written in the form

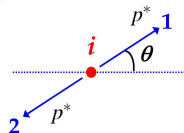
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1) \delta(f(p_1)) dp_1 d\Omega \quad (4)$$

where $g(p_1) = p_1^2 / (E_1 E_2) = p_1^2 (m_1^2 + p_1^2)^{-1/2} (m_2^2 + p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2 + p_1^2)^{1/2} - (m_2^2 + p_1^2)^{1/2}$.

Note:

- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products i.e. $f(p_1) = 0$ for $p_1 = p^*$.
- Eq. (4) can be integrated using the property of δ -function derived earlier (equation (1))



$$\int g(p_1) \delta(f(p_1)) dp_1 = \frac{1}{|df/dp_1|_{p_1=p^*}} \int g(p_1) \delta(p_1 - p^*) dp_1 = \frac{g(p^*)}{|df/dp_1|_{p_1=p^*}}$$

where p^* is the value for which $f(p^*) = 0$.

- All that remains is to evaluate df/dp_1

$$\frac{df}{dp_1} = -\frac{p_1}{(m_1^2 + p_1^2)^{1/2}} - \frac{p_1}{(m_2^2 + p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1 \frac{E_1 + E_2}{E_1 E_2}.$$

Thus (3) becomes:

$$\begin{aligned}\Gamma_{fi} &= \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{E_1 E_2}{p_1 (E_1 + E_2)} \frac{p_1^2}{E_1 E_2} \right|_{p_1=p^*} d\Omega \\ &= \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{E_1 + E_2} \right|_{p_1=p^*} d\Omega.\end{aligned}$$

But from $f(p_1) = 0$, i.e. energy conservation: $E_1 + E_2 = m_i$ so

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i} \int |M_{fi}|^2 d\Omega$$

which, in the particle's rest frame $E_i = m_i$, becomes the important result:

$$\boxed{\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 d\Omega} \quad (5)$$

and this is **VALID FOR ALL TWO-BODY DECAYS !**

- p^* can be obtained by solving $f(p^*) = 0$. This gives:

$$p^* = \frac{1}{2m_i} \sqrt{[m_i^2 - (m_1 + m_2)^2] [m_i^2 - (m_1 - m_2)^2]}. \quad (\text{Question 4})$$

Now try questions 5 and 6.

Cross section definition

$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

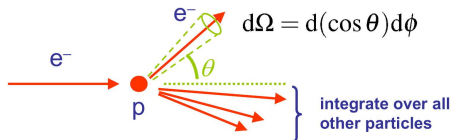
(6)

Flux = number of incident particles/ unit area/unit time

- The “cross section”, σ , can be thought of as the effective crosssectional area of the target particles for the interaction to occur.
- It is a property of the pair of things interacting, rather than of either one singly.
- In general this has nothing to do with the physical size of the target although there are exceptions. (E.g. the cross section for neutron absorption on nuclei is the projective area of the nucleus!)

Differential Cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$



with $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$

Master formula for computing cross sections

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4} \quad (7)$$

- It follows from Fermi's Golden Rule: σ is the transition rate per unit volume and incident flux,

$$\sigma = \frac{\Gamma_{fi}}{v_1 + v_2}, \quad \Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- $\rho(E_f)$: the phase space (density of final states); in the formula above, the δ -functions and the $\frac{d^3\vec{p}}{2E}$ factors.
- M_{fi} : the transition amplitude, normalised to $2E$ particles per unit volume: $M_{fi} = (2E_1 2E_2 2E_3 2E_4)^{1/2} T_{fi}$. It is Lorentz invariant.
- The formula is written for a two-body final state; each additional final-state particle brings one more factor $\frac{d^3\vec{p}}{(2\pi)^3 2E}$.
- $F = 2E_1 2E_2 (v_1 + v_2)$: the flux factor, which can be cast in a Lorentz invariant form:

Full details can be found in Appendix II (page 509)

(Longitudinally) Lorentz Invariant Flux

$$F \stackrel{\text{def}}{=} 4 \sqrt{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - m_1^2 m_2^2} = (2E_1)(2E_2)(\mathbf{v}_1 + \mathbf{v}_2)$$

is called the Lorentz Invariant Flux.

As it is Lorentz Invariant, it can be evaluated easily in different frames to get different looking (but ultimately identically valued) expressions for it. E.g:

Centre of Mass Frame

$$\begin{aligned} F &= 4E_1^* E_2^* (\mathbf{v}_1^* + \mathbf{v}_2^*) \\ &= 4E_1^* E_2^* \left(\frac{|\mathbf{p}_1^*|}{E_1^*} + \frac{|\mathbf{p}_2^*|}{E_2^*} \right) \\ &= 4(E_2^* + E_1^*) |\mathbf{p}^*| \\ &= 4|\mathbf{p}^*| \sqrt{s} \quad \textcircled{Q} \end{aligned}$$

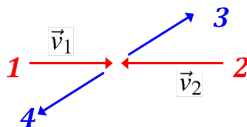
$\uparrow$
 (Mandelstam s)

Fixed Target Frame

$$\begin{aligned} F &= 4E_1 E_2 (\mathbf{v}_1 + \mathbf{v}_2) \\ &= 4E_1 m_2 (\mathbf{v}_1 + 0) \\ &= 4E_1 m_2 \frac{|\mathbf{p}_1|}{E_1} \\ &= 4m_2 |\mathbf{p}_1| \quad \textcircled{R} \end{aligned}$$

2 → 2 Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2 → 2 scattering in C.O.M. frame
- Start from



$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4}$$

- and then note that in the C.o.M.-frame $\vec{p}_1 + \vec{p}_2 = 0$ and $E_1 + E_2 = \sqrt{s}$ so

$$\sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*| \sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s} - E_3 - E_4) \delta^3(\vec{p}_3 + \vec{p}_4) \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4}. \quad (8)$$

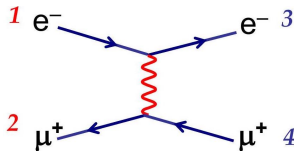
- The integral in (8) is exactly the same as the integral (3) which appeared in the one-to-two particle decay calculation, except that it has m_a replaced by $\sqrt{s}$. So:

$$\sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*| \sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^* \quad (9)$$

- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 d\Omega^*$$

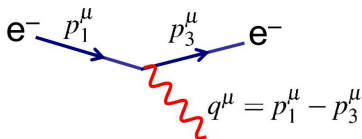


- For calculating the total cross-section (which is Lorentz Invariant) the result (9) on the previous page is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 d\Omega^* \quad (10)$$

because the angles in $d\Omega^* = d(\cos\theta^*) d\phi^*$ refer to the C.o.M frame.

- For the last calculation in this section, we need to find a L.I. expression for $d\sigma$.
- Start by expressing $d\Omega^*$ in terms of Mandelstam $t = (p_1 - p_3)^2$... i.e. the square of the four-momentum transfer:



- Want to express $d\Omega^*$ in terms of Lorentz Invariant dt where
 $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2p_1 \cdot p_3$.
- In C.o.M. frame:

$$p_1^{*\mu} = (E_1^*, 0, 0, |\vec{p}_1^*|)$$

$$p_3^{*\mu} = (E_3^*, |\vec{p}_3^*| \sin \theta^*, 0, |\vec{p}_3^*| \cos \theta^*)$$

$$p_1^\mu p_{3\mu} = E_1^* E_3^* - |\vec{p}_1^*| |\vec{p}_3^*| \cos \theta^*$$

$$t = m_1^2 + m_3^2 - 2E_1^* E_3^* + 2|\vec{p}_1^*| |\vec{p}_3^*| \cos \theta^*$$

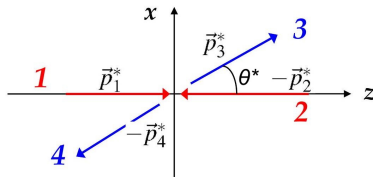
giving $dt = 2|\vec{p}_1^*| |\vec{p}_3^*| d(\cos \theta^*)$

therefore $d\Omega^* = d(\cos \theta^*) d\phi^* = \frac{dt d\phi^*}{2|\vec{p}_1^*| |\vec{p}_3^*|}$

hence $d\sigma = \frac{1}{64\pi^2 s} \left| \frac{\vec{p}_3^*}{|\vec{p}_1^*|} \right| |M_{fi}|^2 d\Omega^* = \frac{1}{2 \cdot 64\pi^2 s |\vec{p}_1^*|^2} |M_{fi}|^2 d\phi^* dt$.

- Finally, integrating over $d\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{64\pi s |\vec{p}_i^*|^2} |M_{fi}|^2}$$

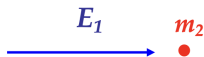


Lorentz Invariant differential cross section

- All quantities in the expression for $d\sigma/dt$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s} \left[s - (m_1 + m_2)^2 \right] \left[s - (m_1 - m_2)^2 \right]$$

- As an example of how to use the invariant expression $d\sigma/dt$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$



e.g. electron or neutrino scattering

In this limit

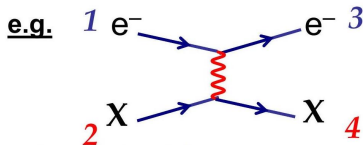
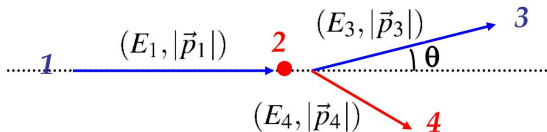
$$|\vec{p}_i^*|^2 = \frac{(s - m_2^2)^2}{4s}$$

and so

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{16\pi (s - m_2^2)^2} |M_{fi}|^2} \quad (m_1 = 0). \quad (11)$$

2 → 2 body fixed-target scattering using lab-frame angles

- A commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0$, $m_2 = m_4 = M$



- Wish to express the cross section in terms of scattering angle of the e^-

$$d\Omega = 2\pi d(\cos \theta)$$

$$\text{therefore} \quad \frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt} \frac{dt}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos \theta)} \frac{d\sigma}{dt}$$

- The rest is some rather tedious algebra.... start from four-momenta:

$$p_1 = (E_1, 0, 0, E_1), p_2 = (M, 0, 0, 0), p_3 = (E_3, E_3 \sin \theta, 0, E_3 \cos \theta), p_4 = (E_4, \vec{p}_4)$$

$$\text{so here } t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 = -2E_1 E_3 (1 - \cos \theta)$$

But from (E, p) conservation $p_1 + p_2 = p_3 + p_4$ and so we can also express t in terms of particles 2 and 4.

$$\begin{aligned}
 t &= (p_2 - p_4)^2 = 2M^2 - 2p_2 \cdot p_4 = 2M^2 - 2ME_4 \\
 &= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)
 \end{aligned}$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{dt}{d(\cos \theta)} = 2M \frac{dE_3}{d(\cos \theta)}$$

Equating the two expressions for t gives $E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos \theta}$ and so

$$\frac{dE_3}{d(\cos \theta)} = \frac{E_1^2 M}{(M + E_1 - E_1 \cos \theta)^2} = E_1^2 M \left(\frac{E_3}{E_1 M} \right)^2 = \frac{E_3^2}{M}$$

$$\text{and } \frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos \theta)} \frac{d\sigma}{dt} = \frac{1}{2\pi} 2M \frac{E_3^2}{M} \frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \frac{1}{16\pi (s - M^2)^2} |M_{fi}|^2$$

(using (11) in the last step). Thus given that $s = (p_1 + p_2)^2 = M^2 + 2p_1 \cdot p_2 = M^2 + 2ME_1$ we have $(s - M^2) = 2ME_1$ and so finally we have the important result:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2 \quad (\text{in limit } m_1 \rightarrow 0).$$

In the last equation, E_3 is a function of θ :

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos \theta}$$

so we could instead write the following if we preferred to be explicit about θ dependence:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{M + E_1 - E_1 \cos \theta} \right)^2 |M_{fi}|^2}.$$

General form for $2 \rightarrow 2$ Body Scattering in Lab. Frame

★ The calculation of the differential cross section for the case where m_1 cannot be neglected is longer and contains no more 'physics' (see Appendix IV). It gives:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{m_2 |\vec{p}_1|} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3| (E_1 + m_2) - E_3 |\vec{p}_1| \cos \theta} \cdot |M_{fi}|^2}$$

Again, there is only one independent variable, θ , which can be seen from the conservation of energy:

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e., $|\vec{p}_3|$ is a function of θ .

Summary

We used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the Lorentz Invariant Matrix Element and wave-functions normalised to $2E$ particles per unit volume. our main results were:

★ Particle decay in C.o.M. frame:

$$(12) \quad \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 d\Omega$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i} \sqrt{[m_i^2 - (m_1 + m_2)^2] [m_i^2 - (m_1 - m_2)^2]}$$

★ Scattering cross section in C.o.M. frame:

$$(13) \quad \sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*$$

★ Invariant differential cross section (valid in all frames):

$$(14) \quad \frac{d\sigma}{dt} = \frac{1}{64\pi s |\vec{p}_i^*|^2} |M_{fi}|^2$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s} [s - (m_1 + m_2)^2] [s - (m_1 - m_2)^2]$$

and (continued next slide ...)

Summary continued ...

★ Differential cross section in the lab. frame ($m_1=0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$

(15)



$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{M + E_1 - E_1 \cos \theta} \right)^2 |M_{fi}|^2$$

(16)

★ Differential cross section in the lab. frame ($m_1 \neq 0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{|m_2|\vec{p}_1|} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$

(17)

with $E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta} + m_4^2$

Summary of the summary:

- We should not need to do any more kinematic calculations, as all the kinematics we need for particle decays and two-to-two scatterings and cross sections are now embedded in the results above.
- Different physical theories change the matrix element, so we can re-use the above for many different sorts of interaction.
- Many later calculations will take one of the above results as a starting point.