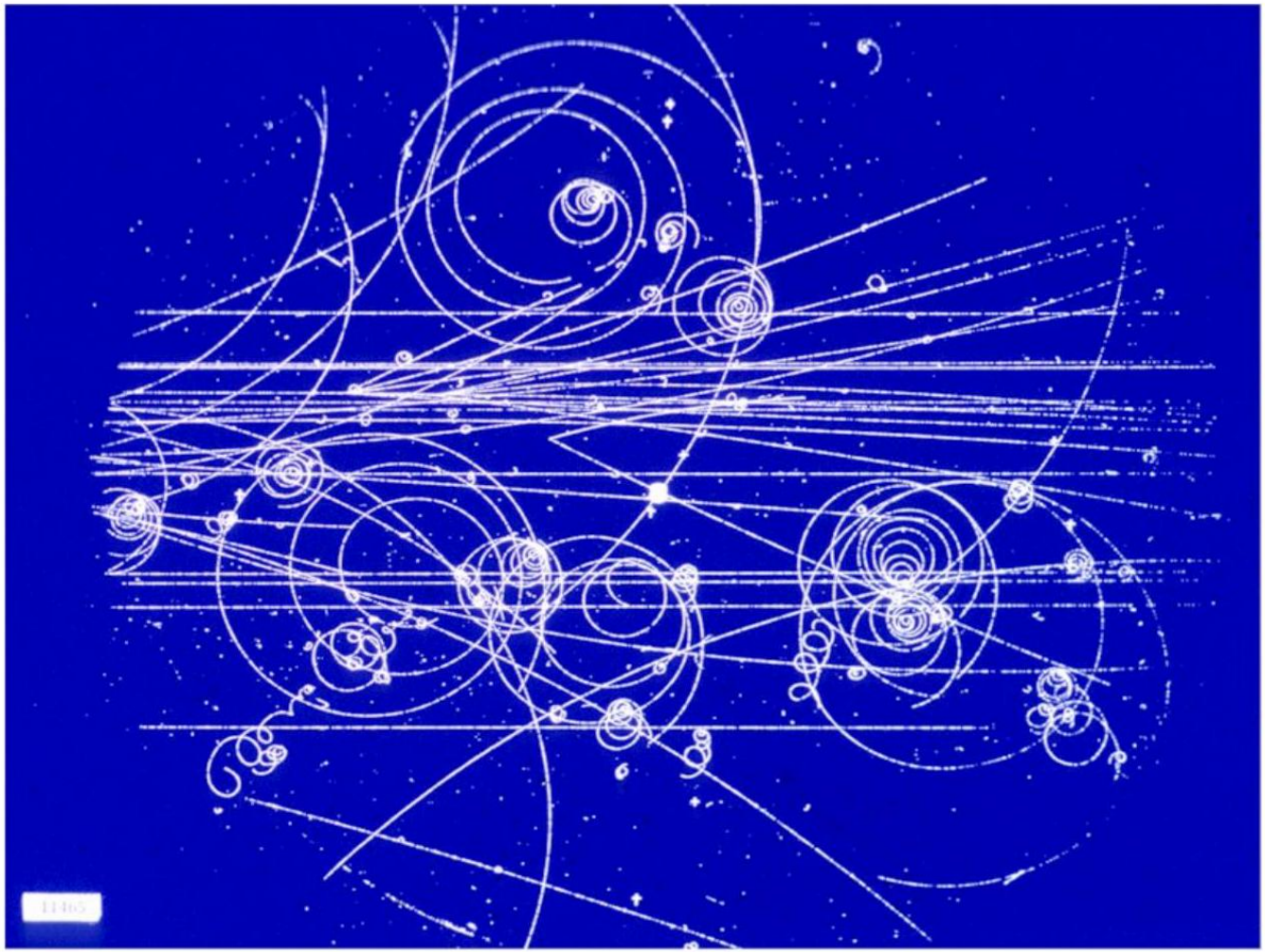
Prof. Alex Mitov, 2026

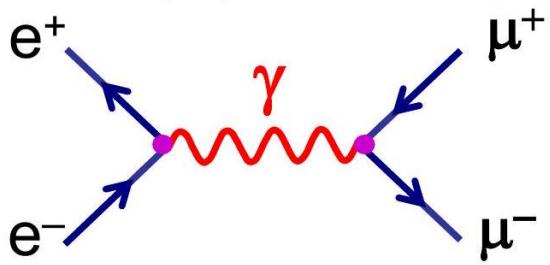


H4: Electron-Positron Annihilation

QED Calculations I

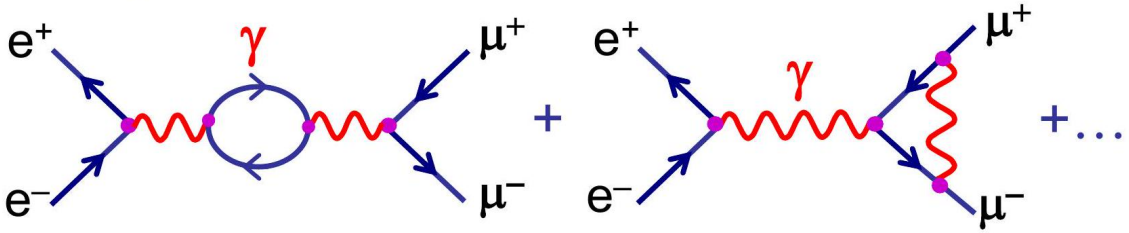
To calculate a cross section using QED (e.g. $e^+e^- \to \mu^+\mu^-$):

1. Draw all possible Feynman Diagrams
   - For $e^+e^- \to \mu^+\mu^-$ there is just one lowest order diagram



$$M \propto e^2 \propto \alpha_{em}$$

   - There are many second order diagrams ...



$$M \propto e^4 \propto \alpha_{em}^2$$

2. For each diagram calculate the matrix element using Feynman rules derived in the previous handout.

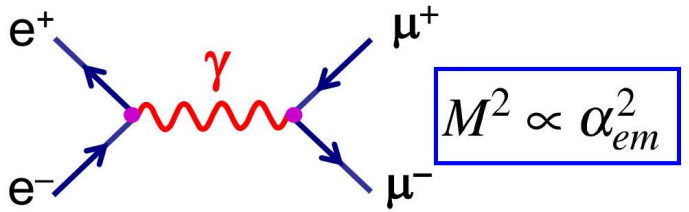
QED Calculations II

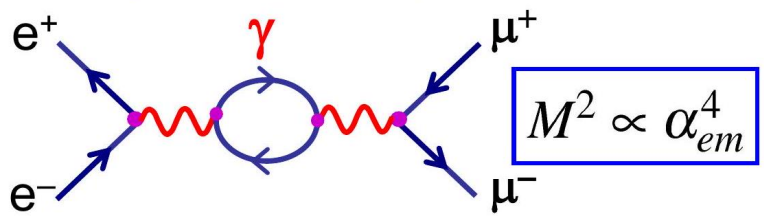
3. Sum the individual matrix elements (i.e. sum the amplitudes)

$$M_{fi} = M_1 + M_2 + M_3 + \ldots$$

Note: in this sum of amplitudes (every term of which has the same initial and the same final state) interference can be both constructive or destructive! On account of this interference one can no more ask 'Which virtual particle was involved?' than one can ask 'Through which of Young's slits did the photon pass?'

4. Find the square of the modulus:
$|M_{fi}|^2 = (M_1 + M_2 + M_3 + \ldots.)\,(M_1^* + M_2^* + M_3^* + \ldots.)$
$\rightarrow$ this gives the full perturbation expansion in α_{em}.
   - For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to neglect higher order diagrams.

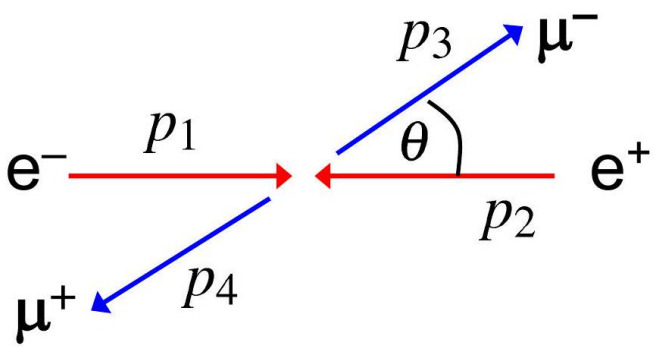




QED Calculations III

5. Calculate decay rate/cross section using formulae from Handout 1. E.g.:
   - For a decay
   $$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 \ \mathrm{d}\Omega.$$
   - For scattering in the centre-of-mass frame
   $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{\left|\vec{p}_f^{\,*}\right|}{\left|\vec{p}_i^{\,*}\right|} |M_{fi}|^2 .$$
   - For scattering in lab. frame (neglecting mass of scattered particle)
   $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 .$$

Electron Positron Annihilation: $e^+e^- \to \mu^+\mu^-$

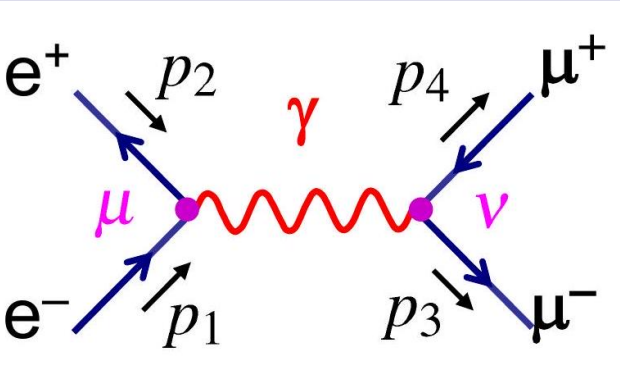


Work in C.o.M. frame so:

$$p_1 = (E, 0, 0, p) \quad p_2 = (E, 0, 0, -p)$$
$$p_3 = (E, \vec{p}_f) \quad p_4 = (E, -\vec{p}_f) \quad \text{and}$$

$$s = (p_1 + p_2)^2 = (2E)^2.$$

Feynman rules for lowest order Feynman diagram give:



$$-iM = [\bar{v}(p_2)\, ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{(p_1+p_2)^2} [\bar{u}(p_3)\, ie\gamma^\nu v(p_4)]$$
$$= [\bar{v}(p_2)\, ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{s} [\bar{u}(p_3)\, ie\gamma^\nu v(p_4)]$$
$$= \frac{ie^2}{s} g_{\mu\nu} [\bar{v}(p_2)\, \gamma^\mu u(p_1)] [\bar{u}(p_3)\, \gamma^\nu v(p_4)].$$

Conclusion

$$M = -\frac{e^2}{s} g_{\mu\nu} [\bar{v}(p_2)\, \gamma^\mu u(p_1)] [\bar{u}(p_3)\, \gamma^\nu v(p_4)]. \tag{54}$$

On the next page we try to write this M more succinctly!

Writing the Annihilation Matrix Element as a Lorentz product of Electron and Muon Currents

In Handout 2 we introduced a four-vector currents like:

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi.$$

These have a similar form to the two terms in [ ] in the matrix element M of (54):

$$M = -\frac{e^2}{s} g_{\mu\nu} \left[\bar{v}\left(p_2\right)\gamma^{\mu} u\left(p_1\right)\right]\left[\bar{u}\left(p_3\right)\gamma^{\nu} v\left(p_4\right)\right]. \qquad (54)$$

Therefore we DEFINE the electron and muon currents as follows:

$$\left(j_e\right)^{\mu} = \bar{v}\left(p_2\right)\gamma^{\mu} u\left(p_1\right) \quad \text{and} \quad \left(j_{\mu}\right)^{\nu} = \bar{u}\left(p_3\right)\gamma^{\nu} v\left(p_4\right).$$

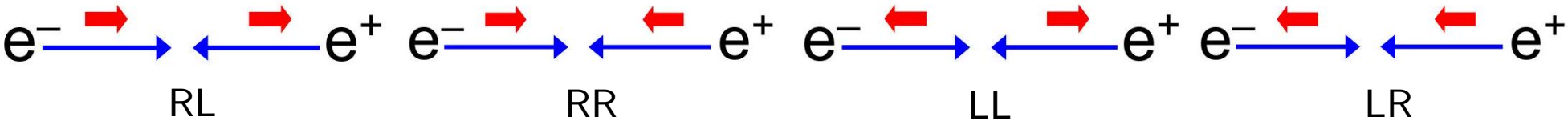
In terms of these electron and muon currents we have the more succint:

$$M = -\frac{e^2}{s} g_{\mu\nu} \left(j_e\right)^{\mu} \left(j_{\mu}\right)^{\nu} \quad \text{or} \quad \boxed{M = -\frac{e^2}{s} j_e \cdot j_{\mu}}. \qquad (55)$$

Matrix element is a four-vector scalar product – confirming it is Lorentz Invariant.

Spin in e^+e^- Annihilation

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the initial state:



- Similarly there are four possible helicity combinations in the final state
- In total there are 16 combinations e.g. $RL \to RR, RL \to RL, \ldots$
- To account for these states we need to sum over all 16 possible helicity combinations and then average over the number of initial helicity states:

$$\left\langle |M|^2 \right\rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4} \left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \ldots \right)$$

i.e. need to evaluate:

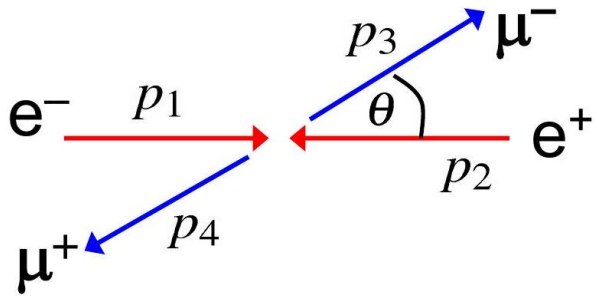
$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

for all 16 helicity combinations !

- Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements - we will see that this is an important feature of QED/QCD.

- In the C.o.M. frame in the limit $E \gg m$

$$p_1 = (E, 0, 0, E); p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



- Left- and right-handed helicity spinors (Handout 2) for particles/anti-particles are:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix} \quad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

where $s = \sin\frac{\theta}{2}$; $\quad c = \cos\frac{\theta}{2}$ and $N = \sqrt{E+m}$

- In the limit $E \gg m$ these become:

$$u_\uparrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; u_\downarrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; v_\uparrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; v_\downarrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}. \tag{56}$$

- The initial-state electron can either be in a left- or right-handed helicity state

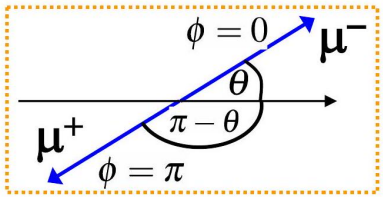
$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix};$$

- For the initial state positron ($\theta = \pi$) can have either:

$$v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}; v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- Similarly for the final state μ^-which has polar angle θ and choosing $\phi = 0$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix}; u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix}$$

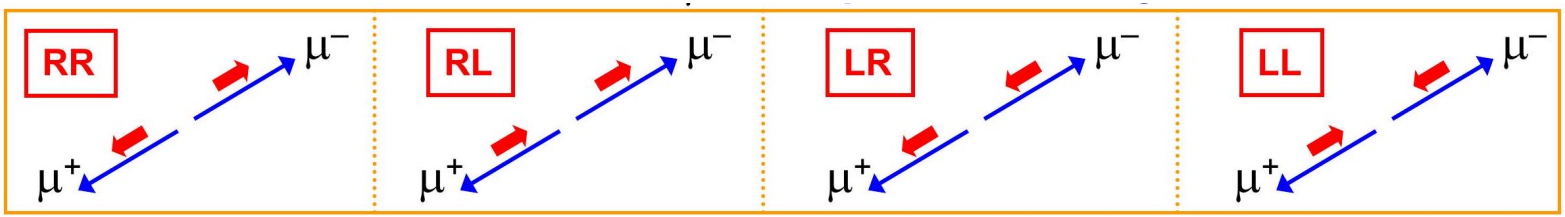


- And for the final state μ^+replacing $\theta \to \pi - \theta$; $\phi \to \pi$ obtain

$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix}; v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix};$$

using $\begin{cases}\sin\left(\frac{\pi-\theta}{2}\right) = \cos\frac{\theta}{2}\\ \cos\left(\frac{\pi-\theta}{2}\right) = \sin\frac{\theta}{2}\\ e^{i\pi} = -1\end{cases}$

- We wish to calculate the matrix element $M = -\frac{e^2}{s} j_e \cdot j_\mu$
- first consider the muon current j_μ for 4 possible helicity combinations



The Muon Current

- We want to evaluate $(j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$ for all four helicity combinations.
- For arbitrary spinors ψ, ϕ one may show that the components of $\bar{\psi}\gamma^\mu\phi$ are:

$$\bar{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4$$
$$\bar{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1$$
$$\bar{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i\left(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1\right)$$
$$\bar{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2$$

- Consider the $\mu_R^-\mu_L^+$ combination using $\psi = u_\uparrow$ and $\phi = v_\downarrow$ with

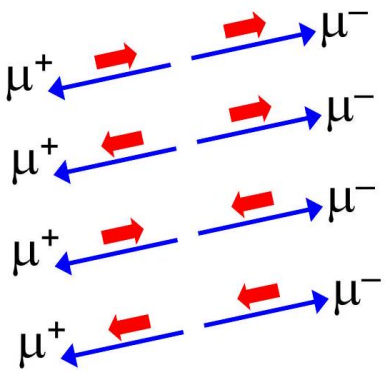
$$v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}; u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}.$$ From these we can compute:

$$\bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0,$$
$$\bar{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E\left(-c^2 + s^2 - c^2 + s^2\right) = 2E\left(s^2 - c^2\right) = -2E\cos\theta,$$
$$\bar{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE\left(-c^2 - s^2 - c^2 - s^2\right) = 2iE, \quad \text{and}$$
$$\bar{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta.$$

- Hence the four-vector muon current for the RL combination is

$$\bar{u}_{\uparrow}(p_3)\,\gamma^{\nu}\,v_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta).$$

- The results for the 4 helicity combinations (obtained in the same manner) are:



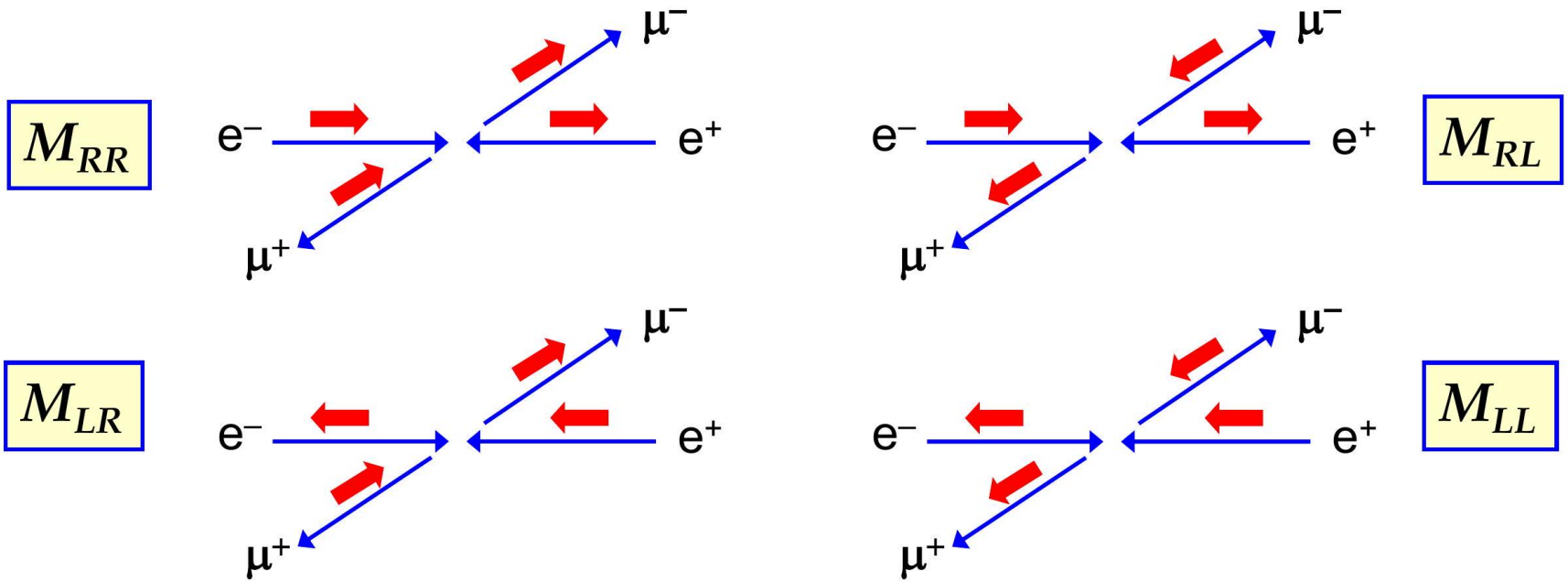
$$\bar{u}_{\uparrow}(p_3)\,\gamma^{\nu}\,v_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$$
$$\bar{u}_{\uparrow}(p_3)\,\gamma^{\nu}\,v_{\uparrow}(p_4) = (0,0,0,0)$$
$$\bar{u}_{\downarrow}(p_3)\,\gamma^{\nu}\,v_{\downarrow}(p_4) = (0,0,0,0)$$
$$\bar{u}_{\downarrow}(p_3)\,\gamma^{\nu}\,v_{\uparrow}(p_4) = 2E(0, -\cos\theta, -i, \sin\theta).$$

... in the limit $E \gg m$ which was used above only two helicity combinations are non-zero!

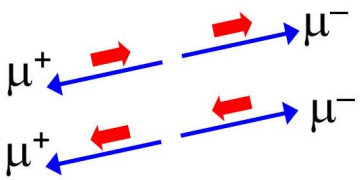
- This is an important feature of QED. It applies equally to QCD.
- In the Weak interaction only **one** helicity combination contributes when $E \gg m$.
- The origin of this will be discussed later (page 120).
- As a consequence: of the 16 possible helicity combinations only 4 give non-zero matrix elements ((2 initial) $\times$ (2 final)).

Electron Positron Annihilation cont.

- For $e^+e^- \to \mu^+\mu^-$ now only have to consider the 4 matrix elements:



- Previously we derived the muon currents for the allowed helicities:



$$\begin{aligned} \mu_R^- \mu_L^+ &: \quad \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta) \\ \mu_L^- \mu_R^+ &: \quad \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta) \end{aligned}$$

- Now need to consider the electron current

The Electron Current

- The incoming electron and positron spinors (L and R helicities) are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}; u_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}; \quad v_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}; v_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- The electron current can either be obtained from equations (3)-(6) as before or it can be obtained directly from the expressions for the muon current.

$$(j_e)^\mu = \bar{v}(p_2)\gamma^\mu u(p_1) \quad (j_\mu)^\mu = \bar{u}(p_3)\gamma^\mu v(p_4)$$

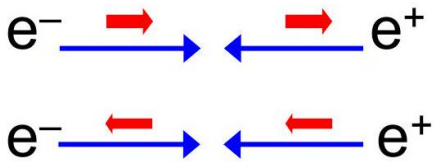
- Taking the Hermitian conjugate of the muon current gives

$$\begin{aligned}
[\bar{u}(p_3)\gamma^\mu v(p_4)]^\dagger &= \left[u(p_3)^\dagger \gamma^0\gamma^\mu v(p_4)\right]^\dagger \\
&= v(p_4)^\dagger \gamma^{\mu\dagger}\gamma^{0\dagger} u(p_3) && (AB)^\dagger = B^\dagger A^\dagger \\
&= v(p_4)^\dagger \gamma^{\mu\dagger}\gamma^0 u(p_3) && \gamma^{0\dagger} = \gamma^0 \\
&= v(p_4)^\dagger \gamma^0\gamma^\mu u(p_3) && \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu \\
&= \bar{v}(p_4)\gamma^\mu u(p_3)
\end{aligned}$$

- Taking the complex conjugate of the muon currents for the two non-zero helicity configurations:

$$\bar{v}_{\downarrow}(p_4)\,\gamma^{\mu}\,u_{\uparrow}(p_3) = [\bar{u}_{\uparrow}(p_3)\,\gamma^{\nu}\,v_{\downarrow}(p_4)]^{*} = 2E(0,-\cos\theta,-i,\sin\theta)$$
$$\bar{v}_{\uparrow}(p_4)\,\gamma^{\mu}\,u_{\downarrow}(p_3) = [\bar{u}_{\downarrow}(p_3)\,\gamma^{\nu}\,v_{\uparrow}(p_4)]^{*} = 2E(0,-\cos\theta,i,\sin\theta)$$

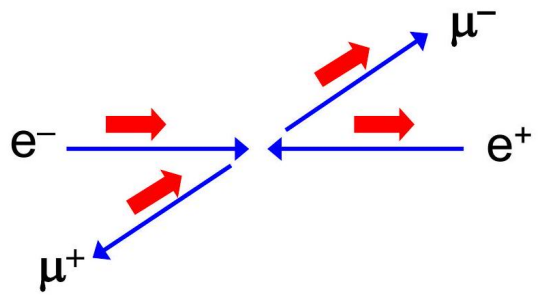
To obtain the electron currents we simply need to set $\theta = 0$



$$e_R^- e_L^+ : \quad \bar{v}_{\downarrow}(p_2)\gamma^{\nu}u_{\uparrow}(p_1) = 2E(0,-1,-i,0)$$
$$e_L^- e_R^+ : \quad \bar{v}_{\uparrow}(p_2)\gamma^{\nu}u_{\downarrow}(p_1) = 2E(0,-1,i,0)$$

Matrix Element Calculation

- We can now calculate $M = -\frac{e^2}{s} j_e \cdot j_\mu$ for the four possible helicity combinations M_{RR}, M_{RL}, M_{LR} and M_{LL}.
- In the above matrix element names, the first subscript refers to the helicity of the e^- and the second to the helicity of the μ^-. We don't need to specify other helicities due to 'helicity conservation', only certain chiral combinations are non-zero.
- E.g. the matrix element for $e_R^- e_L^+ \to \mu_R^- \mu_L^+$ will be denoted by M_{RR}



Using:

$$e_R^- e_L^+ : \quad (j_e)^\mu = \bar{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) = 2E(0,-1,-i,0)$$
$$\mu_R^- \mu_L^+ : \quad (j_\mu)^\nu = \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta, i, \sin\theta)$$

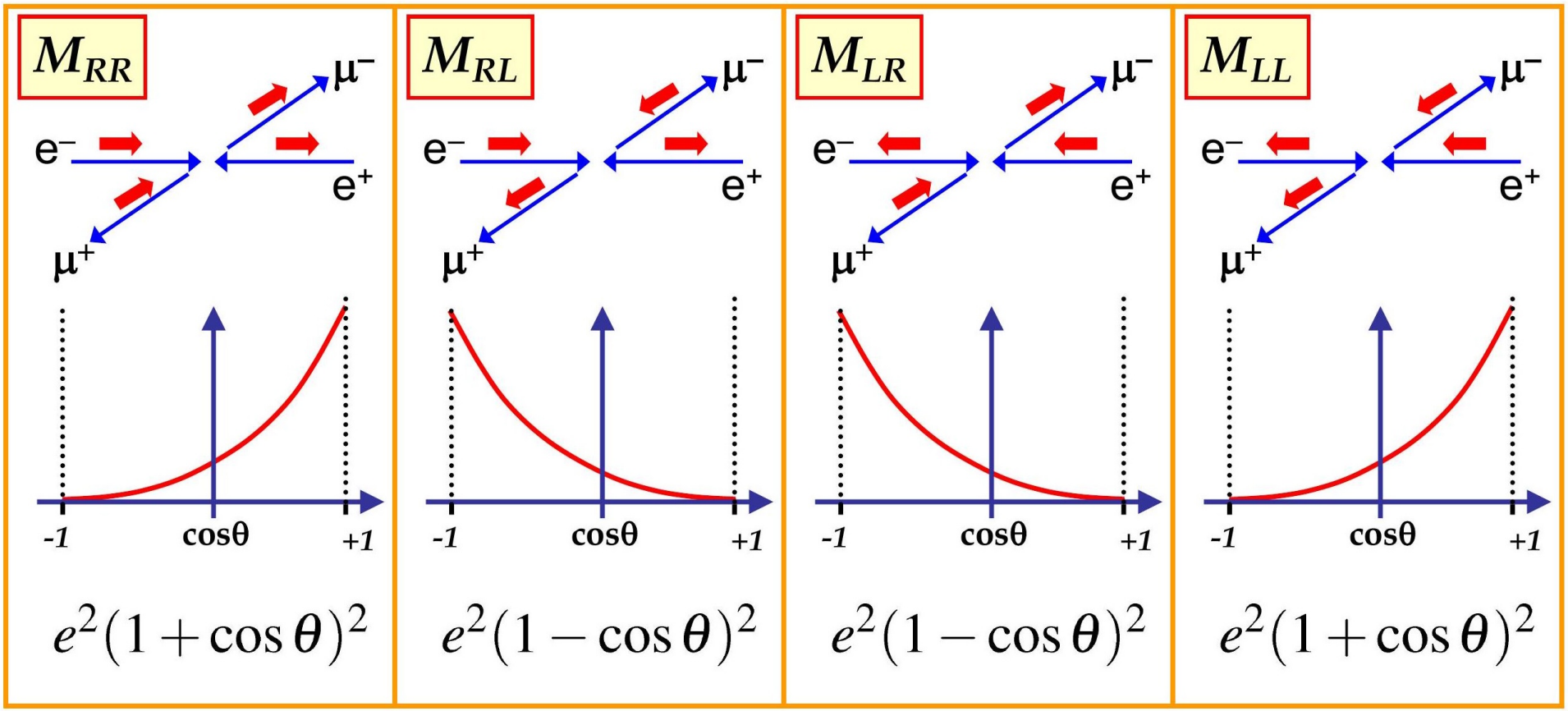
gives

$$\begin{aligned} M_{RR} &= -\frac{e^2}{s}[2E(0,-1,-i,0)] \cdot [2E(0,-\cos\theta, i, \sin\theta)] \\ &= e^2(1+\cos\theta) \\ &= 4\pi\alpha(1+\cos\theta) \end{aligned} \tag{57}$$

where $\alpha = e^2/4\pi \approx 1/137$.

Similarly

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$
$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$



- Assuming that the incoming electrons and positrons are unpolarized, all 4 possible initial helicity states are equally likely.

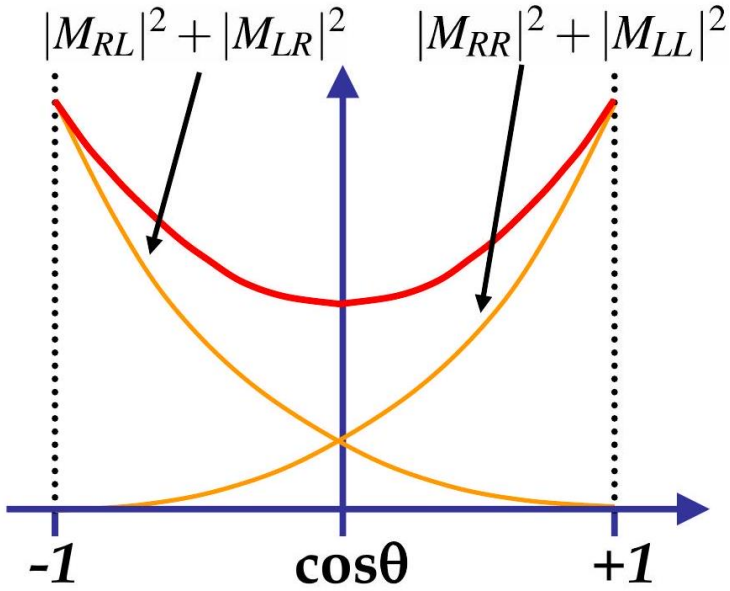
Differential Cross Section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

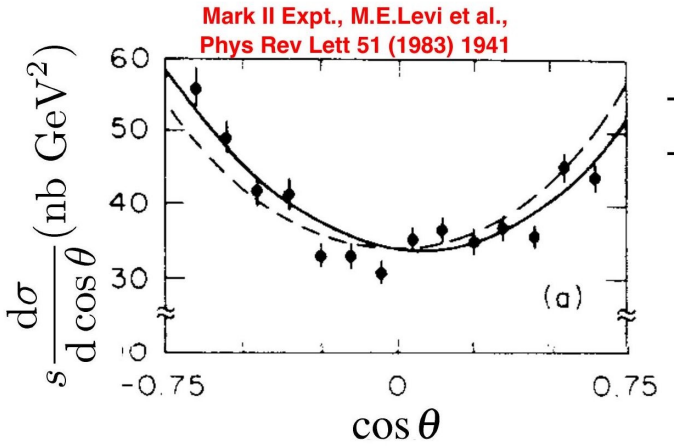
$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \times \frac{1}{64\pi^2 s}\left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + \left|M_{LL}^2\right|\right)$$

$$= \frac{(4\pi\alpha)^2}{256\pi^2 s}\left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$



Example: $e^+e^- \to \mu^+\mu^-$ $\sqrt{s} = 29$ GeV



- dashed line = pure QED
- solid line = QED plus Z contribution
- Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

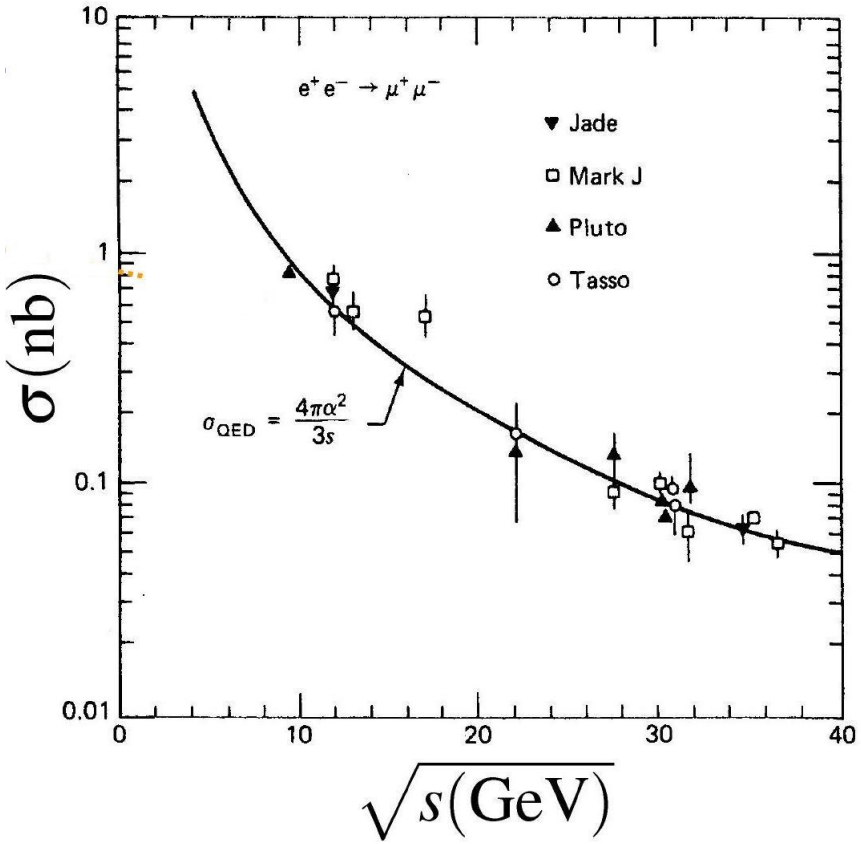
- The total cross section is obtained by integrating over θ, ϕ using

$$\int \left(1+\cos^2\theta\right) \mathrm{d}\Omega = 2\pi \int_{-1}^{+1} \left(1+\cos^2\theta\right) \mathrm{d}\cos\theta = \frac{16\pi}{3}.$$

- The QED total cross-section for the process $e^+e^- \to \mu^+\mu^-$ is found to be

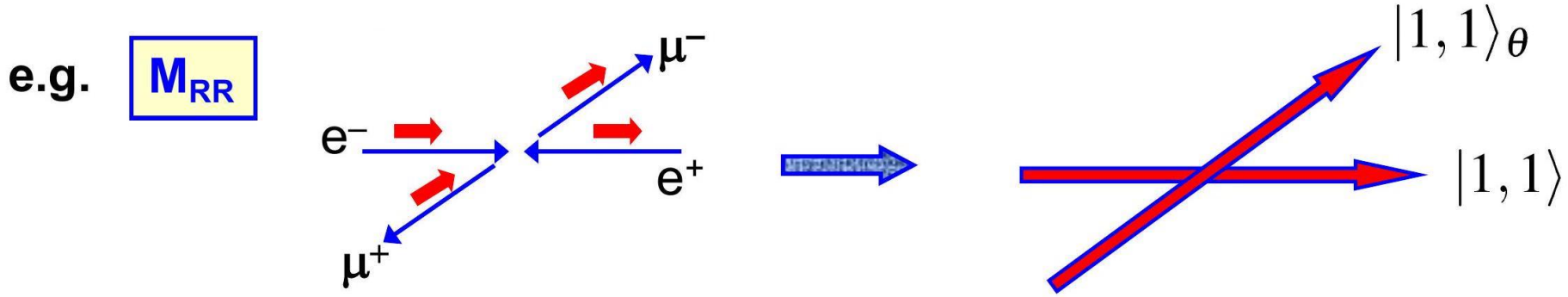
$$\sigma = \frac{4\pi\alpha^2}{3s}.$$

- This is an impressive result. From first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to $\sim 1\%$.



Spin Considerations ($E \gg m$)

- The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum
- Because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$, i.e. in a total spin 1 state aligned along the z axis: $|1, +1\rangle$ or $|1, -1\rangle$
- Similarly the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle θ



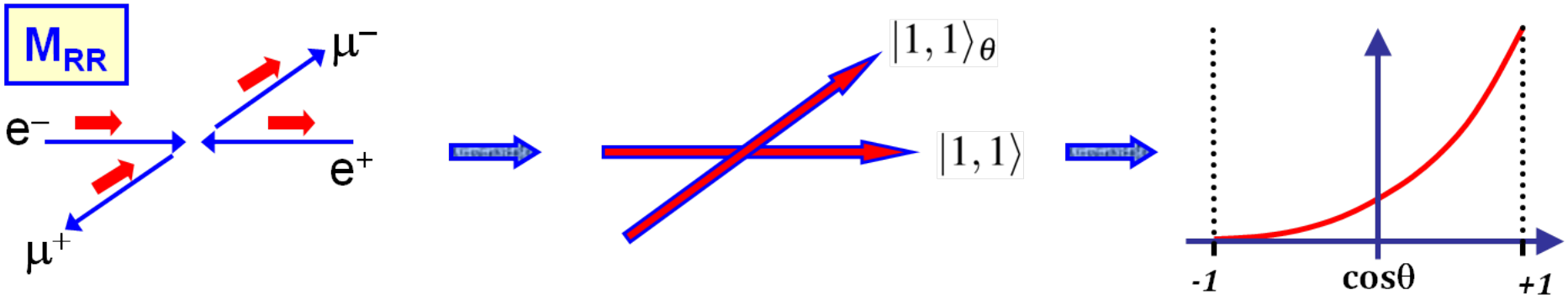
- Hence $M_{\mathrm{RR}} \propto \langle \psi \mid 1, 1 \rangle$ where ψ corresponds to the spin state, $|1, 1\rangle_\theta$, of the muon pair.
- To evaluate this need to express $|1, 1\rangle_\theta$ in terms of eigenstates of S_z
- In Appendix IX (and also in IB QM) it is shown that:

$$|1, 1\rangle_\theta = \frac{1}{2}(1 - \cos\theta)|1, -1\rangle + \frac{1}{\sqrt{2}} \sin\theta |1, 0\rangle + \frac{1}{2}(1 + \cos\theta)|1, +1\rangle.$$

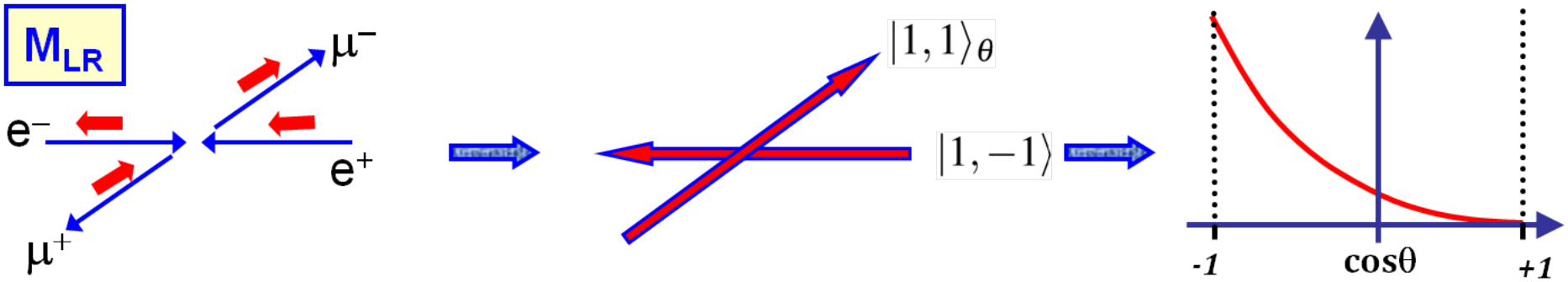
- Using the wave-function for a spin 1 state along an axis at angle θ

$$\psi = |1,1\rangle_\theta = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle$$

we can immediately understand the angular dependence:



$$|M_{\mathrm{RR}}|^2 \propto |\langle\psi \mid 1,+1\rangle|^2 = \frac{1}{4}(1+\cos\theta)^2$$

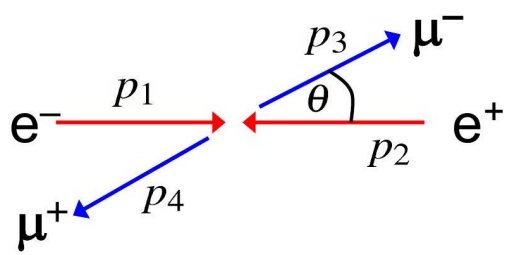


$$|M_{\mathrm{LR}}|^2 \propto |\langle\psi \mid 1,-1\rangle|^2 = \frac{1}{4}(1-\cos\theta)^2$$

Lorentz Invariant form of Matrix Element

- Before concluding this discussion, note that the spin-averaged Matrix Element derived above is written in terms of the muon angle in the C.o.M. frame.

$$\begin{aligned}\left\langle |M_{fi}|^2 \right\rangle &= \frac{1}{4} \times \left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + \left| M_{LL}^2 \right| \right) \\ &= \frac{1}{4} e^4 \left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2 \right) \\ &= e^4 \left(1 + \cos^2\theta \right)\end{aligned}$$



- The matrix element is Lorentz Invariant (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz Invariant 4-vector scalar products.
- In the C.o.M. $p_1 = (E, 0, 0, E)$ $p_2 = (E, 0, 0, -E)$

$$p_3 = (E, E\sin\theta, 0, E\cos\theta) \quad \text{and} \quad p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

giving: $p_1 \cdot p_2 = 2E^2$; $p_1 \cdot p_3 = E^2(1-\cos\theta)$; $p_1 \cdot p_4 = E^2(1+\cos\theta)$

- Hence we can write

$$\left\langle |M_{fi}|^2 \right\rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \left(\frac{t^2 + u^2}{s^2} \right)$$

which is valid in any frame!

Tidying up a source of frustration

QED scattering processes seem to look 'simpler' in the $E \gg m$ limit, but ugly when $E \sim m$. [E.g. page 109 showed that certain helicity combinations disappear when $E \gg m$.]

This motivates the following question:

Is there a better way of categorising particles so that scattering looks simple for **all energies**? (Not just nice for $E \gg m$!)

There is a nice answer:

QED scattering at all energies looks nice in the so-called **Chiral Basis**.

- The next few slides will explore what Chirality is.
- Even though we will see that Chirality is completely different to Helicity, we will see that Chirality happens to coincide with Helicity in the limit $E \gg m$.

$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and the CHIRAL projector operators P_R and P_L

Example sheet Question 10 should persuade you that the matrix γ^5 defined thus:

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3} \tag{58}$$

has the following properties for **any** representation of gamma matrices:

$$(\gamma^5)^2 = 1, \tag{59}$$

$$\gamma^{5\dagger} = \gamma^5, \qquad \text{(i.e. } \gamma^5 \text{ is Hermitian, } \textbf{and so might be observable}\text{)}, \tag{60}$$

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5, \quad \text{(i.e. } \gamma^5 \text{ anticommuites with all other } \gamma\text{-matrices)} \tag{61}$$

and that therefore the operators

$$\boxed{P_R = \frac{1}{2}(1+\gamma^5)} \text{ and } \boxed{P_L = \frac{1}{2}(1-\gamma^5)} \tag{62}$$

are complementary projection operators ($P_R^2 = P_R$, $P_L^2 = P_L$ and $P_R + P_L = 1$) satisfying

$$P_R^\dagger = P_R \tag{63}$$

$$P_L^\dagger = P_L \tag{64}$$

$$P_R\gamma^\mu = \gamma^\mu P_L \qquad \text{and} \tag{65}$$

$$P_L\gamma^\mu = \gamma^\mu P_R. \tag{66}$$

Aside: in the Dirac Representation:

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$P_R = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} \qquad \text{and} \tag{67}$$

$$P_L = \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{68}$$

The complementary projection operators P_R and P_L allow us to write any spinor ψ as the sum of **RIGHT** and **LEFT CHIRAL** parts ψ_R and ψ_L.

Definition:

$$\psi_R \equiv P_R\psi, \qquad \psi_L \equiv P_L\psi, \tag{69}$$

$$\bar{\psi}_R \equiv \overline{P_R\psi} \quad \text{and} \quad \bar{\psi}_L \equiv \overline{P_L\psi}. \tag{70}$$

Check that the chiral parts of each spinor satisfy

$$\psi \equiv \psi_R + \psi_L, \quad \text{and} \tag{71}$$

$$\bar{\psi} \equiv \bar{\psi}_R + \bar{\psi}_L \tag{72}$$

and are eigenstates of γ^5 with opposite chirality:

$$\gamma^5\psi_R = +\psi_R \qquad \text{and} \qquad \gamma^5\psi_L = -\psi_L. \tag{73}$$

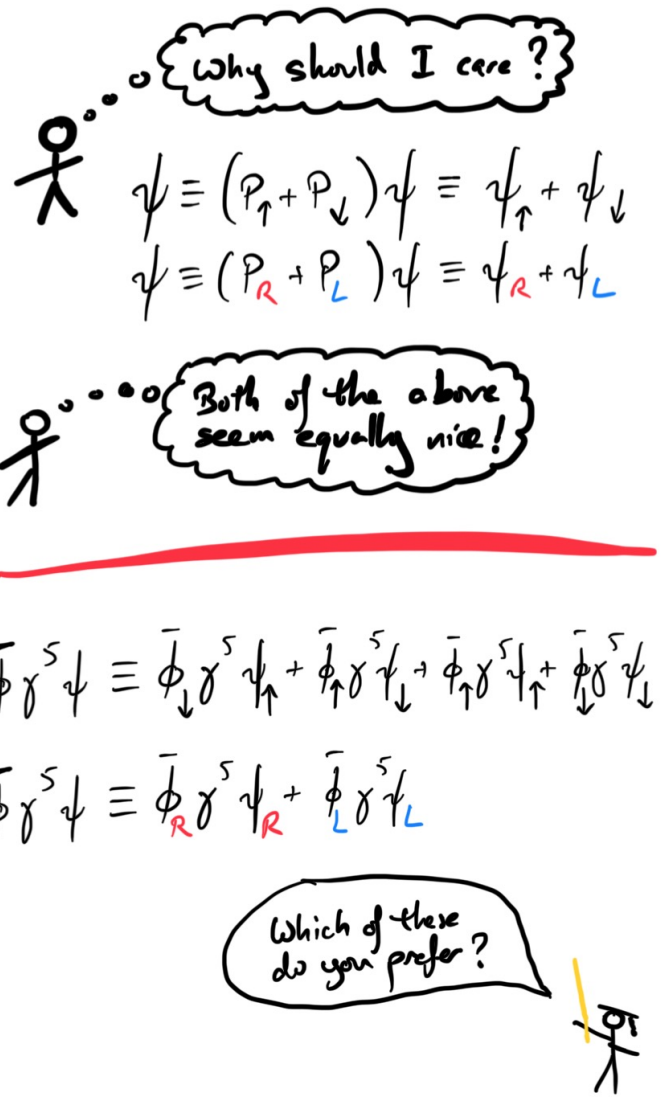
Example sheet Q10 asks you to prove that

$$\bar{\psi}_R \equiv \bar{\psi}P_L \tag{74}$$

$$\bar{\psi}_L \equiv \bar{\psi}P_R \tag{75}$$

and that therefore $\bar{\phi}_R\gamma^\mu\psi_L = \bar{\phi}_L\gamma^\mu\psi_R = 0$ and so

$$\boxed{\bar{\phi}\gamma^\mu\psi \equiv \bar{\phi}_R\gamma^\mu\psi_R + \bar{\phi}_L\gamma^\mu\psi_L}. \tag{76}$$



Link between HELICITY and CHIRALITY when $E \gg m$

- The helicity eigenstates for particles and anti-particles shown in (56) were:

$$u_\uparrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; u_\downarrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; v_\uparrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; v_\downarrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}.$$

- From P_R and P_L ((67), (68) and right) we note:

$$u_\uparrow\Big|_{E\gg m} = P_R u_\uparrow\Big|_{E\gg m}, \quad u_\downarrow\Big|_{E\gg m} = P_L u_\downarrow\Big|_{E\gg m},$$

$$v_\uparrow\Big|_{E\gg m} = P_L v_\uparrow\Big|_{E\gg m}, \quad v_\downarrow\Big|_{E\gg m} = P_R v_\downarrow\Big|_{E\gg m}.$$

Dirac Representation has:

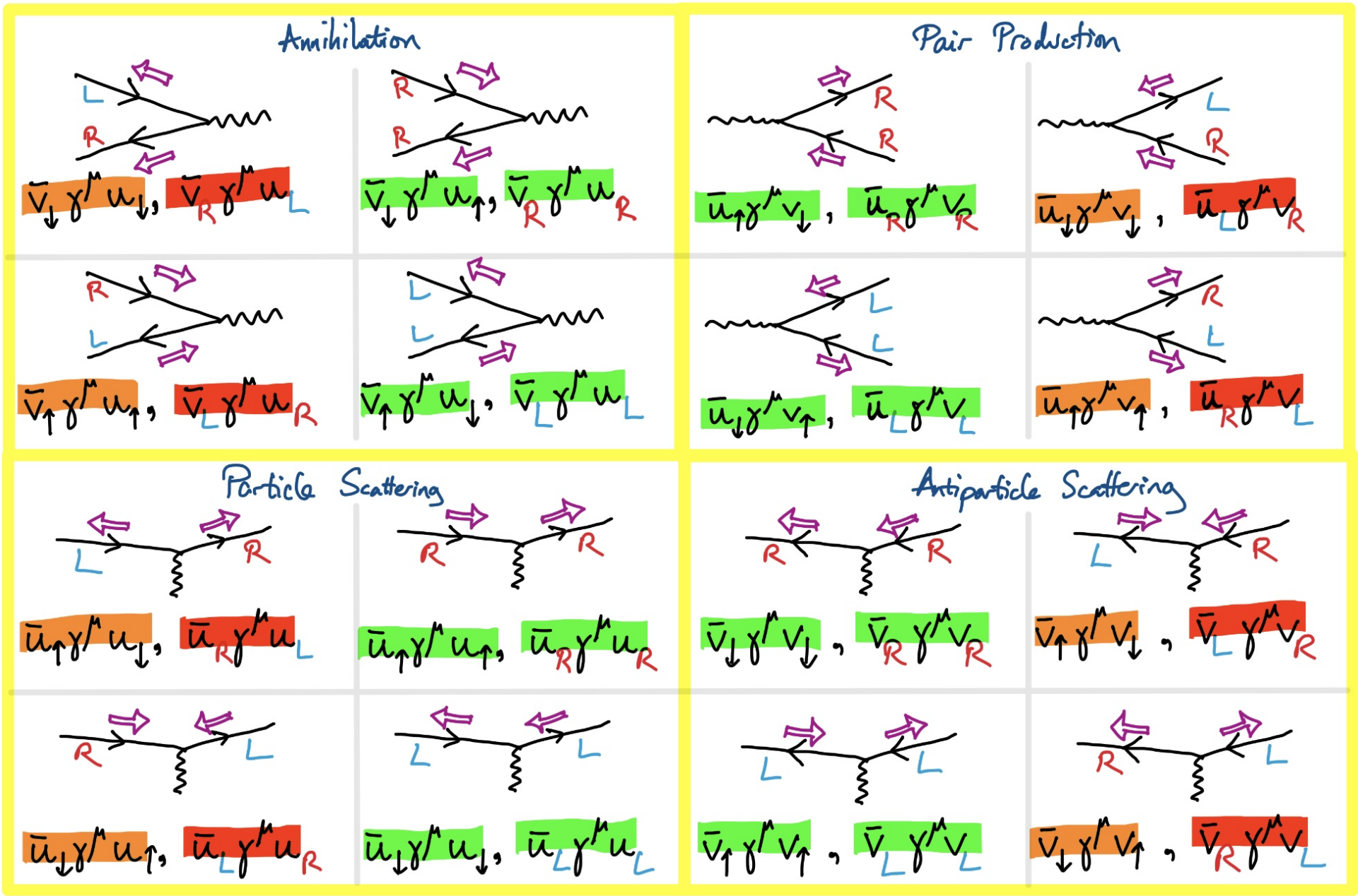
$$P_R = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} \quad \text{and}$$

$$P_L = \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Summary: in the high energy limit $E \gg m$ ...

Right Helicity Particles and Left Helicity Anti-Particles are Right Chiral, and
Left Helicity Particles and Right Helicity Anti-Particles are Left Chiral!

Favoured/Disfavoured/Forbidden QED Helicity and Chirality Combinations



Summary

- In the centre-of-mass frame the $e^+e^- \to \mu^+\mu^-$ differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right).$$

  NOTE: neglected masses of the muons, i.e. assumed $E \gg m_\mu$.
- In QED only certain combinations of LEFT- and RIGHT-HANDED CHIRAL states give non-zero matrix elements.
- CHIRAL states defined by chiral projection operators:

$$P_R = \frac{1}{2}\left(1+\gamma^5\right); \quad P_L = \frac{1}{2}\left(1-\gamma^5\right).$$

- In limit $E \gg m$ the chiral eigenstates correspond to the HELICITY eigenstates and only certain HELICITY combinations give non-zero matrix elements:

