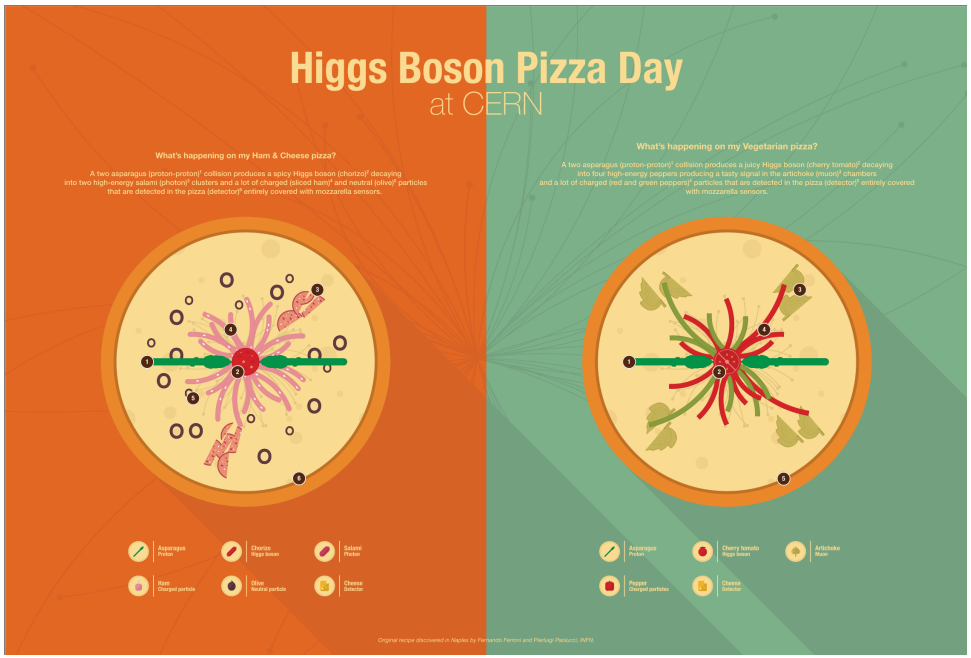
Part III Physics

Particle Physics

Prof. Alex Mitov

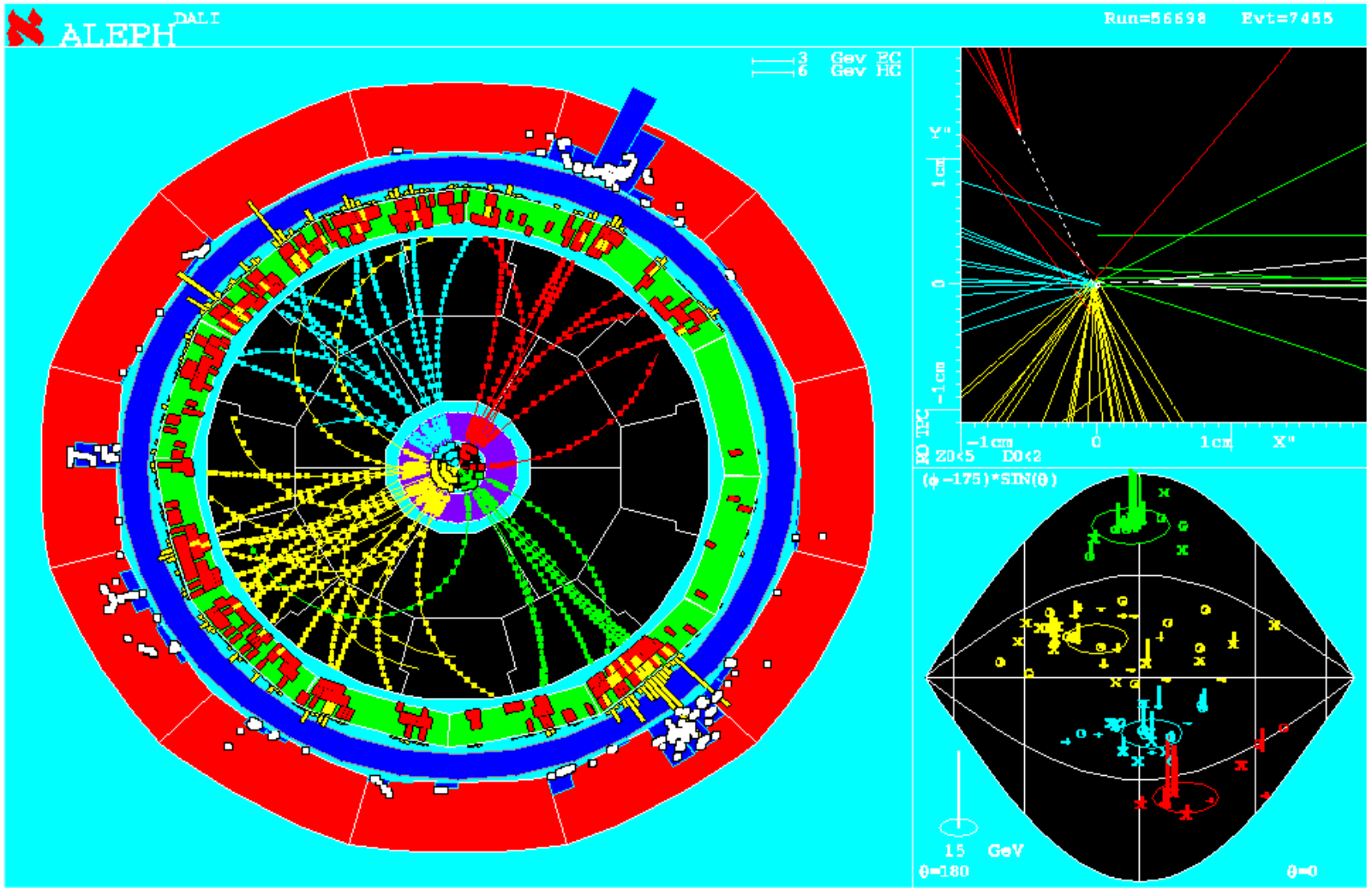




Sub-divisions (Handouts)

1. H1: Introduction
2. H2: The Dirac Equation
3. H3: Interaction by Particle Exchange and QED
4. H4: Electron-Positron Annihilation
5. H5: Electron-Proton Elastic Scattering
6. H6: Deep Inelastic Scattering
7. H7: Symmetries and the Quark Model
8. H8: Quantum Chromodynamics
9. H9: The Weak Interaction and $V - A$
10. H10: Leptonic Weak Interactions and Neutrino Deep Inelastic Scattering
11. H11: Neutrino Oscillations
12. H12: The CKM Matrix and CP Violation
13. H13: Electroweak Unification and the W and Z Bosons
14. H14: Precision Tests of the Standard Model
15. References
16. Appendices

Prof. Alex Mitov, 2026



H1: Introduction

Preliminaries

Web-page / Classes / Feedback / Correction

`https://www.precision.hep.phy.cam.ac.uk/people/mitov/teaching-pp`

- Course web-page above contains examples sheet, old exam papers and solutions, errata, lecturer's contact details, and (when ready) arrangements for examples classes ,*etc*.
- Please send corrections on course and/or feedback to me: `adm74@cam.ac.uk`

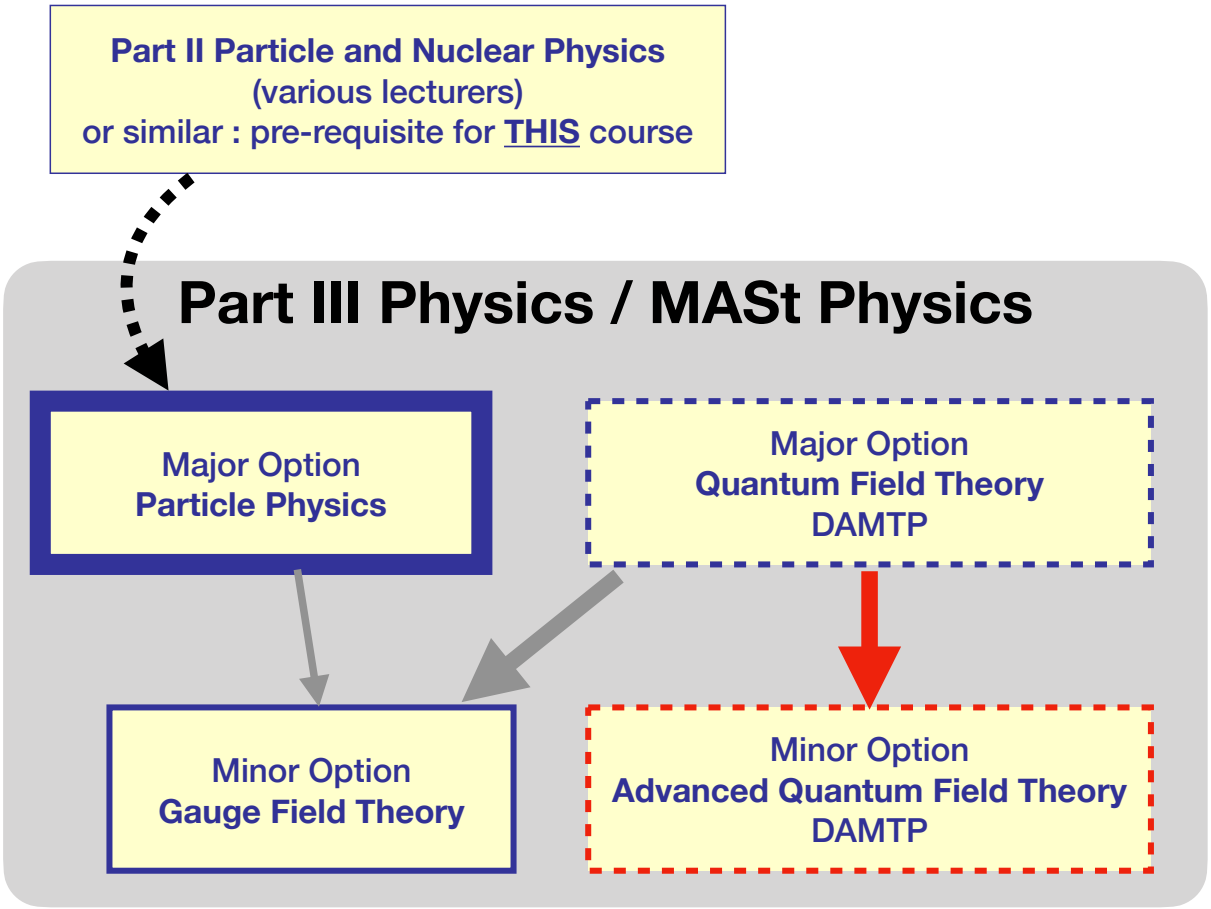
Format

- For historical reasons, the fourteen sections of the course are called 'handouts'. (List on page 2.)
- Some handouts contain additional theoretical background in non-examinable appendices.

Books

- **"Modern Particle Physics"**, Mark Thomson (Cambridge) BASED ON THIS COURSE!
- **"Particle Physics"**, Martin and Shaw (Wiley): fairly basic but good.
- **"Introductory High Energy Physics"**, Perkins (Cambridge): slightly below level of the course but well written.
- **"Introduction to Elementary Physics"**, Griffiths (Wiley): about right level but doesn't cover the more recent material.
- **"Quarks and Leptons"**, Halzen & Martin (Wiley): good graduate level textbook (slightly above level of this course).

Cambridge Particle Physics Courses



Aims of this course

The course is intended as an overview-style course.

It aims to provide:

- a context for the other more rigourous courses (QFT, AQFT, Gauge Field Theory),
- examples of the experiments and types of experimental evidence which have lead to our current underfstanding of The Standard Model, and
- 'just enough' of the theory to understand how/why the experiments constrain theory.

Level of rigour

Since the QFT, AQFT and Gauge Field Theory courses are either not yet lectured or are lectured in parallel, it is necessary for many results in this course to be presented without proof, or with only plausibility arguments, or with outline theoretical motivations. That will be dissatisfying for some taking the course – but are a necessary evil if this course is to complement those other courses.

Non-examinable material

What material is deemed to be **non-examinable** ?

- any slide content explicitly labelled as '**non-examinable**', or
- any slide content explicitly labelled as '**not examinable**', or
- any material located in an appendix of the course, whether or not it carries either of the labels above, unless it is explicitly labelled '**examinable**'.

What does it mean if material is deemed to be **non-examinable** ?

no student is expected or required to read, learn or revise any such material for Tripos; the exam should not *require* prior knowledge of such material.

Terms and conditions, small print, *etc.*

(i) It is not the case that a Tripos question could never have a domain overlapping with or including material deemed to be **non-examinable**. Such overlaps, in the rare cases in which they appear, simply indicate that an examiner has judged that material in the overlap can be reasonably deduced from material which was deemed to be examinable. Therefore, a more specific (though wordier) name for the material could be 'material-which-does-not-need-to-be-learned-or-revised'. (ii) In the event that material has been mis-labelled, a correction would be issued to the class (either by email or verbally in lectures) before the end of Michaelmas Term and then recorded on the course website. (iii) If in doubt about the status of any material, ask the lecturer for clarifications before the end of Michaelmas Term. (iv) Occasionally material from non-examinable sections will be discussed in lectures if the lecturer judges that it could be of benefit to some members of the audience. The discussion of such material in lectures does not change its status unless an official announcement to that effect is given.

Lecture Zero

The course proper begins on Wednesday!
Before then, here are a few things which fit nowhere else:

- Units.
- Assumed knowledge about Dirac δ-Functions.
- Standard Model - review.
- Special Relativity - things you should be familiar with.
- Why Mandelstam variables matter.

Units in Particle Physics

S.I. Units measure:

mass in kg, length in m, time in s, charge in C.

In principle, particle physicists using 'natural' units measure:

mass in GeV/c^2, length in $\hbar c/GeV$, time in $\hbar/GeV$, charge in $(\varepsilon_0 \hbar c)^{\frac{1}{2}}$.

NB: You could change GeV to MeV, TeV or any other eV-based energy unit without upsetting anyone at CERN.

Heaviside-Lorentz convention:

$c = \hbar = \varepsilon_0 = 1$ (and $\mu_0 = 1$ too since $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$)

In practice, particle physicists using natural units measure:

mass in GeV, length in $1/GeV$, time in $1/GeV$, and charge is *dimensionless* on account of using that Heaviside-Lorentz convention!

$1 = \hbar c = 197$ MeV fm : (the one unit-based fact memorized in CERN)

Lazy particle physicists correct their omissions of $\hbar$ and c factors with the following tricks: They get the "*GeV*"s on the bottom/top correctly they remembering that:

- they like **high** energies and **high** momenta: ..... so these are GeV
- they like **small** lengths and **small** times: ..... so these are GeV^{-1}

To help them recover c and $\hbar$ factors omitted from energies they (mostly) use the following *aides-mémoire*:

- (for mass): $E \sim mc^2$, ... so mass is $\propto GeV/c^2$,
- (for momentum): $E \sim (mc)(c) \sim pc$, ... so momentum is $\propto GeV/c$,
- (for time): $\Delta E \Delta t \sim \hbar$, ... so time is $\propto \hbar/GeV$,
- (for length): $1 = \hbar c \approx 197\ MeV \cdot fm$, ... so lenth is $\propto \hbar c/GeV$,

and to get proper S.I. units like Joules and meters:

- (energies to Joules): $eV \approx 1.60 \times 10^{-19} J$,
- (lenths to meters): $1 = \hbar c \approx 197\ MeV \cdot fm$.

Easy to remember: the previous lecturer's height in centimeters ...

Assumed knowledge concerning Dirac δ-functions of one real variable:

It is assumed you know this:

$$\int_X g(x)\delta(u(x))dx = \int g(x(u))\delta(u)\left|\frac{dx}{du}\right| du = \sum_{x\in X \text{ s.t. } u(x)=0} \frac{g(x)}{\left|\frac{du}{dx}\right|}, \tag{1}$$

If the above is not familiar to you, convince yourself of it using other sources.

Examples of the above:

$$\int_{-\infty}^{\infty} \sin x\ \delta(x-a)dx = \sin a, \qquad \text{(trivial example)}$$

$$\int_{-\infty}^{\infty} e^x \delta(x^2-a^2)dx = \sum_{x=\pm a} \frac{e^x}{|2x|} = \frac{e^a}{|2(a)|} + \frac{e^{-a}}{|2(-a)|} = \frac{1}{2|a|}(e^a+e^{-a}) = \frac{1}{|a|}\cosh a.$$

The second example is presumably non-trivial, as many past students of the course have mistakenly produced other answers for it like $e^a + e^{-a}$ or $2\cosh a$.

The generalisation to two or more variables ...

Not examinable

This is not examinable, but the generlisations of the result (1) to Dirac δ-functions of more than one variable would be as follows:

Two variables:

$$\int_X g(x,y)\delta(u(x,y))\delta(v(x,y))dxdy = \int g(x(u,v),y(u,v))\delta(u)\delta(v)\left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| dudv$$
$$= \sum_{(x,y)\in X \text{ s.t. } u(x,y)=v(x,y)=0} \frac{g(x,y)}{\left|\left|\frac{\partial(u,v)}{\partial(x,y)}\right|\right|}.$$

Many variables:

$$\int_X g(\vec{x})\delta^n(\vec{u}(\vec{x}))d^n x = \int g(\vec{x}(\vec{u}))\delta^n(\vec{u})\left|\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)}\right|\right| d^n u = \sum_{\vec{x}\in X \text{ s.t. } \vec{u}(\vec{x})=0} \frac{g(\vec{x})}{\left|\left|\frac{\partial(u_1,\ldots,u_n)}{\partial(x_1,\ldots,x_n)}\right|\right|}.$$

Not examinable

Reminder: Matter in the Standard Model

The Standard Model's point-like spin-1/2 fermions:

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	-1	0.0005	d	$-1/3$	~ 0.3
	ν_1	0	≈ 0	u	$+2/3$	~ 0.3
Second Generation	μ^-	-1	0.106	s	$-1/3$	~ 0.5
	ν_2	0	≈ 0	c	$+2/3$	~ 1.5
Third Generation	τ^-	-1	1.77	b	$-1/3$	~ 4.5
	ν_3	0	≈ 0	t	$+2/3$	173

Open Questions (not addressed in this course!) :

- Why are there three generations, rather than some other number?
- Why are there generations at all?
- Why are the neutrinos so light compared to the others? (e.g. ν_1 has a non-zero mass below ~ 3 eV)
- Are the neutrinos Majorana?
- Why are there no particles that could explain Dark Matter?

Reminder: Forces in the Standard Model

Spin-1 Gauge Bosons mediate the four forces of the SM:

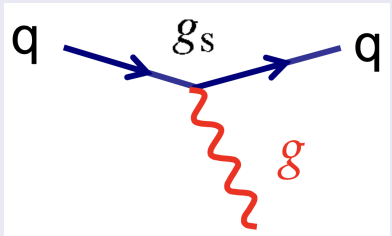
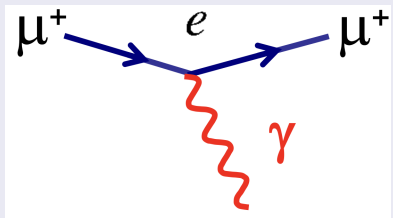
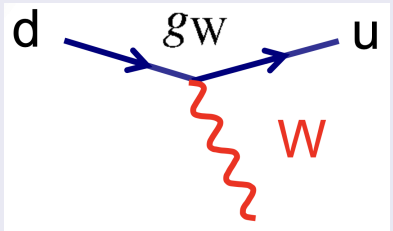
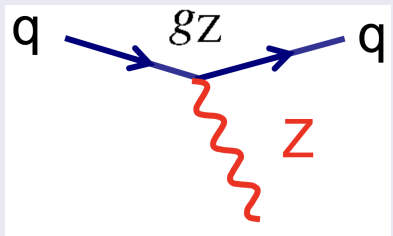
Force	Boson(s)	J^P	m/GeV	Charge "g"	coupling constant "α"	Notes
EM (QED)	Photon γ	1^-	0	e	α	Handout 3
Weak	$W^{\pm}/Z$	1^-	80/91	g_W, g_Z	α_W, α_Z	Handout 13
Strong (QCD)	8 Gluons g	1^-	0	g_s	α_s	Handout 8
Gravity	Graviton?	2^+ ?	0 ?	?	?	Not in SM!

Charges g vs Coupling Constants α

- Coupling constants α are always dimensionless whereas charges g might have strange units.
- $g = \sqrt{4\pi\alpha}$ is true for all forces, but only in natural units with Heaviside Lorentz convention.
- $g = \sqrt{4\pi\alpha\epsilon_0\hbar c} = e$ for QED without Heaviside Lorentz convention.
- Since always dimensionless, the coupling constant α sets the intrinsic strength of each force and does not depend on choice of units.

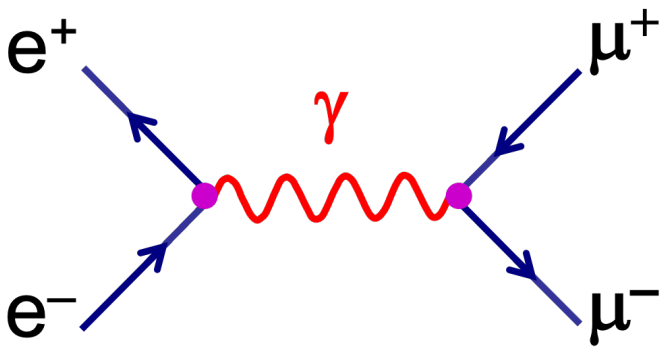
Reminder: Standard Model Vertices

Although much more detail will be provided later, it will be assumed that you have some partial knowledge of the following properties of the SM interactions:

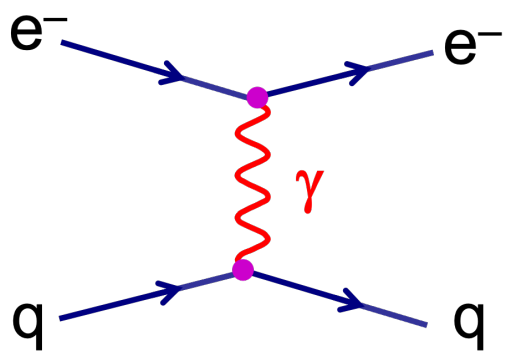
Strong	Electromagnetic	Weak Charged Current (CC)	Weak Neutral Current (NC)
			
Only quarks	All charged fermions	All fermions	All fermions
Never changes flavour	Never changes flavour	Always changes flavour	Never changes flavour
$\alpha_S \sim 1$	$\alpha \simeq 1/137$	$\alpha_W \sim 1/40$	$\alpha_Z \sim 1/40$

Feynman Diagram conventions

electron-positron annihilation example:



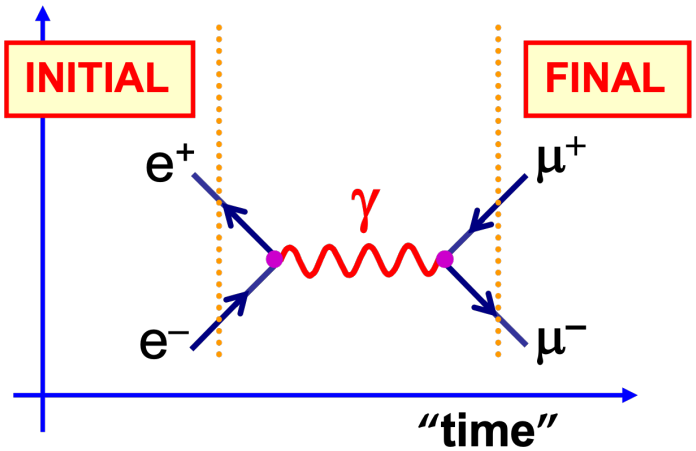
electron-quark scattering example:



Usually in this course: 'time' runs from left to right, but only in the sense that:

- LHS of diagram is initial state
- RHS of diagram is final state
- Middle is how process 'might' have happened

Anti-particles are indicated by arrows in the negative 'time' direction.



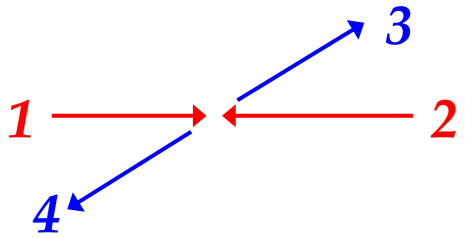
Much more detail appears in Handout 3 and onwards!

Special Relativity and 4-Vector Notation

- Will use 4-vector notation with t as the time-like component, e.g.
  $p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\}$ (contravariant)
  $p_\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\}$ (covariant)
  with
  $$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
- In particle physics, usually deal with relativistic particles. Require all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.
  $p^\mu p_\mu = E^2 - p_x^2 - p_y^2 - p_z^2 = E^2 - p^2 = m^2$ (Invariant Mass)
  $x^\mu p_\mu = Et - p_x x - p_y y - p_z z = Et - \vec{p} \cdot \vec{r}$ (Phase)
- A few words on NOTATION: Four vectors are written as either: p^μ or p. Four vector scalar products as either: $p^\mu p_\mu$ or $p \cdot r$. Three vectors are written as: $\vec{p}$. Quantities evaluated in the centre of mass frame are written: $\vec{p}^*$ or p^* etc.

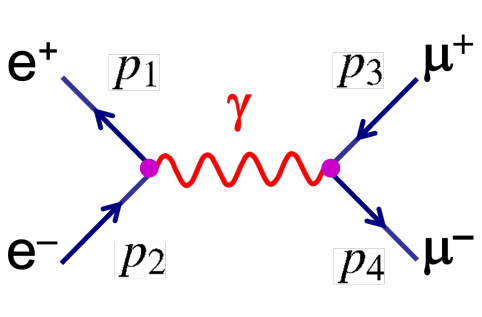
The Mandelstam Variables: s, t, and u

Scattering processes like $1 + 2 \to 3 + 4$ are often characterised by three useful real Lorentz Invariant quantities: s, t, and u. These are built from the squares of particular sums or differences of the Lorentz four-vectors p_1, p_2, p_3 and p_4:

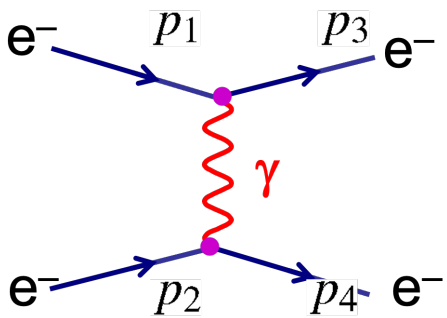


$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2$$

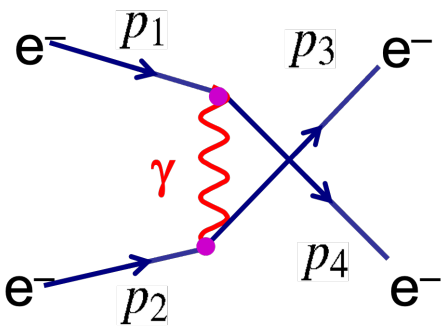
Simple Feynman diagrams for two-to-two processes are named s-, t- or u-channel according to the squared four-momentum of the exchanged particle:



s-channel



t-channel



u-channel

Why care about s, t and u ?

Only two of s, t and u are independent if the masses of the four external particles are known.

Proof is in example sheet question 2 which will ask you to show that

$$s + t + u \equiv m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Without loss of generality, take s and t to be the pair of independent variables.

s contains information about the initial state, and t contains information about the final state.

Therefore, t encodes information about the interaction that occurred.

Hardly surprising then that energy in centre-of-mass frame is $\sqrt{s}$

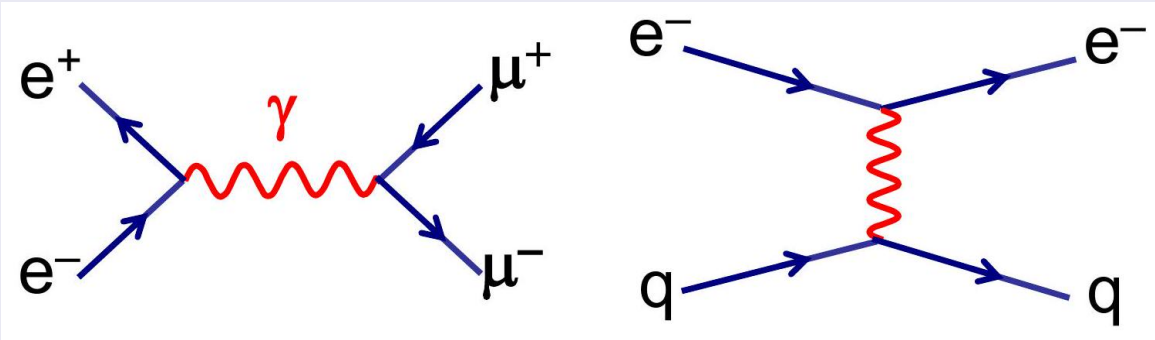
s is Lorentz invariant so takes same value in any frame. In the c.o.m. frame $p_1^* = (E_1, \vec{p}^*)$ and $p_2^* = (E_2, -\vec{p}^*)$ so $s = (E_1 + E_2, \vec{0})^2 = (\text{centre-of-mass energy})^2$.

The first five lectures

Aiming towards proper calculations of decay rates and scattering probabilities

Will concentrate on:

- $e^+e^- \to \mu^+\mu^-$
- $e^-q \to e^-q$ (to probe proton structure)



We will need:

- relativistic calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{ phase space })$$

- relativistic treatment of spin-half particles: (see Dirac Equation, etc, Handout 2)
- relativistic calculation of interaction Matrix Element

Master formula for computing decay rates

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \, \delta\left(E_i - E_1(p_1) - E_2(p_2)\right) \delta^3\left(\vec{p}_i - \vec{p}_1 - \vec{p}_2\right) \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}. \tag{2}$$

- It follows from Fermi's Golden Rule:

$$\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho\left(E_f\right)$$

- $\rho\left(E_f\right)$: the phase space (density of final states); in the formula above, the δ-functions and the $\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E}$ factors.
- M_{fi}: the transition amplitude, normalised to $2E$ particles per unit volume: $M_{fi} = \left(2E_i\, 2E_1\, 2E_2\right)^{1/2} T_{fi}$. It is Lorentz invariant.
- The formula is written for a two-body final state; each additional final-state particle brings one more factor $\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E}$.

Full details can be found in Appendix I (page 489)

Decay Rate Calculations

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \, \delta\left(E_i - E_1(p_1) - E_2(p_2)\right) \delta^3\left(\vec{p}_i - \vec{p}_1 - \vec{p}_2\right) \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}.$$

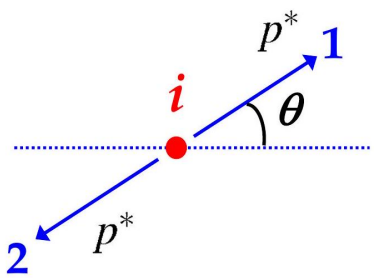
The integral is Lorentz invariant (i.e. frame independent). Therefore, it can be evaluated in any frame we choose. The C.o.M. frame is most convenient.

- In the C.o.M. frame $E_i = m_i$ and $\vec{p}_i = 0 \Rightarrow$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \, \delta\left(m_i - E_1(p_1) - E_2(p_2)\right) \delta^3\left(\vec{p}_1 + \vec{p}_2\right) \frac{\mathrm{d}^3\vec{p}_1}{2E_1(p_1)} \frac{\mathrm{d}^3\vec{p}_2}{2E_2(p_2)}.$$

- Integrating over $\vec{p}_2$ using the δ-function:

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \, \delta\left(m_i - E_1(p_1) - E_2(p_1)\right) \frac{\mathrm{d}^3\vec{p}_1}{4E_1(p_1)E_2(p_1)}$$



(note $E_2(p_1) = \sqrt{m_2^2 + |\vec{p}_1|^2}$ since the δ-function imposes $\vec{p}_2 = -\vec{p}_1$) ... and so writing $|\vec{p}_1|$ as p_1:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \, \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 \, \mathrm{d}p_1 \, \mathrm{d}\Omega}{E_1(p_1)E_2(p_1)}. \quad (3)$$

- The last expression for Γ_{fi} can be written in the form

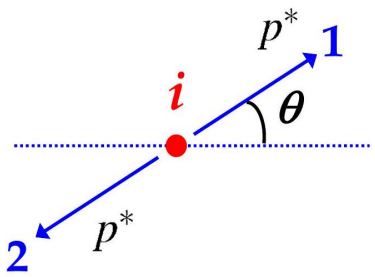
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \, g(p_1) \, \delta(f(p_1)) \, \mathrm{d}p_1 \, \mathrm{d}\Omega \tag{4}$$

where $g(p_1) = p_1^2/(E_1 E_2) = p_1^2 \left(m_1^2 + p_1^2\right)^{-1/2} \left(m_2^2 + p_1^2\right)^{-1/2}$

and $f(p_1) = m_i - \left(m_1^2 + p_1^2\right)^{1/2} - \left(m_2^2 + p_1^2\right)^{1/2}$.

Note:

- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products i.e. $f(p_1) = 0$ for $p_1 = p^*$.
- Eq. (4) can be integrated using the property of δ-function derived earlier (equation (1))



$$\int g(p_1) \, \delta(f(p_1)) \, \mathrm{d}p_1 = \frac{1}{|\,\mathrm{d}f/\mathrm{d}p_1|_{p_1=p^*}} \int g(p_1) \, \delta(p_1 - p^*) \, \mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p_1=p^*}}$$

where p^* is the value for which $f(p^*) = 0$.

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{(m_1^2 + p_1^2)^{1/2}} - \frac{p_1}{(m_2^2 + p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1 \frac{E_1 + E_2}{E_1 E_2}.$$

Thus (3) becomes:
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{E_1 E_2}{p_1 (E_1 + E_2)} \frac{p_1^2}{E_1 E_2} \right|_{p_1 = p^*} \mathrm{d}\Omega$$
$$= \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{E_1 + E_2} \right|_{p_1 = p^*} \mathrm{d}\Omega.$$

But from $f(p_1) = 0$, i.e. energy conservation: $E_1 + E_2 = m_i$ so

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i} \int |M_{fi}|^2 \ \mathrm{d}\Omega$$

which, in the particle's rest frame $E_i = m_i$, becomes the important result:

$$\boxed{\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \ \mathrm{d}\Omega} \tag{5}$$

and this is VALID FOR ALL TWO-BODY DECAYS !

- p^* can be obtained by solving $f(p^*) = 0$. This gives:

$$p^* = \frac{1}{2m_i} \sqrt{\left[m_i^2 - (m_1 + m_2)^2\right] \left[m_i^2 - (m_1 - m_2)^2\right]}. \quad \text{(Question 4)}$$

Now try questions 5 and 6.

Cross section definition

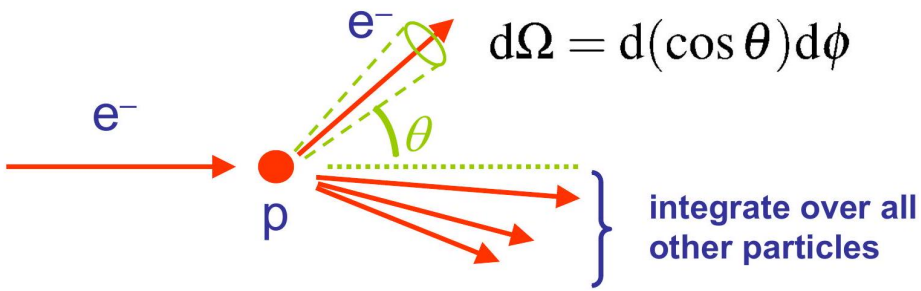
$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}} \tag{6}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the effective crosssectional area of the target particles for the interaction to occur.
- It is a property of the pair of things interacting, rather than of either one singly.
- In general this has nothing to do with the physical size of the target although there are exceptions. (E.g. the cross section for neutron absorption on nucleii is the projective area of the nucleus!)

Differential Cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$



$$d\Omega = d(\cos\theta)d\phi$$

with $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$

Master formula for computing cross sections

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \, \delta(E_1 + E_2 - E_3 - E_4) \, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \, \frac{\mathrm{d}^3\vec{p}_3}{2E_3} \frac{\mathrm{d}^3\vec{p}_4}{2E_4} \tag{7}$$

- It follows from Fermi's Golden Rule: σ is the transition rate per unit volume and incident flux,
$$\sigma = \frac{\Gamma_{fi}}{v_1 + v_2}, \qquad \Gamma_{fi} = 2\pi \, |T_{fi}|^2 \, \rho(E_f)$$
- $\rho(E_f)$: the phase space (density of final states); in the formula above, the δ-functions and the $\frac{\mathrm{d}^3\vec{p}}{2E}$ factors.
- M_{fi}: the transition amplitude, normalised to $2E$ particles per unit volume: $M_{fi} = (2E_1\, 2E_2\, 2E_3\, 2E_4)^{1/2} \, T_{fi}$. It is Lorentz invariant.
- The formula is written for a two-body final state; each additional final-state particle brings one more factor $\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E}$.
- $F = 2E_1 2E_2 (v_1 + v_2)$: the flux factor, which can be cast in a Lorentz invariant form:

Full details can be found in Appendix II (page 509)

(See also Appendix III, page 515)

(Longitudinally) Lorentz Invariant Flux

$$F \underset{\text{def}}{\equiv} 4\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2} = (2E_1)(2E_2)(v_1 + v_2)$$

is called the Lorentz Invariant Flux.

As it is Lorentz Invariant, it can be evaluated easily in different frames to get different looking (but ultimately identically valued) expressions for it. E.g:

Centre of Mass Frame

$$\begin{aligned} F &= 4E_1^* E_2^* (v_1^* + v_2^*) \\ &= 4E_1^* E_2^* \left(\frac{|\underline{p}_1^*|}{E_1^*} + \frac{|\underline{p}_2^*|}{E_2^*}\right) \\ &= 4\left(E_2^* + E_1^*\right)|\underline{p}^*| \\ &= 4|\underline{p}^*|\sqrt{s} \end{aligned}$$

(Q)

($\sqrt{s}$: Mandlestam s)

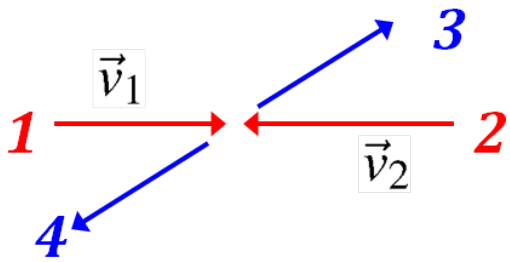
Fixed Target Frame

$$\begin{aligned} F &= 4E_1 E_2 (v_1 + v_2) \\ &= 4E_1 m_2 (v_1 + 0) \\ &= 4E_1 m_2 \frac{|\underline{p}_1|}{E_1} \\ &= 4m_2 |\underline{p}_1| \end{aligned}$$

(R)

$2 \to 2$ Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider $2 \to 2$ scattering in C.O.M. frame



- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \, \delta(E_1 + E_2 - E_3 - E_4) \, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \, \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4}$$

- and then note that in the C.o.M.-frame $\vec{p}_1 + \vec{p}_2 = 0$ and $E_1 + E_2 = \sqrt{s}$ so

$$\sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \, \delta(\sqrt{s} - E_3 - E_4) \, \delta^3(\vec{p}_3 + \vec{p}_4) \, \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4}. \tag{8}$$

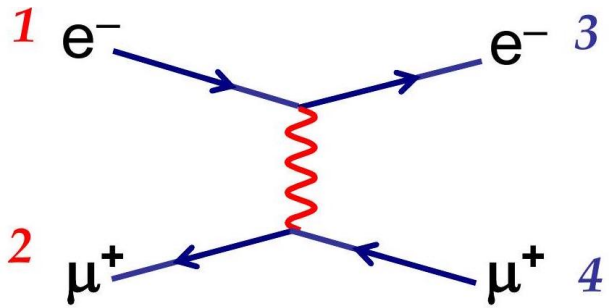
- The integral in (8) is exactly the same as the integral (3) which appeared in the one-to-two particle decay calculation, except that it has m_a replaced by $\sqrt{s}$. So:

$$\sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 \, d\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 \, d\Omega^*} \tag{9}$$

- In the case of elastic scattering $|\vec{p}_i^{\,*}| = |\vec{p}_f^{\,*}|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 \; \mathrm{d}\Omega^*$$

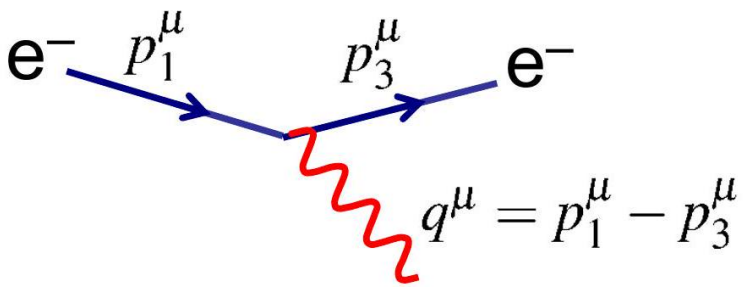


- For calculating the total cross-section (which is Lorentz Invariant) the result (9) on the previous page is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |M_{fi}|^2 \; \mathrm{d}\Omega^* \tag{10}$$

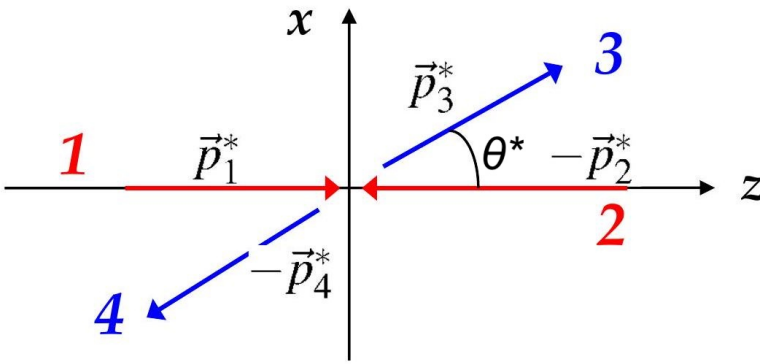
because the angles in $\mathrm{d}\Omega^* = \mathrm{d}\left(\cos\theta^*\right)\mathrm{d}\phi^*$ refer to the C.o.M frame.

- For the last calculation in this section, we need to find a L.I. expression for $d\sigma$.
- Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam $t = (p_1 - p_3)^2$ ... i.e. the square of the four-momentum transfer:



- Want to express $\mathrm{d}\Omega^*$ in terms of Lorentz Invariant $\mathrm{d}t$ where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2p_1 \cdot p_3$.
- In C.o.M. frame:

$$
\begin{aligned}
p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^{\,*}|) \\
p_3^{*\mu} &= (E_3^*, |\vec{p}_3^{\,*}| \sin\theta^*, 0, |\vec{p}_3^{\,*}| \cos\theta^*) \\
p_1^{\mu} p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^{\,*}|\,|\vec{p}_3^{\,*}| \cos\theta^* \\
t &= m_1^2 + m_3^3 - 2E_1^* E_3^* + 2\,|\vec{p}_1^{\,*}|\,|\vec{p}_3^{\,*}| \cos\theta^*
\end{aligned}
$$



giving $\mathrm{d}t = 2\,|\vec{p}_1^{\,*}|\,|\vec{p}_3^{\,*}|\,\mathrm{d}(\cos\theta^*)$

therefore $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\,\mathrm{d}\phi^* = \frac{\mathrm{d}t\ \mathrm{d}\phi^*}{2|\vec{p}_1^{\,*}||\vec{p}_3^{\,*}|}$

hence $\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3^{\,*}|}{|\vec{p}_1^{\,*}|}\,|M_{fi}|^2\ \mathrm{d}\Omega^* = \frac{1}{2\cdot 64\pi^2 s|\vec{p}_1^{\,*}|^2}\,|M_{fi}|^2\ \mathrm{d}\phi^*\ \mathrm{d}t.$

- Finally, integrating over $\mathrm{d}\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s\,|\vec{p}_i^{\,*}|^2}\,|M_{fi}|^2}.$$

Lorentz Invariant differential cross section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^{\,*}|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^{\,*}|^2 = \frac{1}{4s}\left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$



e.g. electron or neutrino scattering

In this limit

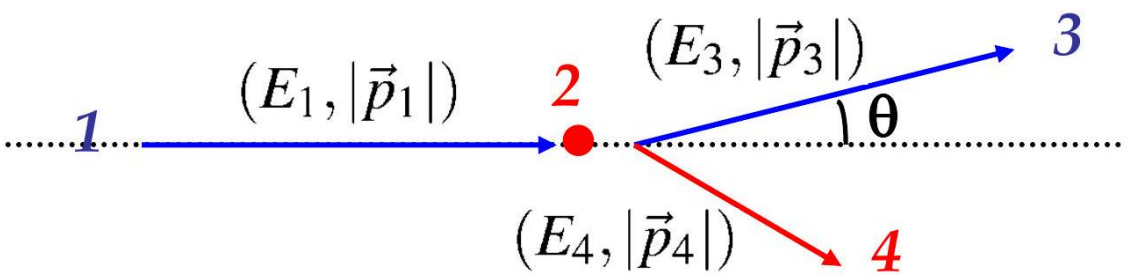
$$|\vec{p}_i^{\,*}|^2 = \frac{\left(s-m_2^2\right)^2}{4s}$$

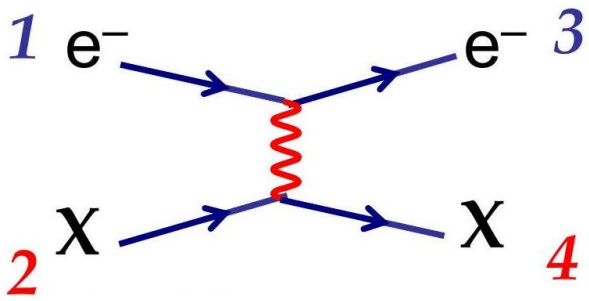
and so

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi\left(s-m_2^2\right)^2}\left|M_{fi}\right|^2} \quad (m_1=0)\,. \tag{11}$$

$2 \to 2$ body fixed-target scattering using lab-frame angles

- A commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, \quad m_2 = m_4 = M$





- Wish to express the cross section in terms of scattering angle of the e^-

$$\mathrm{d}\Omega = 2\pi \mathrm{d}(\cos\theta)$$

$$\text{therefore} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

- The rest is some rather tedious algebra.... start from four-momenta:

$$p_1 = (E_1, 0, 0, E_1)\,, p_2 = (M, 0, 0, 0), p_3 = (E_3, E_3 \sin\theta, 0, E_3\cos\theta)\,, p_4 = (E_4, \vec{p}_4)$$

$$\text{so here } t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 = -2E_1E_3(1 - \cos\theta)$$

But from (E, p) conservation $p_1 + p_2 = p_3 + p_4$ and so we can also express t in terms of particles 2 and 4.

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2 \cdot p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

Equating the two expressions for t gives $\boxed{E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}}$ and so

$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

and $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s - M^2)^2}|M_{fi}|^2$$

(using (11) in the last step). Thus given that $s = (p_1 + p_2)^2 = M^2 + 2p_1 \cdot p_2 = M^2 + 2ME_1$ we have $(s - M^2) = 2ME_1$ and so finally we have the important result:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2} \qquad \text{(in limit } m_1 \to 0\text{)}.$$

In the last equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

so we could instead write the following if we preferred to be explicit about θ dependence:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2}.$$

General form for $2 \rightarrow 2$ Body Scattering in Lab. Frame

$\star$ The calculation of the differential cross section for the case where m_1 cannot be neglected is longer and contains no more 'physics' (see Appendix IV). It gives:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{m_2 \left|\vec{p}_1\right|} \cdot \frac{\left|\vec{p}_3\right|^2}{\left|\vec{p}_3\right| \left(E_1 + m_2\right) - E_3 \left|\vec{p}_1\right| \cos\theta} \cdot \left|M_{fi}\right|^2}$$

Again, there is only one independent variable, θ, which can be seen from the conservation of energy:

$$E_1 + m_2 = \sqrt{\left|\vec{p}_3\right|^2 + m_3^2} + \sqrt{\underbrace{\left|\vec{p}_1\right|^2 + \left|\vec{p}_3\right|^2 - 2\left|\vec{p}_1\right|\left|\vec{p}_3\right|\cos\theta + m_4^2}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3}}$$

i.e., $\left|\vec{p}_3\right|$ is a function of θ.

Summary

We used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the Lorentz Invariant Matrix Element and wave-functions normalised to $2E$ particles per unit volume. our main results were:

★**Particle decay in C.o.M. frame:**

(12)
$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \mathrm{d}\Omega$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

★**Scattering cross section in C.o.M. frame:**

(13)
$$\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

★**Invariant differential cross section (valid in all frames):**

(14)
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

and (continued next slide ...)

Summary continued ...

★Differential cross section in the lab. frame (m_1=0)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \quad (15) \qquad \Leftrightarrow \qquad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2 |M_{fi}|^2 \quad (16)$$

★Differential cross section in the lab. frame ($m_1 \neq 0$)

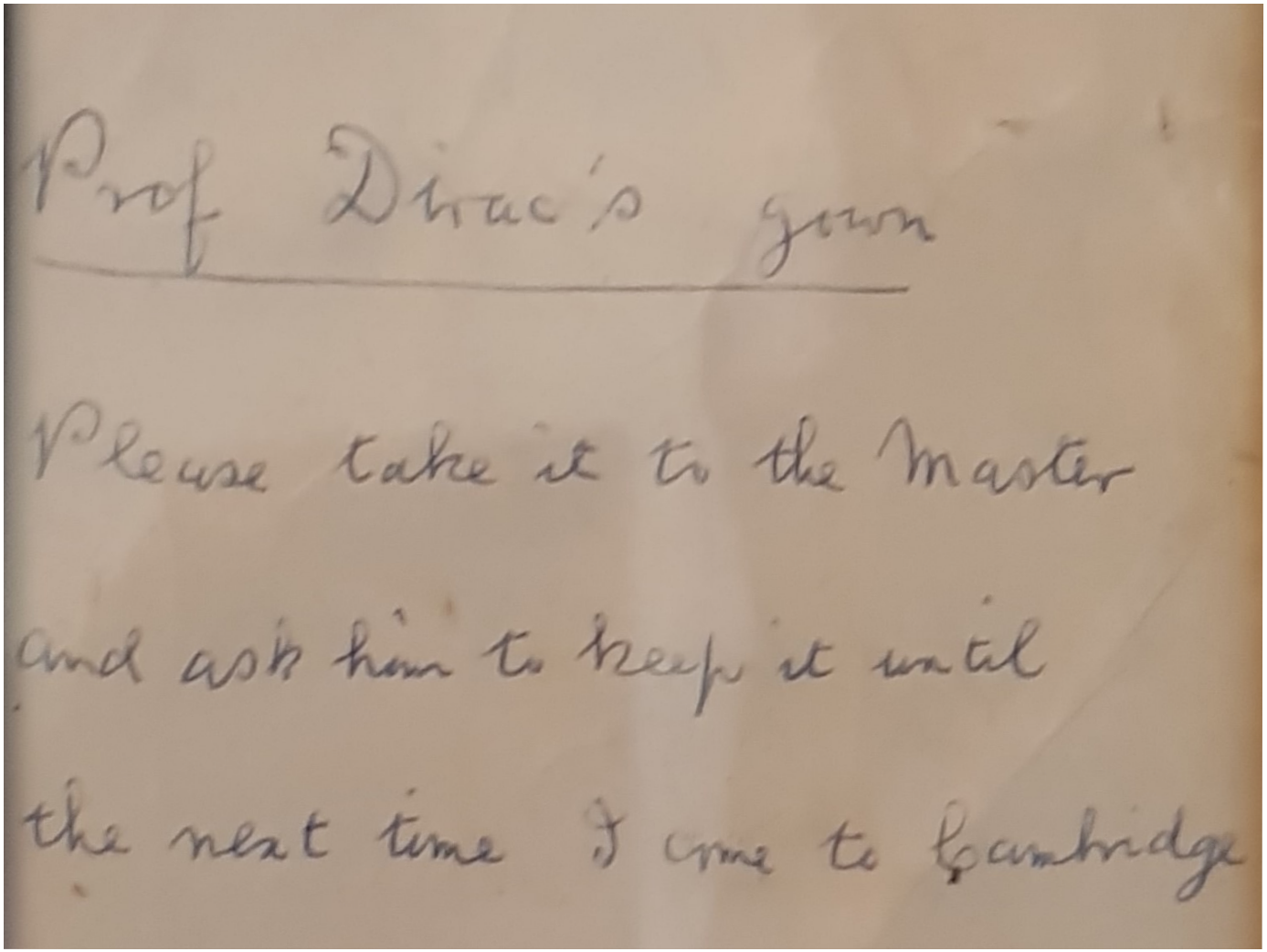
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{|m_2|\vec{p}_1|}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2 \quad (17)$$

with $E_1+m_2 = \sqrt{|\vec{p}_3|^2+m_3^2}+\sqrt{|\vec{p}_1|^2+|\vec{p}_3|^2-2|\vec{p}_1||\vec{p}_3|\cos\theta+m_4^2}$

Summary of the summary:

- We should not need to do any more kinematic calculations, as all the kinematics we need for particle decays and two-to-two scatterings and cross sections are now embedded in the results above.
- Different physical theories change the matrix element, so we can re-use the above for many different sorts of interaction.
- Many later calculations will take one of the above results as a starting point.

Prof. Alex Mitov, 2026



H2: The Dirac Equation

Non-Relativistic QM (Revision)

- For particle physics need a relativistic formulation of quantum mechanics
- Take as the starting point non-relativistic energy:

$$E = T + V = \frac{\vec{p}^2}{2m} + V$$

- In QM we identify the energy and momentum operators:

$$\vec{p} \to -i\vec{\nabla}, \quad E \to i\frac{\partial}{\partial t}$$

- which gives the time dependent Schrödinger equation (take V=0 for simplicity)

$$-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t} \tag{18}$$

with plane wave solutions: $\psi = Ne^{i(\vec{p}.\vec{r}-Et)}$, where $i\frac{\partial\psi}{\partial t} = E\psi$, $-i\nabla\psi = \vec{p}\psi$

- The SE is first order in the time derivatives and second order in spatial derivatives – and is manifestly not Lorentz invariant.
- In what follows we will use probability density/current extensively. For the non-relativistic case these are derived as follows. Firstly, (18) implies that

$$-\frac{1}{2m}\vec{\nabla}^2\psi^* = -i\frac{\partial\psi^*}{\partial t} \tag{19}$$

- $\psi^* \times (18) - \psi \times (19)$:

$$-\frac{1}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = i\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right)$$
$$-\frac{1}{2m}\vec{\nabla}.\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = i\frac{\partial}{\partial t}\left(\psi^*\psi\right)$$

- Which by comparison with the continuity equation

$$\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

leads to the following expressions for probability density and current: $\rho = \psi^*\psi = |\psi|^2$,

$$\vec{j} = \frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

- For a plane wave $\psi = Ne^{i(\vec{p}.\vec{r}-Et)}$

$$\rho = |N|^2 \qquad \text{and} \qquad \vec{j} = |N|^2\frac{\vec{p}}{m} = |N|^2\vec{v}.$$

- The number of particles per unit volume is $|N|^2$.
- For $|N|^2$ particles per unit volume moving at velocity $\vec{v}$, have $|N|^2|\vec{v}|$ passing through a unit area per unit time (particle flux). Therefore $\vec{j}$ is a vector in the particle's direction with magnitude equal to the flux.

The Klein-Gordon Equation

- Applying $\vec{p} \to -i\vec{\nabla}, \quad E \to i\partial/\partial t$ to the relativistic equation for energy:

$$E^2 = |\vec{p}|^2 + m^2 \tag{20}$$

gives the Klein-Gordon equation:

$$\frac{\partial^2 \psi}{\partial t^2} = \vec{\nabla}^2 \psi - m^2 \psi. \tag{21}$$

- Using $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$ the KG can be expressed more compactly as:

$$(\partial^\mu \partial_\mu + m^2)\psi = 0. \tag{22}$$

  - For plane wave solutions, $\psi = Ne^{i(\vec{p}.\vec{r} - Et)}$, the KG equation needs $-E^2\psi = -|\vec{p}|^2\psi - m^2\psi$ to be true, so requires $E = \pm\sqrt{|\vec{p}|^2 + m^2}$.
- Not surprisingly, the KG equation has negative energy solutions – they are allowed by our starting equation (20).
  - Historically the –ve energy solutions were viewed as problematic. But for the KG there is also a problem with the probability density ...

- Proceeding as before to calculate the probability and current densities, the complex conjugate of (21) is:

$$\frac{\partial^2 \psi^*}{\partial t^2} = \vec{\nabla}^2 \psi^* - m^2 \psi^* \tag{23}$$

$\psi^* \times (21) - \psi \times (23)$:

$$\begin{aligned} \psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} &= \psi^* (\nabla^2 \psi - m^2 \psi) - \psi (\nabla^2 \psi^* - m^2 \psi^*) \\ \frac{\partial}{\partial t} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) &= \vec{\nabla} . (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \end{aligned}$$

- Which, again, by comparison with the continuity equation allows us to identify

$$\rho = i \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \qquad \text{and} \qquad \vec{j} = i (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

- For a plane wave $\psi = N e^{i(\vec{p}.\vec{r} - Et)}$ so that

$$\rho = 2E|N|^2 \text{ and } \vec{j} = 2\vec{p}|N|^2$$

- Particle densities are proportional to E and thus to γ. We might have anticipated this from the previous discussion of Lorentz invariant phase space (i.e. density of 1 in the particle's rest frame will appear as γ in a frame where the particle has energy E due to length contraction).

The Dirac Equation

Historically, it was thought that there were two main problems with the Klein-Gordon equation:

- Negative energy solutions
- The negative particle densities associated with these solutions $\rho = 2E|N|^2$

We now know that **in Quantum Field Theory these problems do not arise** and the KG equation is used to describe spin-0 particles (inherently single particle description $\rightarrow$ multi-particle quantum excitations of a scalar field).



Nevertheless:

- These problems motivated Dirac (1928) to search for a different formulation of relativistic quantum mechanics in which all particle densities are positive.
- As we will see, the solutions of the resulting wave equation solved not only this problem but also explained intrinsic spin and why antiparticles exist.
- [Not examinable: The magnetic moment of the electron is also explained by the Dirac Equation. See Appendix VI.]

The Dirac Equation :

- Schrödinger eqn: $-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t}$ is 1st order in time and 2nd order in space.
- Klein-Gordon $(\partial^\mu\partial_\mu + m^2)\psi = 0$ is 2nd order in both time and space.
- Dirac looked for an alternative which was 1st order throughout:

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t} \tag{24}$$

  where $\hat{H}$ is the Hamiltonian operator and, as usual $\vec{p} = -i\vec{\nabla}$.
- Writing (24) in full:

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = \left(i\frac{\partial}{\partial t}\right)\psi$$

and then applying $\frac{\partial}{\partial t}$ to both sides gives

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = -\frac{\partial^2\psi}{\partial t^2}$$

- Which can be expanded in gory details as...

$$
\begin{aligned}
-\frac{\partial^2\psi}{\partial t^2} &= -\alpha_x^2\frac{\partial^2\psi}{\partial x^2} - \alpha_y^2\frac{\partial^2\psi}{\partial y^2} - \alpha_z^2\frac{\partial^2\psi}{\partial z^2} + \beta^2 m^2\psi \\
&\quad -(\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y} - (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z} - (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} \\
&\quad -(\alpha_x\beta + \beta\alpha_x)m\frac{\partial\psi}{\partial x} - (\alpha_y\beta + \beta\alpha_y)m\frac{\partial\psi}{\partial y} - (\alpha_z\beta + \beta\alpha_z)m\frac{\partial\psi}{\partial z}
\end{aligned}
$$

- For this to be a reasonable formulation of relativistic QM, a free particle must also obey $E^2 = \vec{p}^2 + m^2$, i.e. it must satisfy the Klein-Gordon equation:

$$-\frac{\partial^2\psi}{\partial t^2} = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} + m^2\psi$$

- Hence for the Dirac Equation to be consistent with the KG equation require:

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1 \tag{25}$$

$$\alpha_j\beta + \beta\alpha_j = 0 \tag{26}$$

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \qquad (j \neq k) \tag{27}$$

- Immediately we see that the α_j and β cannot be numbers. Require four mutually anti-commuting matrices.
- Must be (at least) 4x4 matrices (see Appendix V)

Consequence: these Dirac Spinors have at least four components!

Somewhat surprising conclusion:

A consequence of introducing a Lorentz Covariant wave equation that is first-order in time/space derivatives is that the wave-function has to have extra degrees of freedom we didn't previously know we would need!

Schematically:

$$\text{(first-order Lorentz-covariant wave equation)} \quad \Longrightarrow \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

A representation of Dirac's $\vec{\alpha}$ and β matrices

We also want the Hamiltonian $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi$ to be Hermitian.

This extra constraint forces:

$$\alpha_x = \alpha_x^\dagger; \qquad \alpha_y = \alpha_y^\dagger; \qquad \alpha_z = \alpha_z^\dagger; \qquad \beta = \beta^\dagger; \tag{28}$$

i.e. we need to find four anti-commuting Hermitian 4x4 matrices.

At this point it is convenient to introduce an explicit representation for $\vec{\alpha}, \beta$. It should be noted that physical results do not depend on the particular representation – everything important could be derived from the anti-commutation relations (25)-(27).

A convenient choice is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

based on the Pauli spin matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Equation: Probability Density and Current

- Now consider probability density/current – this is where the perceived problems with the Klein-Gordon equation arose.
- Start with the Dirac equation

$$-i\alpha_x\frac{\partial\psi}{\partial x}-i\alpha_y\frac{\partial\psi}{\partial y}-i\alpha_z\frac{\partial\psi}{\partial z}+m\beta\psi=i\frac{\partial\psi}{\partial t} \tag{29}$$

and its Hermitian conjugate

$$+i\frac{\partial\psi^\dagger}{\partial x}\alpha_x^\dagger+i\frac{\partial\psi^\dagger}{\partial y}\alpha_y^\dagger+i\frac{\partial\psi^\dagger}{\partial z}\alpha_z^\dagger+m\psi^\dagger\beta^\dagger=-i\frac{\partial\psi^\dagger}{\partial t} \tag{30}$$

- Consider $\psi^\dagger\times(29)-(30)\times\psi$ remembering α,β are Hermitian

$$\psi^\dagger\left(-i\alpha_x\frac{\partial\psi}{\partial x}-i\alpha_y\frac{\partial\psi}{\partial y}-i\alpha_z\frac{\partial\psi}{\partial z}+\beta m\psi\right)-\left(i\frac{\partial\psi^\dagger}{\partial x}\alpha_x+i\frac{\partial\psi^\dagger}{\partial y}\alpha_y+i\frac{\partial\psi^\dagger}{\partial z}\alpha_z+m\psi^\dagger\beta\right)\psi$$

$$=i\psi^\dagger\frac{\partial\psi}{\partial t}+i\frac{\partial\psi^\dagger}{\partial t}\psi$$

$\rightarrow$

$$\psi^\dagger\left(\alpha_x\frac{\partial\psi}{\partial x}+\alpha_y\frac{\partial\psi}{\partial y}+\alpha_z\frac{\partial\psi}{\partial z}\right)+\left(\frac{\partial\psi^\dagger}{\partial x}\alpha_x+\frac{\partial\psi^\dagger}{\partial y}\alpha_y+\frac{\partial\psi^\dagger}{\partial z}\alpha_z\right)\psi+\frac{\partial(\psi^\dagger\psi)}{\partial t}=0$$

- Now using the identity:

$$\psi^\dagger\alpha_x\frac{\partial\psi}{\partial x}+\frac{\partial\psi^\dagger}{\partial x}\alpha_x\psi\equiv\frac{\partial(\psi^\dagger\alpha_x\psi)}{\partial x}$$

- ... we find the continuity equation

$$\vec{\nabla}.(\psi^\dagger \vec{\alpha}\psi) + \frac{\partial(\psi^\dagger\psi)}{\partial t} = 0 \tag{31}$$

  where $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$.
- The probability density and current can be identified as:

$$\rho = \psi^\dagger\psi$$

  and

$$\vec{j} = \psi^\dagger \vec{\alpha}\psi$$

  where $\rho = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0$.
- Unlike the KG equation, the Dirac equation has probability densities which are always positive.
- In addition, the solutions to the Dirac equation are the four component Dirac Spinors. A great success of the Dirac equation is that these extra components naturally give rise to the property of intrinsic spin and antiparticles. (See (49) on page 71 for discussion of why Dirac spinors represent spin-half particles.)
- Such particles have an intrinsic magnetic moment of

$$\vec{\mu} = \frac{q}{m}\vec{S} \qquad \text{(see Appendix VI).}$$

Covariant Notation for Dirac Equation using the Dirac Gamma Matrices

- The Dirac equation can be written more elegantly by introducing the four Dirac gamma matrices: $\gamma^0 \equiv \beta; \quad \gamma^1 \equiv \beta\alpha_x; \quad \gamma^2 \equiv \beta\alpha_y; \quad \gamma^3 \equiv \beta\alpha_z$
- Premultiply the Dirac equation (29) by β

$$i\beta\alpha_x \frac{\partial\psi}{\partial x} + i\beta\alpha_y \frac{\partial\psi}{\partial y} + i\beta\alpha_z \frac{\partial\psi}{\partial z} - \beta^2 m\psi = -i\beta \frac{\partial\psi}{\partial t}$$

$$\rightarrow i\gamma^1 \frac{\partial\psi}{\partial x} + i\gamma^2 \frac{\partial\psi}{\partial y} + i\gamma^3 \frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial\psi}{\partial t}$$

- using $\partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ this can be written compactly as:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0} \tag{32}$$

- NOTE: it is important to realise that the Dirac gamma matrices are not four-vectors - they are constant matrices which remain invariant under a Lorentz transformation. However it can be shown that the Dirac equation is itself Lorentz covariant (see page 551 of Appendix VII).

Properties of the gamma matrices

From the properties of the α and β matrices (25)-(27) one immediately obtains:

$$(\gamma^0)^2 = \beta^2 = 1 \qquad \text{and} \qquad (\gamma^1)^2 = \beta\alpha_x\,\beta\alpha_x = -\alpha_x\beta\beta\alpha_x = -\alpha_x^2 = -1.$$

The full set of relations is

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 &= -1 \\ \gamma^0\gamma^j + \gamma^j\gamma^0 &= 0 \\ \gamma^j\gamma^k + \gamma^k\gamma^j &= 0 \qquad (j \neq k) \end{aligned}$$

which can be expressed as:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}} \tag{33}$$

which defines an algebra.

- β is Hermitian so γ^0 is Hermitian.
- The α matrices are also Hermitian, giving

$$\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger = \alpha_x\beta = -\beta\alpha_x = -\gamma^1.$$

hence γ^1, γ^2, γ^3 are anti-Hermitian:

$$\boxed{\gamma^{0\dagger} = \gamma^0, \quad \gamma^{1\dagger} = -\gamma^1, \quad \gamma^{2\dagger} = -\gamma^2, \quad \gamma^{3\dagger} = -\gamma^3}.$$

Pauli-Dirac Representation

- From now on we will use the Pauli-Dirac representation of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

  which when written in full are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Using the gamma matrices $\rho = \psi^\dagger \psi$ and $\vec{j} = \psi^\dagger \vec{\alpha} \psi$ can be written as $j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$ where j^μ is the four-vector current. (The proof that j^μ is indeed a four vector concludes on page 559)
- In terms of the four-vector current the continuity equation becomes $\partial_\mu j^\mu = 0$
- Finally the expression for the four-vector current $j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi$

can be simplified by introducing the adjoint spinor.

The Adjoint Spinor

The adjoint spinor is defined as follows in any representation of the gamma matrices:

$$\overline{\psi} = \psi^\dagger \gamma^0.$$

In the Dirac representation of the gamma matrices:

$$\overline{\psi} = \psi^\dagger \gamma^0 = (\psi^*)^T \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ so}$$

$$\overline{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*).$$

In terms the adjoint spinor the four vector current can be written:

$$j^\mu = \overline{\psi} \gamma^\mu \psi$$

We will use this expression in deriving the Feynman rules for the Lorentz invariant matrix element for the fundamental interactions.

Dirac Equation: Plane Wave Solutions

We are interested in plane wave solutions to the Dirac Equation (32) of the form:

$$\boxed{\psi = e^{\pm i(\vec{p}\cdot\vec{r}-Et)}u_{\pm}(E,\vec{p})} \tag{34}$$

where $u_{\pm}$ are appropriately chosen four-component 'spinors' and $E^2 = m^2 + \vec{p}^2$. [Aside: we will also name these spinors $(u_+, u_-) \leftrightarrow (u, v)$ in equations that feature just one but not the other.] Since the Dirac Equation (32) is:

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

such solutions would need to satisfy: $(i\gamma^{\mu}(\mp ip_{\mu}) - m)e^{\pm i(\vec{p}\cdot\vec{r}-Et)}u_{\pm}(E,\vec{p}) = 0$ or

$$\boxed{(\gamma^{\mu}p_{\mu} \mp m)u_{\pm} = 0} \tag{35}$$

which is written by most sources as two separate expressions:

$$(\gamma^{\mu}p_{\mu} - m)u = 0 \tag{36}$$

$$(\gamma^{\mu}p_{\mu} + m)v = 0 \tag{37}$$

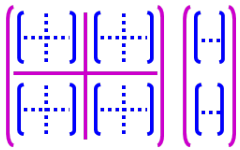
with (36) and (37) referred to as 'the momentum space representation of the Dirac Equation for particle spinors' and 'anti-particle spinors', respectively. [Aside: proper justification for those names will be provided later!]
Note that (36) and (37) are algebraic rather than differential equations.

Dirac Equation: Plane Wave Solutions cont.

We will solve (35) to gain insight into the properties that u and v (i.e. that $u_\pm$) must have. Re-writing (35), i.e. $(\gamma^\mu p_\mu \mp m)u_\pm = 0$, in our representation of the Dirac Algebra gives us:

$$\left(E\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - \vec{p}\cdot\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \mp m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right) u_\pm = 0 \tag{38}$$

or more succinctly:



$$\begin{pmatrix} E \mp m & -\vec{\sigma}\cdot\vec{p} \\ +\vec{\sigma}\cdot\vec{p} & -E \mp m \end{pmatrix} u_\pm = \begin{pmatrix} E \mp m & -\vec{\sigma}\cdot\vec{p} \\ +\vec{\sigma}\cdot\vec{p} & -E \mp m \end{pmatrix}\begin{pmatrix} A_\pm \\ B_\pm \end{pmatrix} = 0 \tag{39}$$

where we have broken the four-spinors $u_\pm$ into the their top two parts $A_\pm$ and their bottom two parts $B_\pm$. The expression (39) may be written as two simultaneous equations:

$$(E \mp m)A_\pm = (\vec{\sigma}\cdot\vec{p})B_\pm \tag{40}$$

$$(E \pm m)B_\pm = (\vec{\sigma}\cdot\vec{p})A_\pm. \tag{41}$$

These two equations turn out to be equivalent to each other (at least when $E^2 \neq m^2$) and so they amount to half as many constraints on $A_\pm$ and $B_\pm$ as one might naively imagine. To prove this statement, note first that $(\vec{\sigma}\cdot\vec{p})^2 = (E-m)(E+m)I$ since

$$(\vec{\sigma}\cdot\vec{p})^2 = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}^2 = (p_x^2 + p_y^2 + p_z^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\vec{p}|^2 I = (E^2 - m^2)I.$$

Dirac Equation: Plane Wave Solutions cont.

Then observe that when $E^2 \neq m^2$:

$$
\begin{aligned}
(40) &\iff [(E \mp m)A_\pm = (\vec{\sigma}\cdot\vec{p})B_\pm] \\
&\iff \left[(E \mp m)(\vec{\sigma}\cdot\vec{p})A_\pm = (\vec{\sigma}\cdot\vec{p})^2 B_\pm\right] \\
&\iff [(E \mp m)(\vec{\sigma}\cdot\vec{p})A_\pm = (E-m)(E+m)B_\pm] \\
&\iff [(\vec{\sigma}\cdot\vec{p})A_\pm = (E \pm m)B_\pm] \\
&\iff (41).
\end{aligned}
$$

[Aside: above we have required $E^2 \neq m^2$ (equivalently $\vec{p} \neq 0$) since it allows us to divide by $E \mp m$ which makes the maths easier to present. While equations (40) and (41) are not equivalent when $E^2 = m^2$, it may still be shown that the $\vec{p} \to 0$ limit of all our future result(s) is indeed the same as the result(s) one *would* have found by considering $E = \pm m$ as a special case at the outset. In other words: there is nothing special or magical about $\vec{p} = 0$; it is the same as the limit $\vec{p} \to 0$. Thus we cheerfully assume that $E^2 \neq m^2$ wherever required hereafter, even though our results are also valid for $E^2 = m^2$.]

Dirac Equation: Plane Wave Solutions cont.

As $\boxed{(E \mp m)A_\pm = (\vec{\sigma}\cdot\vec{p})B_\pm \quad (40)}$ and $\boxed{(E \pm m)B_\pm = (\vec{\sigma}\cdot\vec{p})A_\pm \quad (41)}$

are equivalent (at least when $E^2 \neq m^2$) we may keep whichever one we like and discard the other, and then we may regard one of $A_\pm$ and $B_\pm$ free while the other is fixed by the retained equation. For reasons which may become clear later, we choose the following:

- when considering u_+ we use (41) to fix B_+ in terms of an unconstrained A_+; and
- when considering u_- we use (40) to fix A_- in terms of an unconstrained B_-, so that

$$u_+ = u \in \left\{ \begin{pmatrix} A_+ \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} A_+ \end{pmatrix} \middle| \forall A_+ \right\} \text{ and } u_- = v \in \left\{ \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} B_- \\ B_- \end{pmatrix} \middle| \forall B_- \right\}$$

or, equivalently, we could say $u_+ = u \in \text{span}\{u_1, u_2\}$ and $u_- = v \in \text{span}\{v_1, v_2\}$ where:

$$u_1 = N \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}, \quad u_2 = N \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}, \quad v_1 = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}, \quad v_2 = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

where N may be freely chosen (see next slide).

Comments on the Dirac spinors u_1, u_2, v_1 and v_2.

1. It may be instructive to observe that the last definitions unpack to:

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad v_1 = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

2. If one wishes to choose N so that $\psi^\dagger(u_1)\psi(u_1)$, $\psi^\dagger(u_2)\psi(u_2)$, $\psi^\dagger(v_1)\psi(v_1)$, and $\psi^\dagger(v_2)\psi(v_2)$ are all to equal $2E$, then a suitable choices is

$$N = \sqrt{E+m}.$$

This gives the '$2E$-particles-per-unit-volume' normalisation which we determined was needed in the previous handout. [Check this value of N is correct! You may find it easier to perform this check on the non-unpacked forms of the u and v spinors given on the previous slide.]

Does the Dirac Equation solve the negative energy states problem?

The answer to this question depends on the context.

- Our definition of u and v via $u_\pm$ in (34) placed no explicit requirements on the sign of E. It demanded only that $E^2 = m^2 + \vec{p}^2$.
- Actual observed Dirac Particles clearly have positive energy (see e.g. evidence on page 61). This strongly motivates us taking $E = +\sqrt{m^2 + \vec{p}^2}$.
- Nonetheless, (34) shows that:

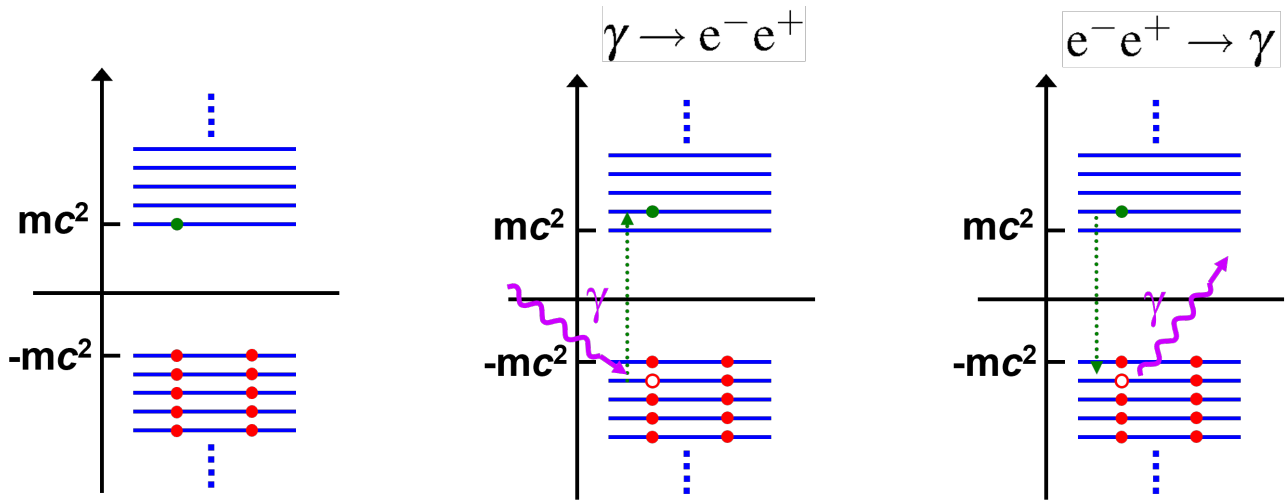
$$\hat{E}\ \psi(u_\pm) = \left(i\frac{\partial}{\partial t}\right)\left(e^{\pm i(\vec{p}\cdot\vec{r}-Et)}u_\pm(E,\vec{p})\right) = \pm E e^{\pm i(\vec{p}\cdot\vec{r}-Et)}u_\pm(E,\vec{p}) = \pm E\ \psi(u_\pm)$$

  implying that taking $E > 0$ would lead the 'usual' energy operator having positive eigenvalues on the $u = u_+$ spinors, but negative eigenvalues on the $v = u_-$ spinors. Dirac initially thought that half of his particles had negative energies. He developed various (now discredited) theories as workarounds (Dirac hole model, Dirac sea, *etc*).
- With the benefit of hindsight, all these were non-problems: the transition from Quantum Mechanics (1st quantisation) to Quantum Field Theories (2nd quantisation) gave a new interpretation to the $i\frac{\partial}{\partial t}$ operator and the energy operator in the contexts above. In Quantum Field Theory, all energies are positive, but the u-spinors have 'positive frequencies' and the v-spinors have 'negative frequencies' ... see QFT course.
- As this course is QM rather than QFT based, we need to apply workaround to some of our operators. See page 66.

Discredited Dirac sea / hole model

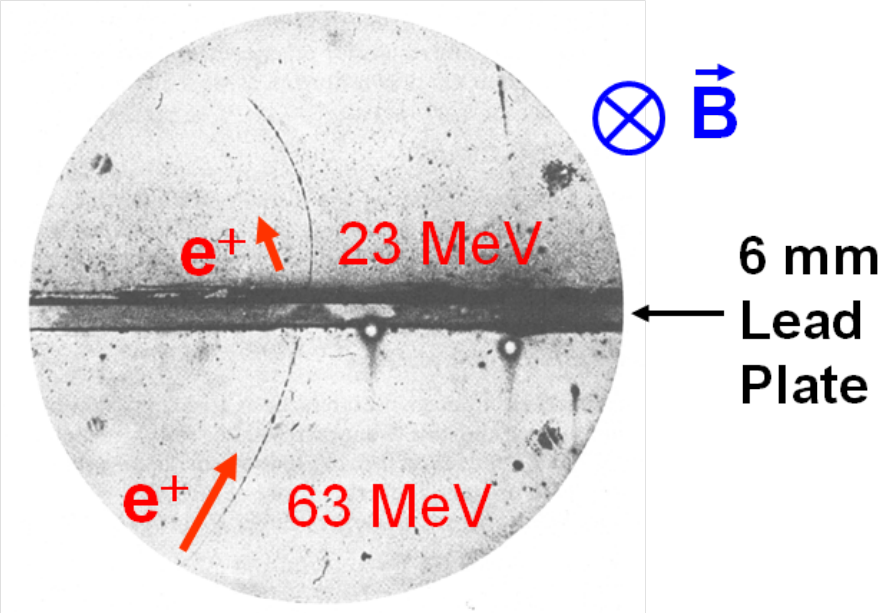
Not examinable

- An attempt to explain why particles able to take negative energies would not fall down to ever lower energies radiating lots of energy in the process.
- Dirac Interpretation: the vacuum corresponds to all –ve energy states being full with the Pauli exclusion principle preventing electrons falling into -ve energy states. Holes in the –ve energy states correspond to +ve energy anti-particles with opposite charge. Provides a picture for pair-production and annihilation.



Not examinable

Discovery of Positron





- e+ enters at bottom, slows down in the lead plate – know direction
- Curvature in B-field shows that it is a positive particle
- Can't be a proton as would have stopped in the lead $\Longrightarrow$ Provided Verification of Predictions of Dirac Equation
- Anti-particle solutions exist ! But the picture of the vacuum corresponding to the state where all –ve energy states are occupied is rather unsatisfactory, what about bosons (no exclusion principle),. . . .

Chronology relating to Negative Energy Solutions

Not examinable

- 1928, Dirac invents his Equation. Probability density is positive, but negative energies are permitted (Proc. Roy. Soc. A117, 610-628) [1].
- 1930, Dirac tries to solve negative energies via the "hole" theory. He relates anti-particles to negative energy eigenstates. (Proc. Cam. Phil. Soc. 26, 376-381) [2].
- 1934, Paulu and Weisskopf present a new interpretation of Klein-Gordon equation: as field equation for a charged spin-0 field. ρ represents the charge density. The energy is given via

$$\frac{1}{2}\int d^3r \left[|\nabla\psi|^2 + m^2|\psi|^2\right]$$

  and thus positive by definition (Helv. Phys. Acta 7, 709-734) [3].
- 1934, The Dirac equation aquired a field-theoretic interpretation. It no longer represented a probability amplitude. Instead it became the field operator of a spin-$\frac{1}{2}$ field in a QFT. See the QFT and AQFT courses.

Not examinable

Charge Conjugation I

- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \vec{A})$ can be obtained by making the minimal substitution $\vec{p} \to \vec{p} - e\vec{A}; \quad E \to E - e\phi$. With $\vec{p} = -i\vec{\nabla}$, $E = i\partial/\partial t$ and $A^\mu = (\phi, \vec{A})$ this can be written

$$\partial_\mu \to \partial_\mu + ieA_\mu$$

and so under the above substitution the Dirac equation becomes:

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0 \tag{42}$$

- Now (for fun, and just because we can!) take the complex conjugate of the above and pre-multiplying by $-i\gamma^2$ to get this:

$$-i\gamma^2\gamma^{\mu *}(\partial_\mu - ieA_\mu)\psi^* - m\gamma^2\psi^* = 0. \tag{43}$$

To simplify (43), note that in our representation of the gamma matrices we have:

$$\gamma^{0*} = \gamma^0;\ \gamma^{1*} = \gamma^1;\ \gamma^{2*} = -\gamma^2;\ \gamma^{3*} = \gamma^3$$

Charge Conjugation II

from which we can show that commuting a γ^2 past any gamma-matrix complex-conjugates-and-negates that gamma-matrix:

$$\gamma^2\gamma^{\mu *} = -\gamma^\mu\gamma^2.$$

We may therefore push the γ^2 past the $\gamma^{\mu *}$ in (43) to get:

$$\gamma^\mu(\partial_\mu - ieA_\mu)i\gamma^2\psi^* + imi\gamma^2\psi^* = 0. \tag{44}$$

Since the expression $i\gamma^2\psi^*$ features twice in the above, we may simplify things further by defining an operator $\hat{C}$ as follows

$$\boxed{\psi' = \hat{C}\psi = i\gamma^2\psi^*}$$

so that (44) becomes:

$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi' + im\psi' = 0. \tag{45}$$

- Comparing (45) to the original equation (42):

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0$$

we see that the spinor ψ' describes a particle of the same mass but with opposite charge, i.e. an anti-particle ! $\hat{C}$: particle spinor $\Longleftrightarrow$ anti-particle spinor

Charge Conjugation III

- It cannot be stressed how revolutionary the result of the last slide was, when discovered. That discovery shows that half of all spinor degrees of freedom relate to particles that 'go the other way' when experiencing an external field. In short, Dirac discovered that every fermion has an anti-fermionic partner.
- Because of the change in the sign of e in (42) compared to (45), we can henceforth name $\hat{C}$ the **Charge Conjugation Operator** for our representation of the gamma-matrices.
- For fun, consider the action of $\hat{C}$ on the free particle wave-function: $\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)}$
  $\psi' = \hat{C}\psi = i\gamma^2\psi^* = i\gamma^2 u_1^* e^{-i(\vec{p}.\vec{r}-Et)}$

  $$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1 \text{ hence}$$

  $\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_1 e^{-i(\vec{p}.\vec{r}-Et)}$.

  Similarly $\psi = u_2 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_2 e^{-i(\vec{p}.\vec{r}-Et)}$.
- Thus, **henceforth we may call v_1 the anti-partner of u_1 and we may call v_2 the anti-partner of u_2.**

Special operators for (some) anti-particle solutions

We noted on page 59 that if we were to apply the usual QM operators for energy and momentum

$$\hat{H} = i\partial/\partial t \qquad \text{and} \qquad \hat{p} = -i\vec{\nabla}$$

to particle or anti-particle solutions of the Dirac Equation of the form

$$\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)} \qquad \text{and} \qquad \psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$$

then we would get eigenvalues for the anti-particles whose signs would be reversed from the physically/desired expected values. The signs for the anti-particles would be 'broken' by the $-$ in their exponent. This is not an issue if working in QFT from the start (different operators!)

- As the course is in limbo, between experiment and theory: our workaround is use different operators (just for anti-particle states) to extract physical energies. I.e. define:

$$\hat{H}^{(v)} = -i\partial/\partial t \qquad \text{and} \qquad \hat{p}^{(v)} = i\vec{\nabla}.$$

- Need to do same for some other operators too. E.g.: under the transformation $(E,\vec{p}) \to (-E,-\vec{p})$ one would necessarily get $\vec{L} = \vec{r} \wedge \vec{p} \to -\vec{L}$. Conservation of total angular momentum is $[H, \vec{L} + \vec{S}] = 0$ so one needs to use $\hat{S}^{(v)} \to -\hat{S}$ to measure physical spin of anti-particles.

Summary of Solutions to the Dirac Equation

- The $2E$-per-unit-vol-normalised free PARTICLE solutions to the Dirac equation $\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)}$ satisfy $\boxed{(\gamma^\mu p_\mu - m)u = 0}$

with $u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$; $\quad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$

- The ANTI-PARTICLE solutions in terms of the physical energy and momentum: $\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$ satisfy $\boxed{(\gamma^\mu p_\mu + m)v = 0}$

with $v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$; $\quad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$

- For the anti-particle states, operators whose eigenvalues are time-odd require reversed forms, e.g. $\hat{S}^{(v)} = -\hat{S}$.
- For both particle and anti-particle solutions $E = \sqrt{|\vec{p}|^2 + m^2}$.

(Now try question 8 – mainly about four-vector current.)

Why do Dirac particles have intrinsic angular momentum? (slide 1 of 3)

The Ehrenfest Theorem: $\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle [H,A]\rangle + \left\langle \frac{\partial A}{\partial t}\right\rangle$.

- For a Dirac spinor is orbital angular momentum a good quantum number? i.e. does $L = \vec{r}\wedge\vec{p}$ commute with the Hamiltonian?

$$\begin{aligned}[H,\vec{L}] &= [\vec{\alpha}.\vec{p}+\beta m, \vec{r}\wedge\vec{p}] \\ &= [\vec{\alpha}.\vec{p}, \vec{r}\wedge\vec{p}]\end{aligned}$$

Consider the x component of L:

$$\begin{aligned}[H,L_x] &= [\vec{\alpha}.\vec{p}, (\vec{r}\wedge\vec{p})_x] \\ &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y]\end{aligned}$$

The only non-zero contributions come from:

$$\begin{aligned}[H,L_x] &= \alpha_y p_z [p_y, y] - \alpha_z p_y [p_z, z] \\ &= -i(\alpha_y p_z - \alpha_z p_y) \\ &= -i(\vec{\alpha}\wedge\vec{p})_x.\end{aligned}$$

Consideration of other components shows that

$$[H,\vec{L}] = -i\vec{\alpha}\wedge\vec{p} \tag{46}$$

- As this is not zero, the orbital angular momentum operator does not commute with the Hamiltonian, and so **orbital angular momentum is not a constant of motion** by Ehrenfest. **This should make you unhappy!**

Why do Dirac particles have intrinsic angular momentum? (slide 2 of 3)

Avoid depression by introducing a new mystery (perhaps even useless??) 4x4 operator:

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli spin matrices: i.e.

$$\Sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \Sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \Sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now consider the commutator

$$[H, \vec{\Sigma}] = [\vec{\alpha}.\vec{p} + \beta m, \vec{\Sigma}]$$

here

$$[\beta, \vec{\Sigma}] = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = 0$$

and hence

$$[H, \vec{\Sigma}] = [\vec{\alpha}.\vec{p}, \vec{\Sigma}]$$

Consider the x component:

$$\begin{aligned} [H, \Sigma_x] &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] \\ &= p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x] \end{aligned}$$

- Taking each of the commutators in turn:

$$[\alpha_x, \Sigma_x] = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = 0,$$

$$\begin{aligned}[\alpha_y, \Sigma_x] &= \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_y\sigma_x - \sigma_x\sigma_y \\ \sigma_y\sigma_x - \sigma_x\sigma_y & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix} \\ &= -2i\alpha_z\end{aligned}$$

and similarly $[\alpha_z, \Sigma_x] = 2i\alpha_y$. Hence:

- $$\begin{aligned}[H, S_x] &= \frac{1}{2}\left(p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x]\right) \\ &= \frac{1}{2}\left(0 - 2ip_y\alpha_x + 2ip_z\alpha_y\right) = i(\vec{\alpha} \wedge \vec{p})_x\end{aligned}$$

and in general

$$\left[H, \vec{S}\right] = i\vec{\alpha} \wedge \vec{p}. \tag{47}$$

Why do Dirac particles have intrinsic angular momentum? (slide 3 of 3)

- Hence the observable corresponding to the operator $\vec{S}$ is also not a constant of motion. However, comparing (46) and (47) we see something quite amazing!

$$[H, \vec{L}] + [H, \vec{S}] = -i\vec{\alpha} \wedge \vec{p} + i\vec{\alpha} \wedge \vec{p} = 0 \tag{48}$$

Therefore an amazing result of this course is that $L + S$ is conserved:

$$\boxed{\frac{d}{dt}(L + S) \propto [H, \vec{L} + \vec{S}] = 0}. \tag{49}$$

- In passing, but less excitingly, one might also note that because

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

the commutation relationships for $\vec{S}$ are the same as for the $\vec{\sigma}$, e.g. $[S_x, S_y] = iS_z$. Furthermore both S^2 and S_z are diagonal

$$S^2 = \tfrac{1}{4}(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) = \tfrac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad S_z = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Consequently $S^2\psi = S(S+1)\psi = \frac{3}{4}$ and for a particle travelling along the z-direction $S_z\psi = \pm\frac{1}{2}\psi$ or $\frac{\hbar}{2}$ in non-natural units and so dirac particles are spin-$\frac{1}{2}$.

Spin States

- In general the spinors u_1, u_2, v_1, v_2 are not Eigenstates of $\hat{S}_z$

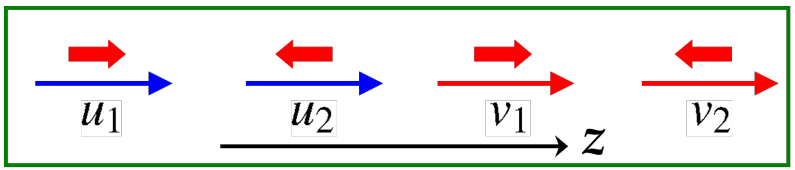
$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \tfrac{1}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

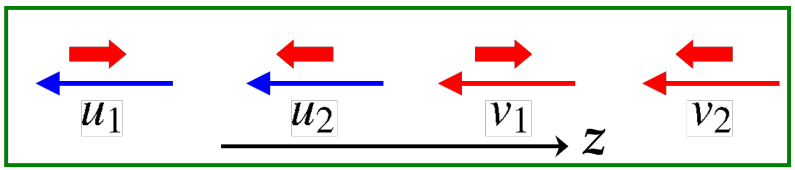
- However, for particles/anti-particles travelling in the z-direction: $p_z = \pm|\vec{p}|$, so:
- $u_1 = N\begin{pmatrix}1\\ 0\\ \frac{\pm|\vec{p}|}{E+m}\\ 0\end{pmatrix}$; $u_2 = N\begin{pmatrix}0\\ 1\\ 0\\ \frac{\mp|\vec{p}|}{E+m}\end{pmatrix}$; $v_1 = N\begin{pmatrix}0\\ \frac{\mp|\vec{p}|}{E+m}\\ 0\\ 1\end{pmatrix}$; $v_2 = N\begin{pmatrix}\frac{\pm|\vec{p}|}{E+m}\\ 0\\ 1\\ 0\end{pmatrix}$.

are Eigenstates of $\hat{S}_z$

$$\hat{S}_z u_1 = +\frac{1}{2}u_1, \qquad \hat{S}_z u_2 = -\frac{1}{2}u_2$$

$$\hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\frac{1}{2}v_1, \qquad \hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\frac{1}{2}v_2.$$





Note the change of sign of $\hat{S}$ when dealing with antiparticle spinors.
Spinors u_1 u_2 v_1 v_2 are only eigenstates of $\hat{S}_z$ for $p_z = \pm|\vec{p}|$.

Pause for Breath...

- We have found solutions to the Dirac equation which are also eigenstates $\hat{S}_z$ but only for particles travelling along the z axis. This is not a particularly useful basis!
- More generally, we want to label our states in terms of "good quantum numbers", i.e. a set of commuting observables.
- We can't use z component of spin since $[\hat{H}, \hat{S}_z] \neq 0$ as see in (48)
- We will introduce a new concept: "HELICITY". Helicity plays an important role in much that follows in the course...

Helicity

- The component of a particles spin along its direction of flight is a good quantum number (unless you overtake the particle) because $[H, \vec{S}.\vec{p}] = 0$. The same would be true even if $\vec{S}$ or $\vec{p}$ were scaled by (possibly different) constants.
- This motivates defining the helicity operator h by:

$$h \equiv \vec{\Sigma} \cdot \hat{\vec{p}} \equiv \vec{\Sigma} \cdot \left(\frac{\vec{p}}{|\vec{p}|} \right)$$



because with this definition:

1. the helicity of a particle will be independent of boosts along that particle's momentum direction (unless you overtake the particle!), and
2. the helicity values will be +1 or -1 for Dirac fermions. [We have already seen that Dirac fermions are spin-$\frac{1}{2}$ and that the S_z operator measures this if the particle is moving in the z-direction. Since $\Sigma = 2S$ the allowed values of helicity will be ± 1.]

- Conventionally, $h = +1$ and $h = -1$ are often referred to as follows:



- These are right and left handed HELICITY eigenstates. In Handout 4 we will discuss right and left handed CHIRAL eigenstates. Only in the limit $v \approx c$ are the HELICITY eigenstates the same as the CHIRAL eigenstates. **Do not confuse them!**

Helicity Eigenstates: $\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\}$. I

- The basis spinors we already found $\{u_1, u_2, v_1, v_2\}$ were nice in that $\hat{C}(u_1, u_2) = (v_1, v_2)$. However, the difference between u_1 and u_2 was physically meaningless. It was set only by an arbitrary choice of two linearly independent vectors $A_+ = (1,0)^T$ or $(0,1)^T$ on page 57.
- We wish to remove that arbitrariness by finding a better spinor basis $\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\}$ whose elements are all eigenstates of the helicity operator, e.g.:

$$(\vec{\Sigma}.\hat{p})u_\uparrow = +u_\uparrow \qquad \text{and}$$
$$(\vec{\Sigma}.\hat{p})u_\downarrow = -u_\downarrow.$$

- Since $\vec{\Sigma} \cdot \hat{p} = \begin{pmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{pmatrix}$ is proportional to $\begin{pmatrix} \vec{\sigma} \cdot \vec{p} & 0 \\ 0 & \vec{\sigma} \cdot \vec{p} \end{pmatrix}$ and since all of the 'chunked' spinor-halves on page 57 are already proportional to some power of $(\vec{\sigma} \cdot \vec{p})$ we can find the new basis we seek $(\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\})$ by replacing the arbitrary choices for A_+ or B_- (namely $(1,0)^T$ or $(0,1)^T$) with the eigenvectors of the (2×2)-matrix $(\vec{\sigma} \cdot \vec{p})$.

Helicity Eigenstates: $\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\}$. II

- If the momentum $\vec{p}$ points in the (θ, ϕ) direction in the usual spherical polar coordinates:

$$\vec{p} = |\vec{p}| \ (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

then

$$\begin{aligned}
\vec{\sigma}.\hat{\vec{p}} &= \frac{1}{|\vec{p}|}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.
\end{aligned}$$

- The eigenvectors of this last matrix (normalised such that $(\vec{e}_+)^\dagger \vec{e}_+ = (\vec{e}_-)^\dagger e_- = 1$) are:

$$\vec{e}_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \qquad \text{and} \qquad \vec{e}_- = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

which may be verified (test your memory of trig identities!) by checking that:

$$(\vec{\sigma}.\hat{\vec{p}})\ \vec{e}_+ = +\vec{e}_+ \qquad \text{and} \qquad (\vec{\sigma}.\hat{\vec{p}})\ \vec{e}_- = -\vec{e}_-.$$

Helicity Eigenstates: $\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\}$. III

Accordingly, we augment/replace our previous definition from page 57 ...

$$u_1 = N \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}, \quad u_2 = N \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}, \quad v_1 = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}, \quad v_2 = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

... with the following new definition (also relativistically normalised with $N = \sqrt{E+m}$):

$$u_\uparrow = N \begin{pmatrix} \vec{e}_+ \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\vec{e}_+ \end{pmatrix}, \quad u_\downarrow = N \begin{pmatrix} \vec{e}_- \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\vec{e}_- \end{pmatrix}, \quad v_\uparrow = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\vec{e}_- \\ \vec{e}_- \end{pmatrix}, \quad v_\downarrow = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\vec{e}_+ \\ \vec{e}_+ \end{pmatrix}$$

which looks as follows if everything (except the N prefactor, due to lack of space!) is written out in full:

$$u_\uparrow \propto \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}\cos\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, u_\downarrow \propto \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}\sin\frac{\theta}{2} \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}, v_\uparrow \propto \begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\frac{\theta}{2} \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}, v_\downarrow \propto \begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}.$$

Helicity Eigenstates: $\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\}$. IV

Aside: for the reasons already given on page 66, the desired (physical) helicity eigenvalues will only be found for the anti-particle helicity states $v_\uparrow$ and $v_\downarrow$ if they are evaluated using $\hat{h}^{(v)} = -\hat{h}$ rather than $\hat{h}$. In other words, the arrows only make sense as follows:

$$\hat{h}u_\uparrow = +u_\uparrow, \qquad \hat{h}u_\downarrow = -u_\downarrow \qquad \text{yet} \qquad \hat{h}^{(v)}v_\uparrow = +v_\uparrow, \qquad \hat{h}^{(v)}v_\downarrow = -v_\downarrow.$$

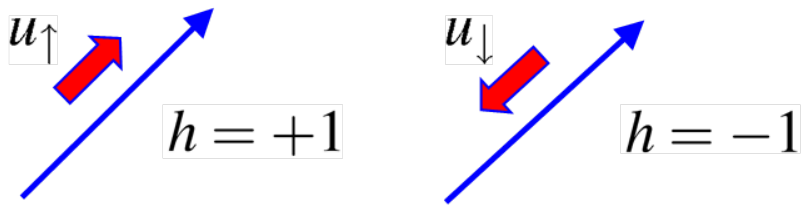
Summary of Helicity Eigenstates of the Dirac Equation:

- The particle and anti-particle helicity eigenstates states are:

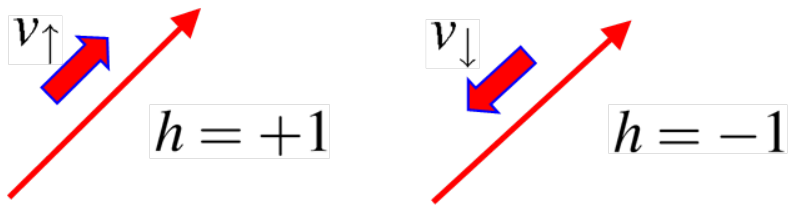
$$u_\uparrow = N \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N \begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N \begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

particles



anti-particles



- For all four states, normalising to $2E$-per-unit-vol still needs $N = \sqrt{E+m}$. These helicity eigenstates will be used extensively in the calculations that follow.

Intrinsic Parity of Dirac Particles I

Before leaving the Dirac equation, consider parity. The parity operation is defined as spatial inversion through the origin:

$$x' \equiv -x; \quad y' \equiv -y; \quad z' \equiv -z; \quad t' \equiv t. \tag{50}$$

Consider a Dirac spinor, $\psi(x, y, z, t)$, which satisfies the Dirac equation

$$i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial \psi}{\partial t}. \tag{51}$$

Pre-multiplying (51) by γ^0 and making the unprimed to primed substitutions defined in (50) then results in:

$$\begin{aligned} & -i\gamma^0\gamma^1 \frac{\partial \psi}{\partial x'} - i\gamma^0\gamma^2 \frac{\partial \psi}{\partial y'} - i\gamma^0\gamma^3 \frac{\partial \psi}{\partial z'} - m\gamma^0\psi = -i\gamma^0\gamma^0 \frac{\partial \psi}{\partial t'} \\ \Longrightarrow \quad & i\gamma^1 \frac{\partial \gamma^0\psi}{\partial x'} + i\gamma^2 \frac{\partial \gamma^0\psi}{\partial y'} + i\gamma^3 \frac{\partial \gamma^0\psi}{\partial z'} - m\gamma^0\psi = -i\gamma^0 \frac{\partial \gamma^0\psi}{\partial t'} \\ \Longrightarrow \quad & i\gamma^1 \frac{\partial \psi'}{\partial x'} + i\gamma^2 \frac{\partial \psi'}{\partial y'} + i\gamma^3 \frac{\partial \psi'}{\partial z'} - m\psi' = -i\gamma^0 \frac{\partial \psi'}{\partial t'} \end{aligned} \tag{52}$$

provided that in the last line we have introduced a new quantity $\psi'(x', y', z', t')$ defined by

$$\psi'(x', y', z', t') = \gamma^0 \psi(x, y, z, t).$$

Intrinsic Parity of Dirac Particles II

Equations (51) and (52) are identical except that one is written with primed quantities and one without. Thus:

- We have found that the parity operator $\hat{P}$ which transforms spinors to their parity-conjugates must take the form: $\hat{P} = \lambda\gamma^0$ for some λ.
- If we further wish to have $(\hat{P})^2 = 1$ we are constrained to take $\lambda = \pm 1$. Either would work, but the most common convention in the literature is:

$$\boxed{\hat{P} = \gamma^0}.$$

- A basis for the spinors of stationary particles and anti-particles can be obtained from the $\vec{p} \to 0$ limit of the spinors on page 57 and yields:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad \text{Also:} \quad \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and so in this convention fermions (u_1 and u_2) have positive parity and anti-fermions (v_1 and v_2) have negative parity when at rest.

- If we had used $\hat{P} = -\gamma^0$ then the parities for both would have reversed, but **it would still be the case that fermions and anti-fermions have opposite parity**. We will use this fact when we come to discuss mesons much later in the course.

Summary (1 or 2)

- The formulation of relativistic quantum mechanics starting from the linear Dirac equation

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$$

  implied new degrees of freedom which were found to describe spin-$\frac{1}{2}$ particles and spin-$\frac{1}{2}$ anti-particles.
- In terms of (4×4) gamma-matrices the Dirac Equation was written:

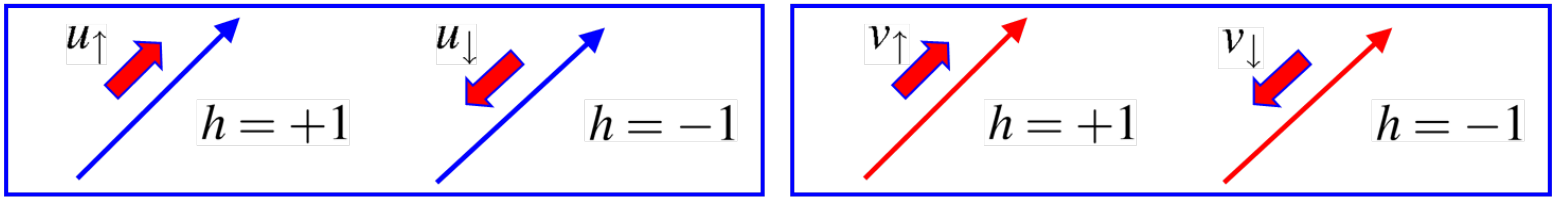
$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0.$$

- We introduced a four-vector current and an adjoint spinor:

$$j^{\mu} = \psi^{\dagger}\gamma^{0}\gamma^{\mu}\psi = \overline{\psi}\gamma^{\mu}\psi$$

Summary (2 of 2)

- A useful helicity-ordered basis for particle and anti-particle spinors $\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\}$ was summarised on page 79.



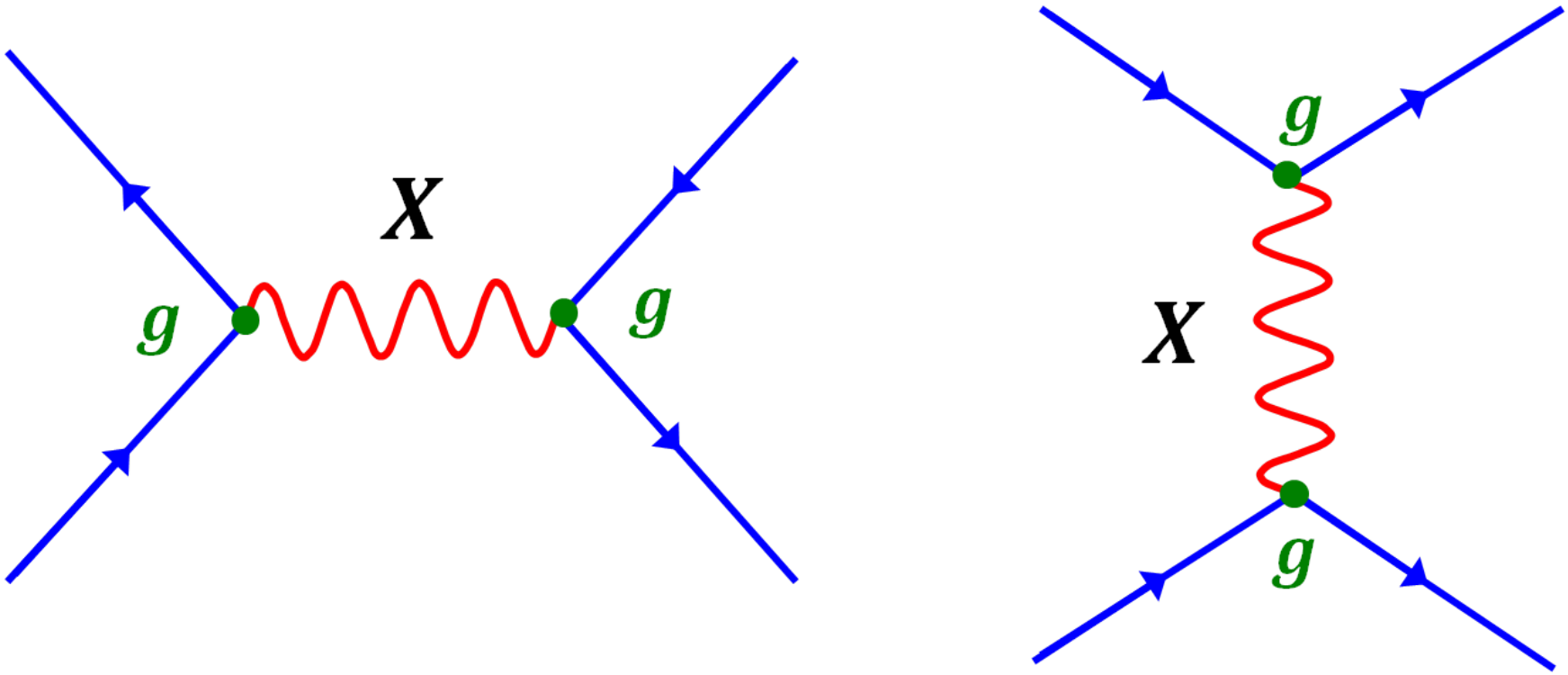
- In terms of 4-component spinors, the charge conjugation and parity operations were found (in our representation of the gamma-matrices) to be:

$$\psi \to \hat{C}\psi = i\gamma^2\psi^\dagger$$

$$\psi \to \hat{P}\psi = \gamma^0\psi$$

- Now that we have all we need to know about a relativistic description of the particles we scatter off each other, we can go on to discuss particle interactions and QED next!

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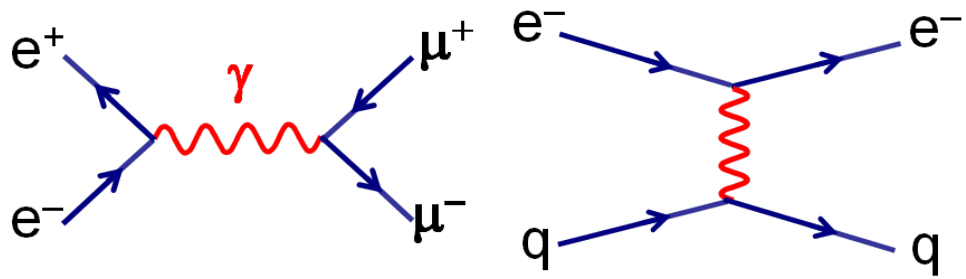


H3: Interaction by Particle Exchange and QED

Reminder, and plan for this handout

We are working towards a proper calculation of decay and scattering processes Initially concentrate on:

- $e^+e^- \to \mu^+\mu^-$
- $e^-q \to e^-q$



This handout concentrates on the Lorentz Invariant Matrix Element M_{fi}.

- Considerably more has to be taken on trust in this handout than in the previous handouts. All motivational information has the status of plausibility arguments, at best. It is not a substitute for the QFT course which is lectured in parallel.

The main areas which will be covered in this handout are:

- the meaning of the phrases 'interaction by particle exchange' and 'virtual particle';
- a take-it-on-trust introduction to Feynman diagrams;
- a description of the Feynman rules for tree-level QED; and
- use of those rules to work out scattering cross sections for processes like those shown above.

Interaction by Particle Exchange

- We previously noted that we **calculate transition rates from Fermi's Golden Rule**

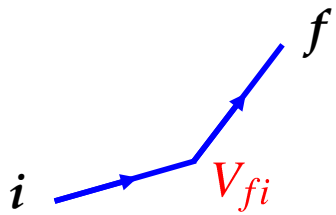
$$\Gamma_{fi} = 2\pi \left| T_{fi} \right|^2 \rho \left(E_f \right)$$

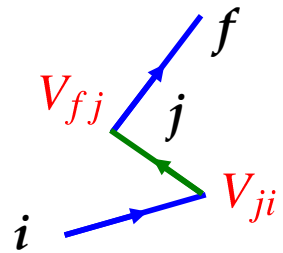
where T_{fi} is perturbation expansion for the Transition Matrix Element

$$T_{fi} = \langle f | V | i \rangle + \sum_{j \neq i} \frac{\langle f | V | j \rangle \langle j | V | i \rangle}{E_i - E_j} + \ldots .$$

- For particle scattering, the first two terms in the perturbation series can be viewed as:

"scattering in a potential"





"scattering via an intermediate state"

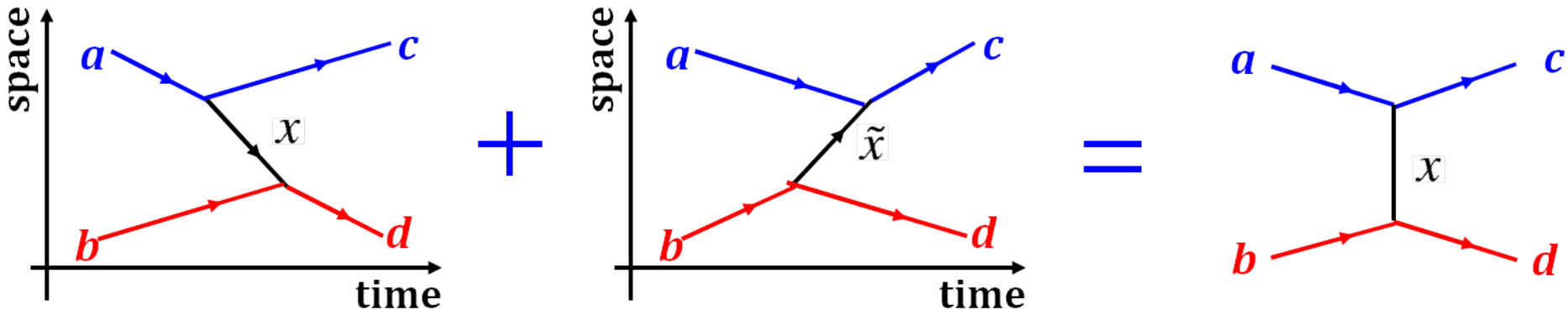
- **"Classical picture"** – particles act as sources for fields which give rise to a potential in which other particles scatter – **"action at a distance"**
- **"Quantum Field Theory picture"** – forces arise due to the exchange of virtual particles. No action at a distance: forces between particles now due to particles
- The above ingredients result in M_{fi} containing terms of the form

$$M_{fi} \sim \frac{g_a g_b}{q^2 - m_X^2}$$

involving the exchange of a virtual (i.e. off-shell) particle X with mass m_X and virtuality $q^2 \neq m_X^2$.

Summary: the sum of all time-ordered momentum-non-conserving 'exchanged' real particles looks like one momentum-conserving 'virtual' particle.

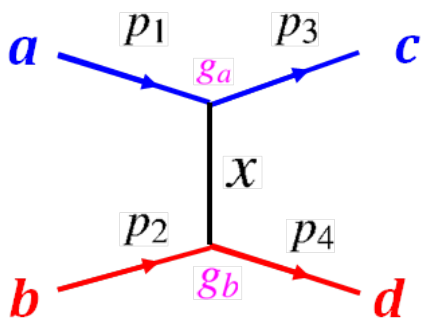
- The sum over all possible time-orderings is represented by a Feynman diagram

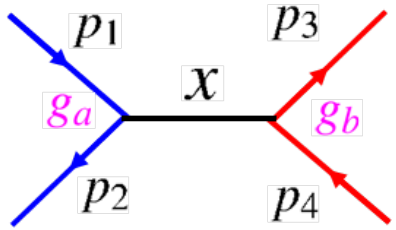


- Momentum is not conserved at any vertex in the **time-ordered diagram** (FGR did not ask for this in its sums over states!), but the exchanged particles there all have their 'real' masses, ($E_X^2 - p_X^2 = m_X^2$ or 'on mass shell') but ...
- ... the **virtual particles in the Feynman Diagrams** have ended up conserving momentum at each vertex, albeit at the cost of having the 'wrong' masses ($E_X^2 - p_X^2 = q^2 \neq m_X^2$ or "off mass shell").
- A 'propagator', i.e. a factor like $\frac{1}{q^2 - m_X^2}$ arises naturally in association with each virtual particle in a Feynman diagram.

Sign of q^2

A matrix element like $M_{fi} = \frac{g_a g_b}{q^2 - m_X^2}$ depends on the four-momentum, q, carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices. Note that q^2 can be either positive or negative:





Here $q^2 = (p_1 - p_3)^2 = (p_4 - p_2)^2 = t$.
For elastic scattering: $p_1 = (E, \vec{p}_1)$ and $p_3 = (E, \vec{p}_3)$ so $q^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_3)^2 < 0$ and so this is termed 'space-like' t-channel scattering.

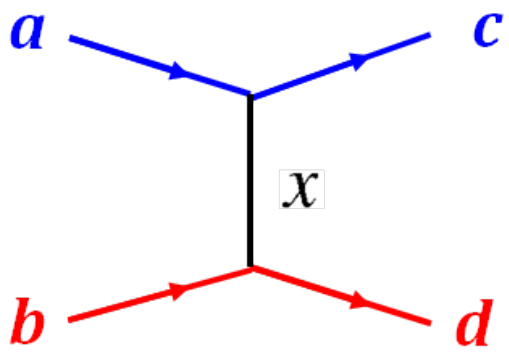
Here $q^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2 = s$.
In C.o.M. $p_1 = (E, \vec{p})$ and $p_2 = (E, -\vec{p})$ so $q^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2 > 0$ and so this is termed 'time-like' s-channel annihilation.

Aside: V(r) from Particle Exchange

Not examinable

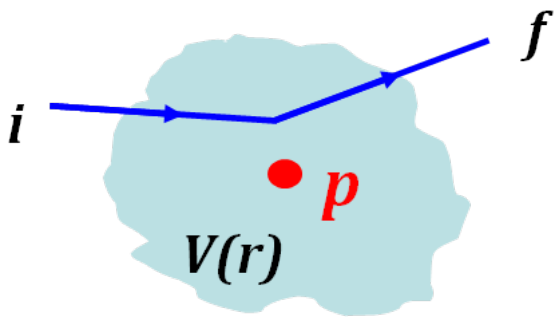
One can view the scattering of an electron by a proton at rest in two ways:

(1) As an interaction by particle exchange in 2nd order perturbation theory.



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}.$$

(2) As a process in first order perturbation theory treating proton as a fixed source of a field which gives rise to a potential $\mathrm{V(r)}$.



$$M_{fi} = \langle \psi_f | V(r) | \psi_i \rangle .$$

One obtains the same expression for M_{fi} in both cases if one uses a 'Yukawa Potential':

$$V(r) = g_a g_b \frac{e^{-mr}}{r}.$$

In this way, one can relate potential and forces to the particle exchange picture. However, scattering from a fixed potential $V(r)$ is not a relativistic invariant view!

Not examinable

Quantum Electrodynamics (QED) from semi-classical / historical perspective

The basic interaction between a photon and a charged particle may have been introduced by making the minimal substitution mentioned earlier (see Part II Electrodynamics) i.e. via:

$$i\partial_\mu \to i\partial_\mu - qA_\mu$$

where $A_\mu = (\phi, -\vec{A})$. As we saw ealier, this leads to the Dirac equation changing from

$$\gamma^\mu \partial_\mu \psi + im\psi = 0$$

to

$$\gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0$$

or equivalently

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\vec{\gamma} \cdot \vec{\nabla}\psi - q\gamma^\mu A_\mu \psi - m\psi = 0$$

thus

$$i\gamma^0 \frac{\partial \psi}{\partial t} = -i\vec{\gamma} \cdot \vec{\nabla}\psi + q\gamma^\mu A_\mu \psi + m\psi$$

and so $(\times \gamma^0)$

$$i\frac{\partial \psi}{\partial t} = -i\gamma^0\vec{\gamma} \cdot \vec{\nabla}\psi + m\gamma^0\psi + q\gamma^0\gamma^\mu A_\mu \psi. \tag{53}$$

We recognise

$$i\frac{\partial\psi}{\partial t} = -i\gamma^0\vec{\gamma}\cdot\vec{\nabla}\psi + m\gamma^0\psi. + q\gamma^0\gamma^\mu A_\mu\psi \qquad (53)$$

as the Schroedinger Equation:

$$i\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

with

$$\hat{H} = \underbrace{\left(\gamma^0 m - i\gamma^0\vec{\gamma}\cdot\vec{\nabla}\right)}_{\text{Combined rest mass + K.E.}} + \underbrace{q\gamma^0\gamma^\mu A_\mu}_{\text{Potential or interaction energy}}$$

i.e. we can infer that the operator associated with the interaction/potential energy of a charged spin-half particle in an electromagnetic field, $\hat{H}_{\text{int}}$, might resemble:

$$\hat{H}_{\text{int}} \sim q\gamma^0\gamma^\mu A_\mu.$$

i.e. we expect the $\left\langle i\middle|\hat{H}_{\text{int}}\middle|j\right\rangle$ terms in F.G.R. to contain expressions like:

$$\left\langle \Psi_i\middle|q\gamma^0\gamma^\mu A_\mu\middle|\Psi_j\right\rangle$$

or, using the definition of the adjoint spinor $\bar{\psi} = \psi^\dagger\gamma^0$:

$$A_\mu\boxed{\left\langle \bar{\Psi}_i\middle|q\gamma^\mu\middle|\Psi_j\right\rangle}.$$

Not examinable

Alas, there are many complications concerning A_μ that are beyond this course (but which relate to the fact that A_μ is able/needed to encode photon polarisations, but can also encode unphysical things on account of gauge invariance). E.g. for a real photon propagating in the z direction

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(p_z z - Et)}$$

we have two orthogonal transverse polarization states in some 'gauges' (see Appendix XIII much later) :

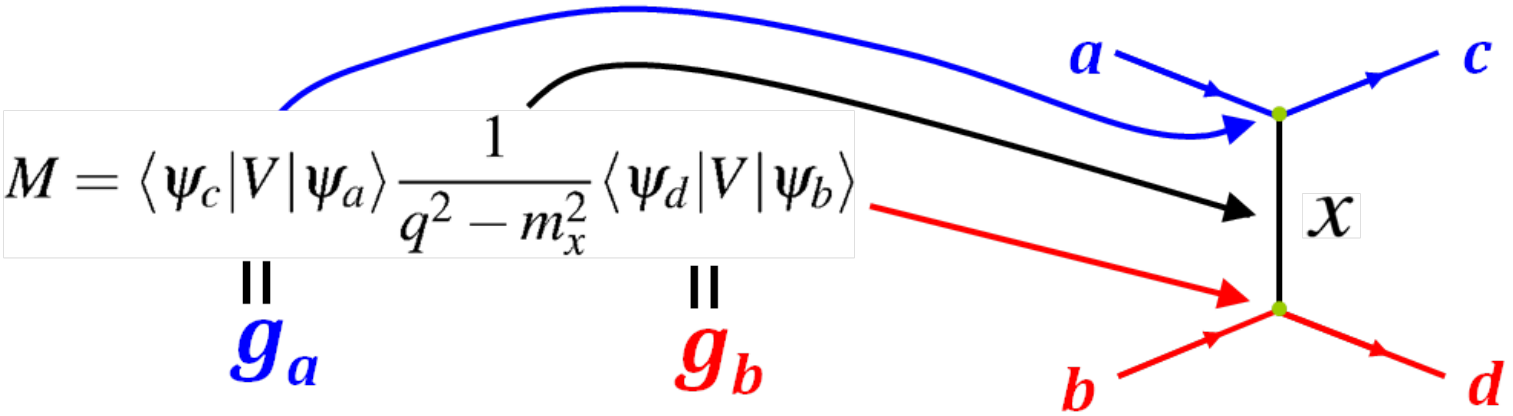
$$\boldsymbol{\varepsilon}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The area concerning photon polarisation spin-sums $\sum_\lambda \varepsilon_\mu^\lambda \left(\varepsilon_\nu^\lambda\right)^*$ is very complicated and needs an entire course on **Gauge Field Theories**. (Consider reading around eq (4.66) in Michio Kaku's "Quantum Field Theory: a modern introduction" if you want an inkling of what is involved ...)

Suffice it to say that after the dust has settled, terms like those shown on the next slide are those which appear in expansions of the M_{fi} terms needed by Fermi's Golden Rule ...

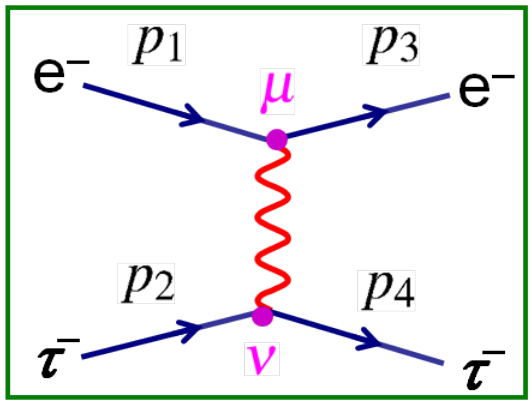
Not examinable

That which schematically was like this:



becomes like this for the case of electron-tau scattering via a photon:

$$M = \underbrace{[\bar{u}_e(p_3)\, q_e \gamma^\mu u_e(p_1)]}_{\text{Interaction of } e \text{ with photon}} \frac{-g_{\mu\nu}}{q^2 - 0^2} \underbrace{[\bar{u}_\tau(p_4)\, q_\tau \gamma^\nu u_\tau(p_2)]}_{\text{Interaction of } \tau \text{ with photon}}$$



Our claimed first-order Matrix Element for electron-tau scattering:

$$M = \left[\bar{u}_e\left(p_3\right) q_e \gamma^\mu u_e\left(p_1\right)\right] \frac{-g_{\mu\nu}}{q^2 - 0^2} \left[\bar{u}_\tau\left(p_4\right) q_\tau \gamma^\nu u_\tau\left(p_2\right)\right]$$

is a remarkably simple expression! It was shown in Appendix VII that $\bar{u}_1 \gamma^\mu u_2$ transforms as a four vector, so writing

$$j_e^\mu = \bar{u}_e\left(p_3\right) \gamma^\mu u_e\left(p_1\right)$$

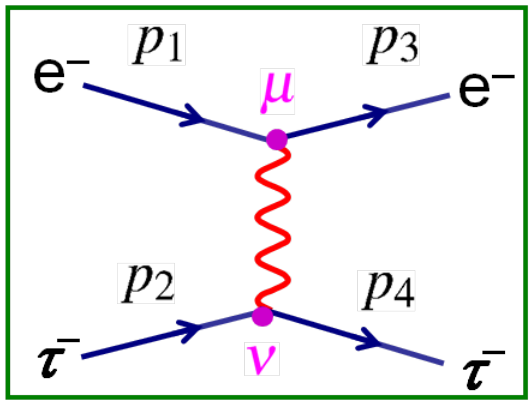
and

$$j_\tau^\nu = \bar{u}_\tau\left(p_4\right) \gamma^\nu u_\tau\left(p_2\right)$$

we have

$$M = -q_e q_\tau \frac{j_e \cdot j_\tau}{q^2}$$

making the Lorentz Invariance of M more easily visible.



Old Fashioned Time-Ordered Perturbation Theory vs Feynman Rules

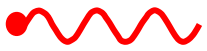
- Even though we did not deduce the following leading-order expression very rigourously:

$$M = [\bar{u}_e(p_3)\, q_e \gamma^\mu u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\bar{u}_\tau(p_4)\, q_\tau \gamma^\nu u_\tau(p_2)]$$

  we made attempts to illuatrate how it was obtained as a sum over all possible time orderings of the virtual photon – and as a sum over the photon polarisations (though these were discussed even less!). Calculations of that sort (but done rigorously) are now called calcuations in 'old fashioned time ordered perturbation theory'.
- Fortunately, an amazing result, first intuited by Feynman, then later proved by Schwinger and Dyson, is that there is a much simpler way that is provably equivalent albeit mysterious. **We can just write down any matrix element by using a set of simple rules, so called 'Feynman Rules' – see next page!**

Basic Feynman Rules for QED

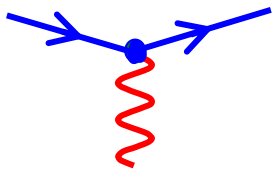
External Lines

spin 1/2	incoming particle	$u(p)$	
	outgoing particle	$\bar{u}(p)$	
	incoming antiparticle	$\bar{v}(p)$	
	outgoing antiparticle	$v(p)$	
spin 1	incoming photon	$\varepsilon_\mu(p)$	
	outgoing photon	$\varepsilon_\mu(p)^*$	

Internal Lines (propagators)

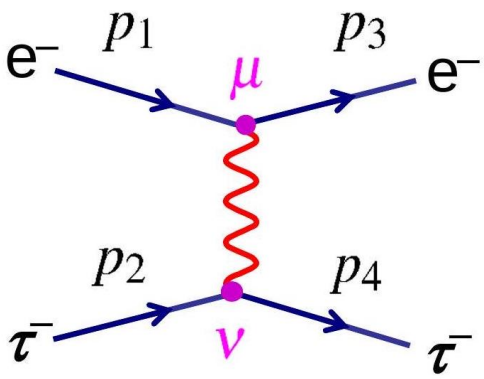
spin 1	photon	$-\frac{ig_{\mu\nu}}{q^2}$	
spin 1/2	fermion	$\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$	

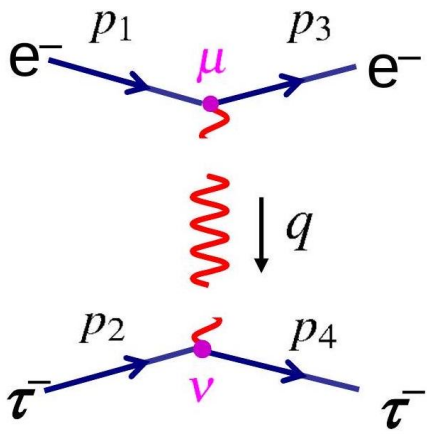
Vertex Factors

spin 1/2	fermion (charge -|e|)	$ie\gamma^\mu$	

Matrix Element $-iM$ = product of all factors

e.g.





$$\bar{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

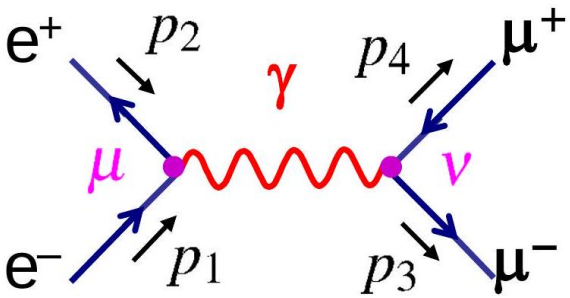
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\bar{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\bar{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

as we obtained previously. Or, we could look at an entirely different process,

e.g.



$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

NOTE:

- Each fermion line is traversed 'backwards' (i.e. against the arrows) from adjoint spinor to ordinary spinor via any vertices inbetween in the order they are encountered!

Summary

- Interaction by particle exchange naturally gives rise to Lorentz Invariant Matrix Element of the form

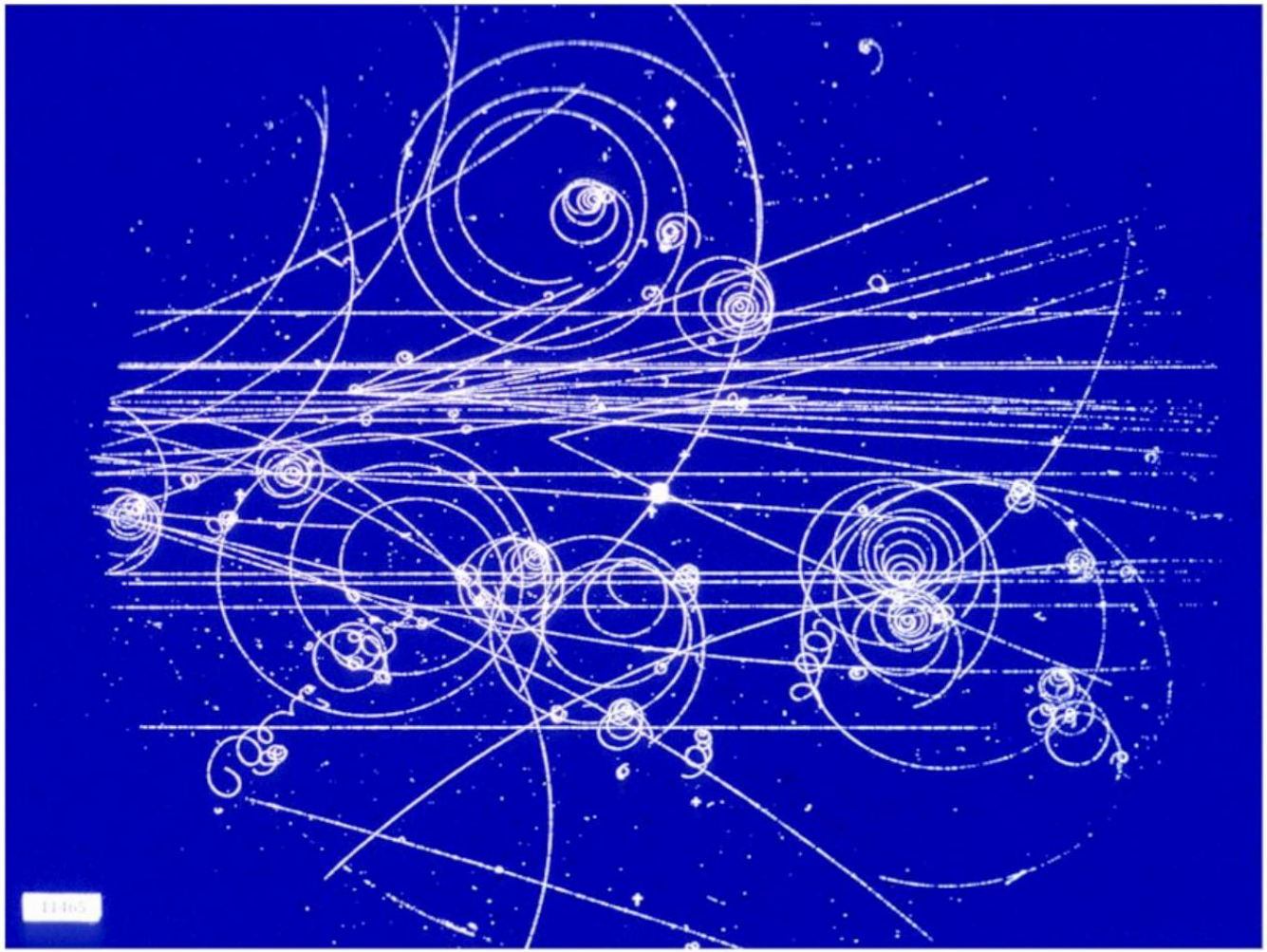
$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Derived the basic interaction in QED taking into account the spins of the fermions and polarization of the virtual photons:

$$-iM = \left[\bar{u}\left(p_3\right) ie\gamma^{\mu} u\left(p_1\right)\right] \frac{-ig_{\mu\nu}}{q^2} \left[\bar{u}\left(p_4\right) ie\gamma^{\nu} u\left(p_2\right)\right]$$

- We now have all the elements to perform proper calculations in QED!

Prof. Alex Mitov, 2026

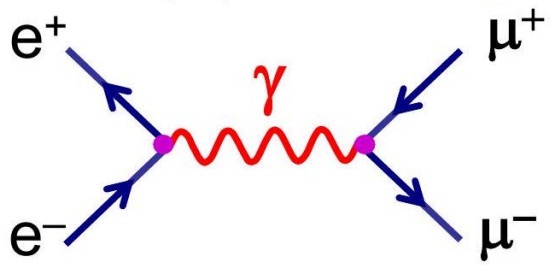


H4: Electron-Positron Annihilation

QED Calculations I

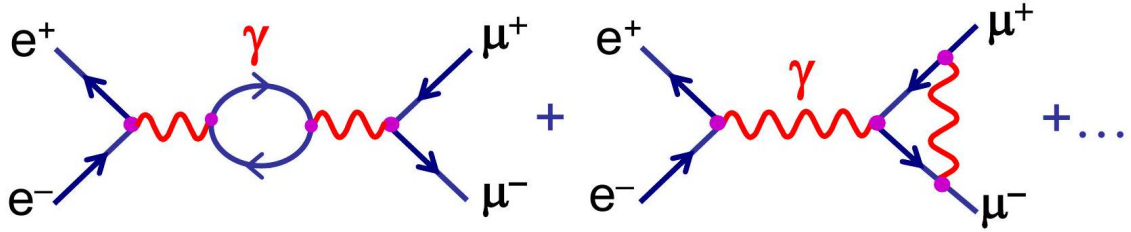
To calculate a cross section using QED (e.g. $e^+e^- \to \mu^+\mu^-$):

1. Draw all possible Feynman Diagrams
   - For $e^+e^- \to \mu^+\mu^-$ there is just one lowest order diagram



$$M \propto e^2 \propto \alpha_{em}$$

   - There are many second order diagrams ...



$$M \propto e^4 \propto \alpha_{em}^2$$

2. For each diagram calculate the matrix element using Feynman rules derived in the previous handout.

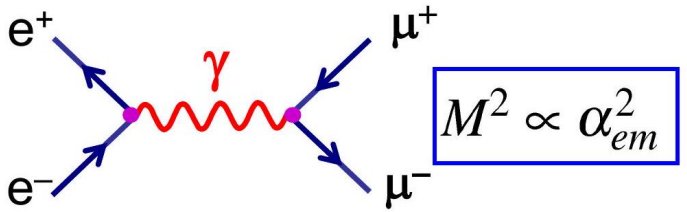
QED Calculations II

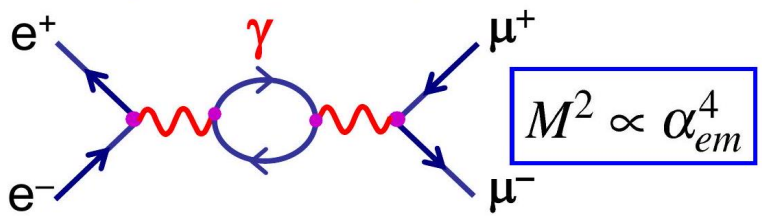
3. Sum the individual matrix elements (i.e. sum the amplitudes)

$$M_{fi} = M_1 + M_2 + M_3 + \ldots$$

Note: in this sum of amplitudes (every term of which has the same initial and the same final state) interference can be both constructive or destructive! On account of this interference one can no more ask 'Which virtual particle was involved?' than one can ask 'Through which of Young's slits did the photon pass?'

4. Find the square of the modulus:
$|M_{fi}|^2 = (M_1 + M_2 + M_3 + \ldots.)\,(M_1^* + M_2^* + M_3^* + \ldots.)$
$\rightarrow$ this gives the full perturbation expansion in α_{em}.
   - For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to neglect higher order diagrams.

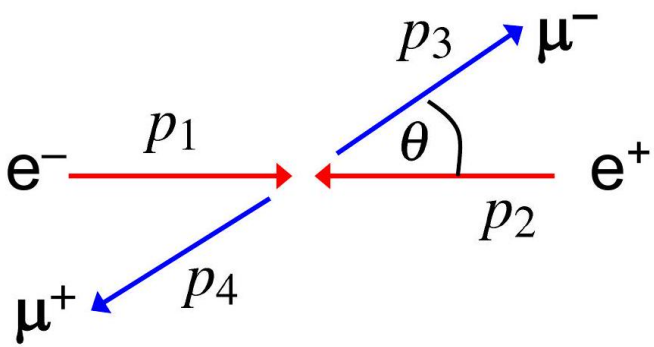




QED Calculations III

5. Calculate decay rate/cross section using formulae from Handout 1. E.g.:
   - For a decay
   $$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 \ \mathrm{d}\Omega.$$
   - For scattering in the centre-of-mass frame
   $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{\left|\vec{p}_f^{\,*}\right|}{\left|\vec{p}_i^{\,*}\right|} |M_{fi}|^2 .$$
   - For scattering in lab. frame (neglecting mass of scattered particle)
   $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 .$$

Electron Positron Annihilation: $e^+e^- \to \mu^+\mu^-$

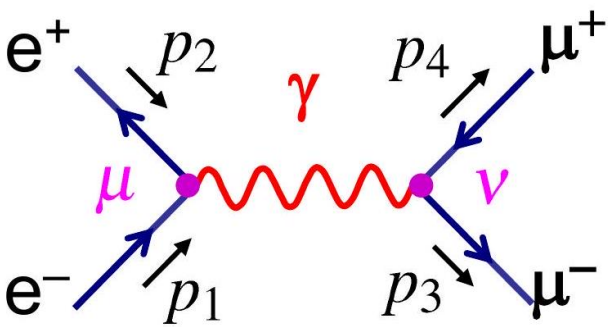


Work in C.o.M. frame so:

$$p_1 = (E, 0, 0, p) \quad p_2 = (E, 0, 0, -p)$$
$$p_3 = (E, \vec{p}_f) \quad p_4 = (E, -\vec{p}_f) \quad \text{and}$$

$$s = (p_1 + p_2)^2 = (2E)^2.$$

Feynman rules for lowest order Feynman diagram give:



$$-iM = [\bar{v}(p_2)\, ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{(p_1+p_2)^2} [\bar{u}(p_3)\, ie\gamma^\nu v(p_4)]$$
$$= [\bar{v}(p_2)\, ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{s} [\bar{u}(p_3)\, ie\gamma^\nu v(p_4)]$$
$$= \frac{ie^2}{s} g_{\mu\nu} [\bar{v}(p_2)\, \gamma^\mu u(p_1)] [\bar{u}(p_3)\, \gamma^\nu v(p_4)].$$

Conclusion

$$M = -\frac{e^2}{s} g_{\mu\nu} [\bar{v}(p_2)\, \gamma^\mu u(p_1)] [\bar{u}(p_3)\, \gamma^\nu v(p_4)]. \tag{54}$$

On the next page we try to write this M more succinctly!

Writing the Annihilation Matrix Element as a Lorentz product of Electron and Muon Currents

In Handout 2 we introduced a four-vector currents like:

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi.$$

These have a similar form to the two terms in [ ] in the matrix element M of (54):

$$M = -\frac{e^2}{s} g_{\mu\nu} \left[\bar{v}\left(p_2\right)\gamma^{\mu} u\left(p_1\right)\right]\left[\bar{u}\left(p_3\right)\gamma^{\nu} v\left(p_4\right)\right]. \qquad (54)$$

Therefore we DEFINE the electron and muon currents as follows:

$$\left(j_e\right)^{\mu} = \bar{v}\left(p_2\right)\gamma^{\mu} u\left(p_1\right) \quad \text{and} \quad \left(j_{\mu}\right)^{\nu} = \bar{u}\left(p_3\right)\gamma^{\nu} v\left(p_4\right).$$

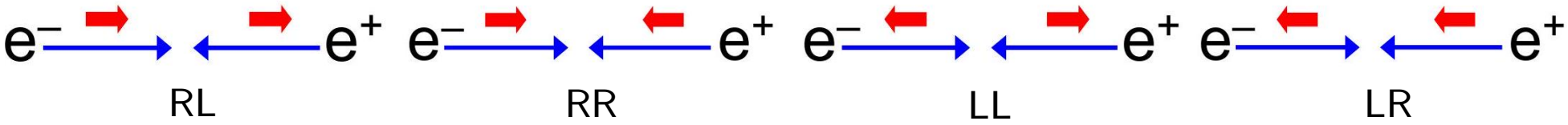
In terms of these electron and muon currents we have the more succint:

$$M = -\frac{e^2}{s} g_{\mu\nu} \left(j_e\right)^{\mu} \left(j_{\mu}\right)^{\nu} \quad \text{or} \quad \boxed{M = -\frac{e^2}{s} j_e \cdot j_{\mu}}. \qquad (55)$$

Matrix element is a four-vector scalar product – confirming it is Lorentz Invariant.

Spin in e^+e^- Annihilation

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the initial state:



- Similarly there are four possible helicity combinations in the final state
- In total there are 16 combinations e.g. $RL \to RR, RL \to RL, \ldots$
- To account for these states we need to sum over all 16 possible helicity combinations and then average over the number of initial helicity states:

$$\left\langle |M|^2 \right\rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4} \left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \ldots \right)$$

i.e. need to evaluate:

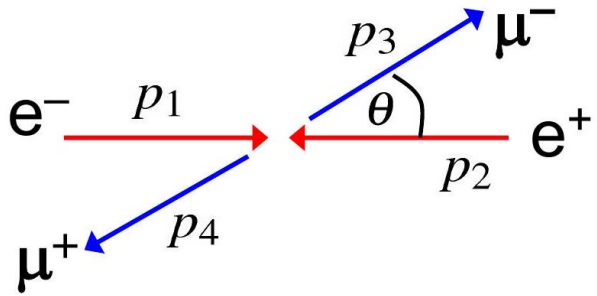
$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

for all 16 helicity combinations !

- Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements - we will see that this is an important feature of QED/QCD.

- In the C.o.M. frame in the limit $E \gg m$

$$p_1 = (E, 0, 0, E); p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



- Left- and right-handed helicity spinors (Handout 2) for particles/anti-particles are:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix} \quad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

where $s = \sin\frac{\theta}{2}$; $\quad c = \cos\frac{\theta}{2}$ and $N = \sqrt{E+m}$

- In the limit $E \gg m$ these become:

$$u_\uparrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; u_\downarrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; v_\uparrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; v_\downarrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}. \tag{56}$$

- The initial-state electron can either be in a left- or right-handed helicity state

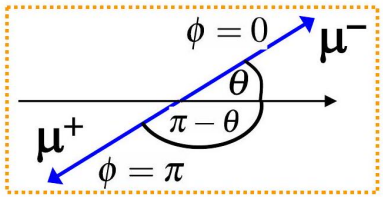
$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix};$$

- For the initial state positron ($\theta = \pi$) can have either:

$$v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}; v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- Similarly for the final state μ^-which has polar angle θ and choosing $\phi = 0$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix}; u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix}$$

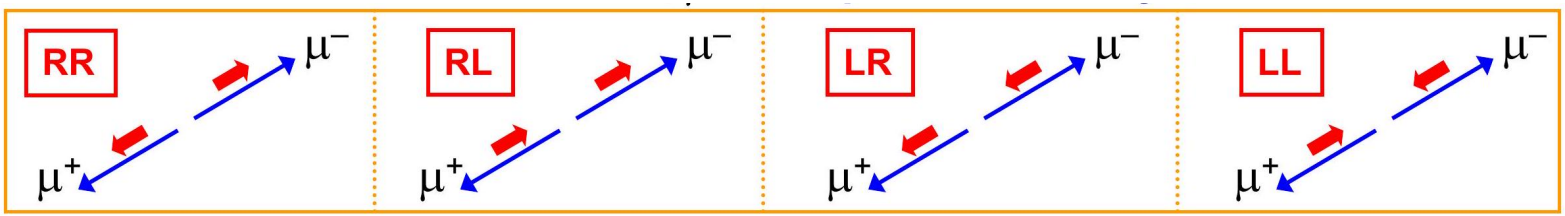


- And for the final state μ^+replacing $\theta \to \pi - \theta$; $\phi \to \pi$ obtain

$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix}; v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix};$$

using $\begin{cases}\sin\left(\frac{\pi-\theta}{2}\right) = \cos\frac{\theta}{2}\\ \cos\left(\frac{\pi-\theta}{2}\right) = \sin\frac{\theta}{2}\\ e^{i\pi} = -1\end{cases}$

- We wish to calculate the matrix element $M = -\frac{e^2}{s} j_e \cdot j_\mu$
- first consider the muon current j_μ for 4 possible helicity combinations



The Muon Current

- We want to evaluate $(j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$ for all four helicity combinations.
- For arbitrary spinors ψ, ϕ one may show that the components of $\bar{\psi}\gamma^\mu\phi$ are:

$$\bar{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4$$
$$\bar{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1$$
$$\bar{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i\left(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1\right)$$
$$\bar{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2$$

- Consider the $\mu_R^-\mu_L^+$ combination using $\psi = u_\uparrow$ and $\phi = v_\downarrow$ with

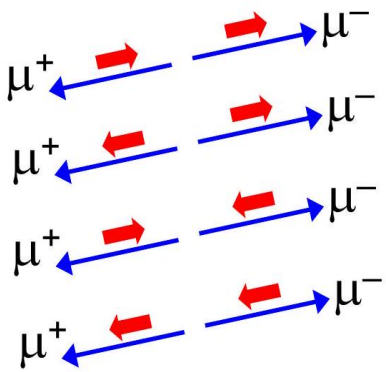
$$v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}; u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}.$$ From these we can compute:

$$\bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0,$$
$$\bar{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E\left(-c^2 + s^2 - c^2 + s^2\right) = 2E\left(s^2 - c^2\right) = -2E\cos\theta,$$
$$\bar{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE\left(-c^2 - s^2 - c^2 - s^2\right) = 2iE, \quad \text{and}$$
$$\bar{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta.$$

- Hence the four-vector muon current for the RL combination is

$$\bar{u}_{\uparrow}(p_3)\,\gamma^{\nu}\,v_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta).$$

- The results for the 4 helicity combinations (obtained in the same manner) are:



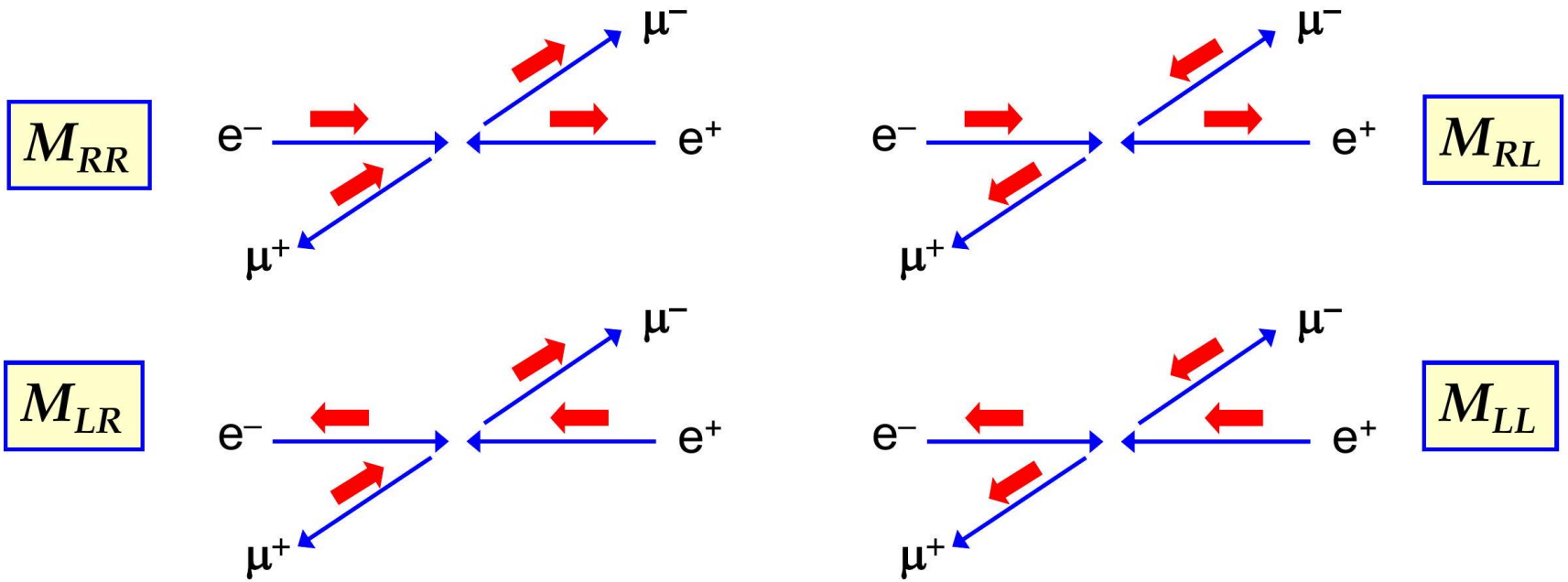
$$\bar{u}_{\uparrow}(p_3)\,\gamma^{\nu}\,v_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$$
$$\bar{u}_{\uparrow}(p_3)\,\gamma^{\nu}\,v_{\uparrow}(p_4) = (0,0,0,0)$$
$$\bar{u}_{\downarrow}(p_3)\,\gamma^{\nu}\,v_{\downarrow}(p_4) = (0,0,0,0)$$
$$\bar{u}_{\downarrow}(p_3)\,\gamma^{\nu}\,v_{\uparrow}(p_4) = 2E(0, -\cos\theta, -i, \sin\theta).$$

... in the limit $E \gg m$ which was used above only two helicity combinations are non-zero!

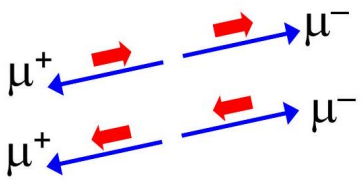
- This is an important feature of QED. It applies equally to QCD.
- In the Weak interaction only **one** helicity combination contributes when $E \gg m$.
- The origin of this will be discussed later (page 120).
- As a consequence: of the 16 possible helicity combinations only 4 give non-zero matrix elements ((2 initial) $\times$ (2 final)).

Electron Positron Annihilation cont.

- For $e^+e^- \to \mu^+\mu^-$ now only have to consider the 4 matrix elements:



- Previously we derived the muon currents for the allowed helicities:



$$\begin{aligned} \mu_R^- \mu_L^+ : \quad \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) &= 2E(0,-\cos\theta,i,\sin\theta) \\ \mu_L^- \mu_R^+ : \quad \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) &= 2E(0,-\cos\theta,-i,\sin\theta) \end{aligned}$$

- Now need to consider the electron current

The Electron Current

- The incoming electron and positron spinors (L and R helicities) are:

$$u_{\uparrow} = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}; u_{\downarrow} = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}; \quad v_{\uparrow} = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}; v_{\downarrow} = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- The electron current can either be obtained from equations (3)-(6) as before or it can be obtained directly from the expressions for the muon current.

$$(j_e)^{\mu} = \bar{v}(p_2)\gamma^{\mu}u(p_1) \quad (j_{\mu})^{\mu} = \bar{u}(p_3)\gamma^{\mu}v(p_4)$$

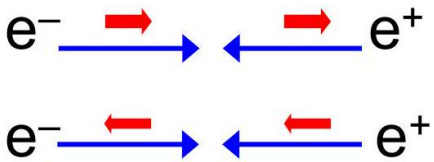
- Taking the Hermitian conjugate of the muon current gives

$$\begin{aligned}
[\bar{u}(p_3)\gamma^{\mu}v(p_4)]^{\dagger} &= \left[u(p_3)^{\dagger}\gamma^0\gamma^{\mu}v(p_4)\right]^{\dagger} \\
&= v(p_4)^{\dagger}\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_3) && (AB)^{\dagger} = B^{\dagger}A^{\dagger} \\
&= v(p_4)^{\dagger}\gamma^{\mu\dagger}\gamma^0 u(p_3) && \gamma^{0\dagger} = \gamma^0 \\
&= v(p_4)^{\dagger}\gamma^0\gamma^{\mu}u(p_3) && \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^{\mu} \\
&= \bar{v}(p_4)\gamma^{\mu}u(p_3)
\end{aligned}$$

- Taking the complex conjugate of the muon currents for the two non-zero helicity configurations:

$$\bar{v}_{\downarrow}(p_4)\,\gamma^{\mu}\,u_{\uparrow}(p_3) = [\bar{u}_{\uparrow}(p_3)\,\gamma^{\nu}\,v_{\downarrow}(p_4)]^{*} = 2E(0,-\cos\theta,-i,\sin\theta)$$
$$\bar{v}_{\uparrow}(p_4)\,\gamma^{\mu}\,u_{\downarrow}(p_3) = [\bar{u}_{\downarrow}(p_3)\,\gamma^{\nu}\,v_{\uparrow}(p_4)]^{*} = 2E(0,-\cos\theta,i,\sin\theta)$$

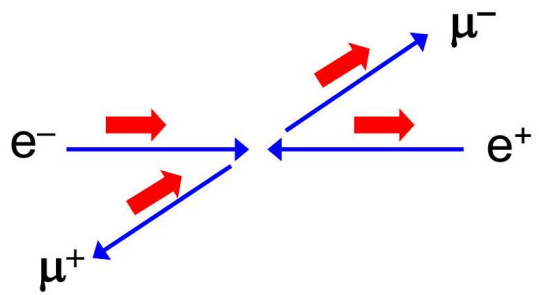
To obtain the electron currents we simply need to set $\theta = 0$



$$\begin{aligned} e_R^- e_L^+ &: \quad \bar{v}_{\downarrow}(p_2)\gamma^{\nu}u_{\uparrow}(p_1) = 2E(0,-1,-i,0) \\ e_L^- e_R^+ &: \quad \bar{v}_{\uparrow}(p_2)\gamma^{\nu}u_{\downarrow}(p_1) = 2E(0,-1,i,0) \end{aligned}$$

Matrix Element Calculation

- We can now calculate $M = -\frac{e^2}{s} j_e \cdot j_\mu$ for the four possible helicity combinations M_{RR}, M_{RL}, M_{LR} and M_{LL}.
- In the above matrix element names, the first subscript refers to the helicity of the e^- and the second to the helicity of the μ^-. We don't need to specify other helicities due to 'helicity conservation', only certain chiral combinations are non-zero.
- E.g. the matrix element for $e_R^- e_L^+ \to \mu_R^- \mu_L^+$ will be denoted by M_{RR}



Using:

$$e_R^- e_L^+ : \quad (j_e)^\mu = \bar{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) = 2E(0,-1,-i,0)$$
$$\mu_R^- \mu_L^+ : \quad (j_\mu)^\nu = \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta,i,\sin\theta)$$

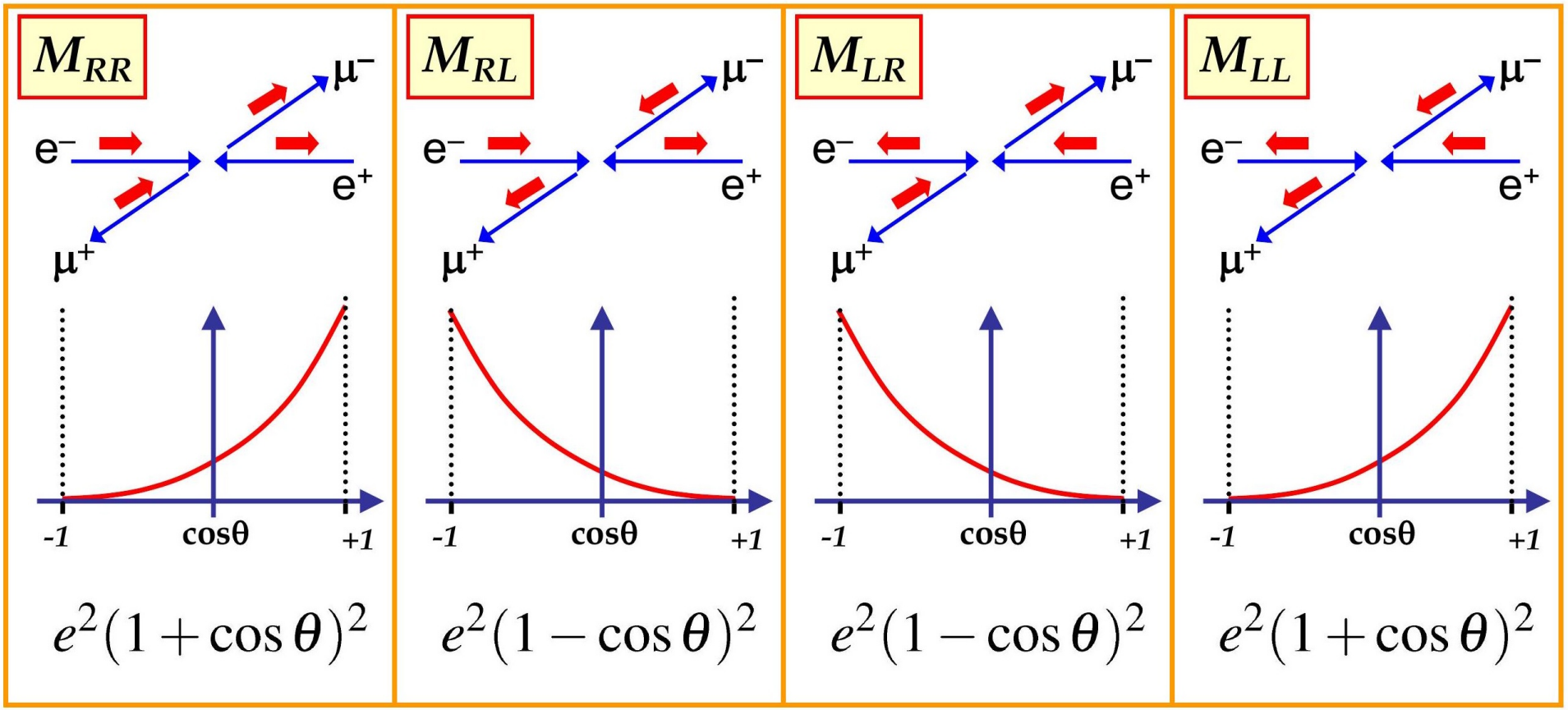
gives

$$\begin{aligned} M_{RR} &= -\frac{e^2}{s}[2E(0,-1,-i,0)] \cdot [2E(0,-\cos\theta,i,\sin\theta)] \\ &= e^2(1+\cos\theta) \\ &= 4\pi\alpha(1+\cos\theta) \end{aligned} \tag{57}$$

where $\alpha = e^2/4\pi \approx 1/137$.

Similarly

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$
$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$



- Assuming that the incoming electrons and positrons are unpolarized, all 4 possible initial helicity states are equally likely.

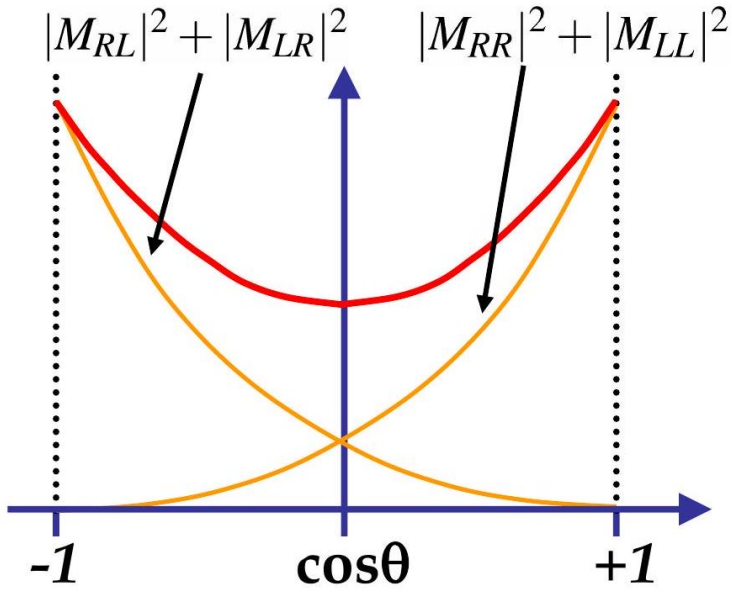
Differential Cross Section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

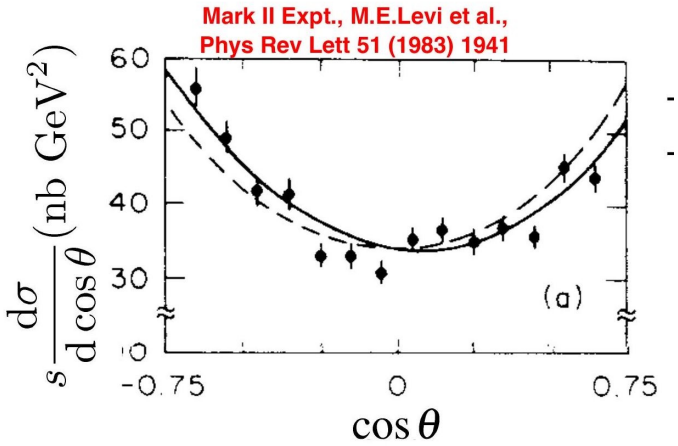
$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \times \frac{1}{64\pi^2 s}\left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + \left|M_{LL}^2\right|\right)$$

$$= \frac{(4\pi\alpha)^2}{256\pi^2 s}\left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$



Example: $e^+e^- \to \mu^+\mu^-$ $\sqrt{s} = 29$ GeV



- dashed line = pure QED
- solid line = QED plus Z contribution
- Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

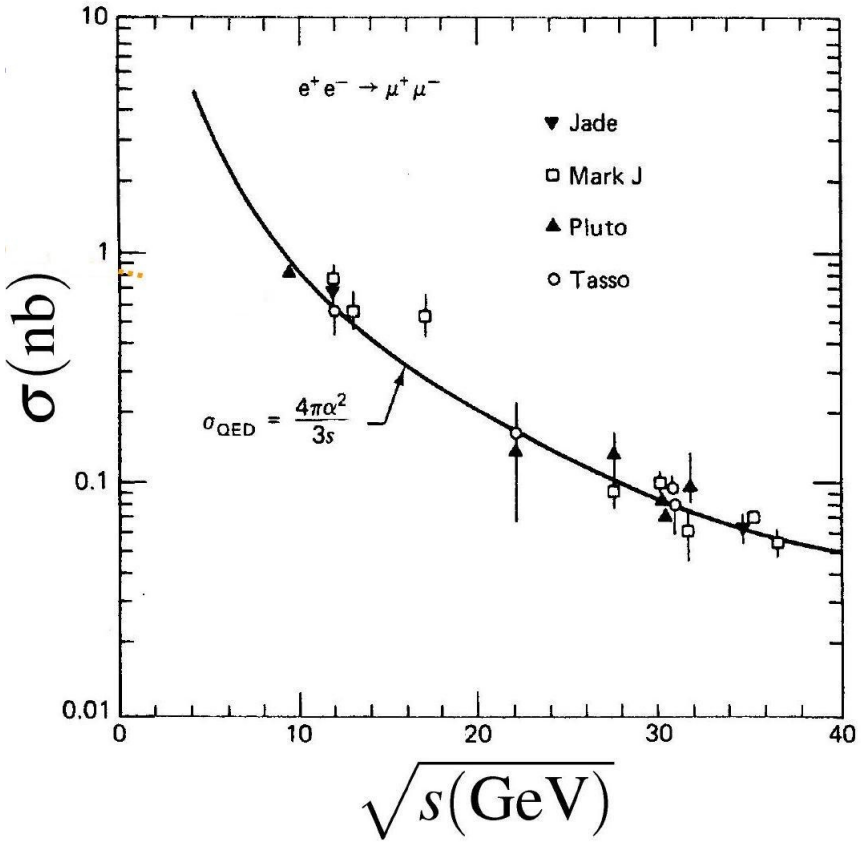
- The total cross section is obtained by integrating over θ, ϕ using

$$\int \left(1+\cos^2\theta\right) \mathrm{d}\Omega = 2\pi \int_{-1}^{+1} \left(1+\cos^2\theta\right) \mathrm{d}\cos\theta = \frac{16\pi}{3}.$$

- The QED total cross-section for the process $e^+e^- \to \mu^+\mu^-$ is found to be

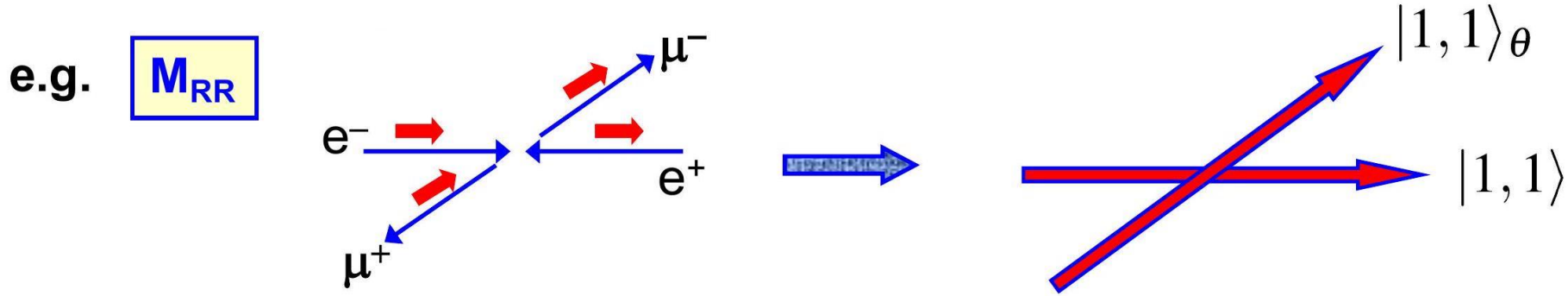
$$\sigma = \frac{4\pi\alpha^2}{3s}.$$

- This is an impressive result. From first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to $\sim 1\%$.



Spin Considerations ($E \gg m$)

- The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum
- Because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$, i.e. in a total spin 1 state aligned along the z axis: $|1, +1\rangle$ or $|1, -1\rangle$
- Similarly the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle θ



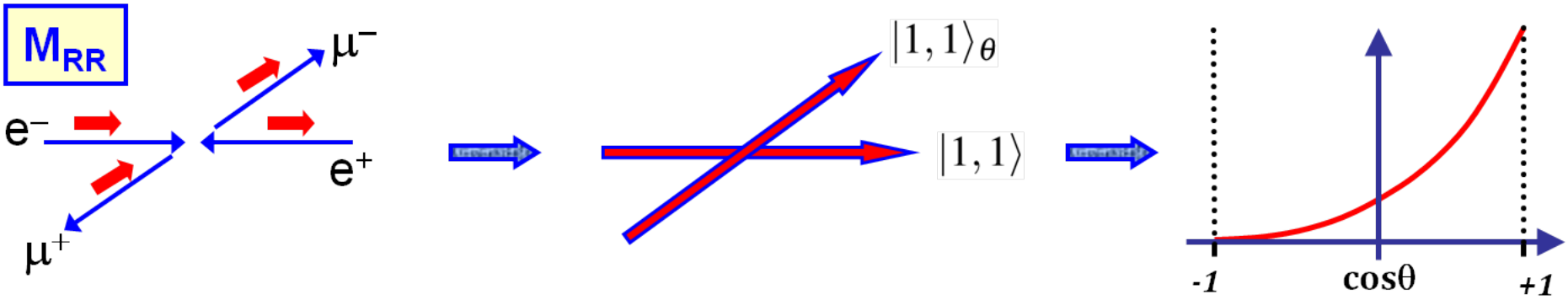
- Hence $M_{\mathrm{RR}} \propto \langle \psi \mid 1, 1 \rangle$ where ψ corresponds to the spin state, $|1, 1\rangle_\theta$, of the muon pair.
- To evaluate this need to express $|1, 1\rangle_\theta$ in terms of eigenstates of S_z
- In Appendix IX (and also in IB QM) it is shown that:

$$|1,1\rangle_\theta = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle.$$

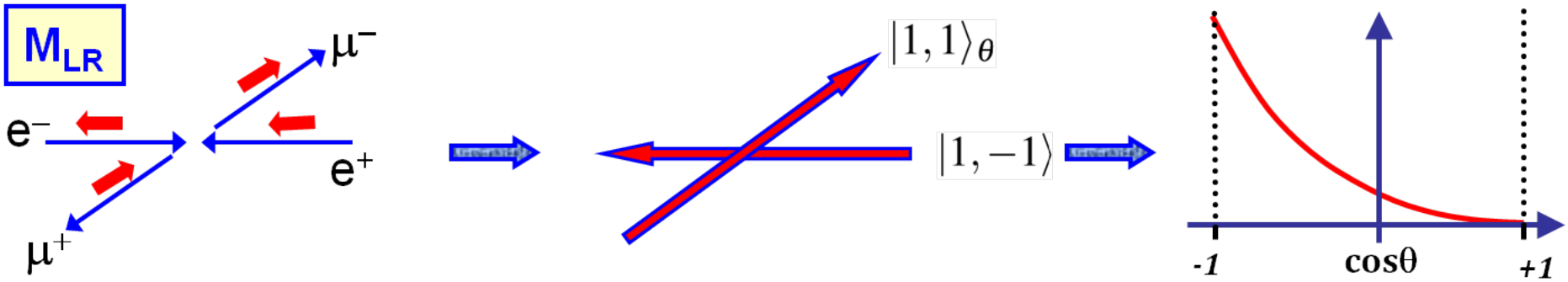
- Using the wave-function for a spin 1 state along an axis at angle θ

$$\psi = |1,1\rangle_\theta = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle$$

we can immediately understand the angular dependence:



$$|M_{\mathrm{RR}}|^2 \propto |\langle\psi \mid 1,+1\rangle|^2 = \frac{1}{4}(1+\cos\theta)^2$$

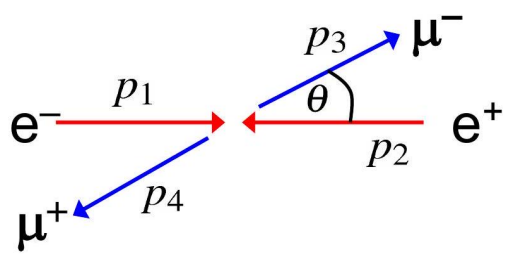


$$|M_{\mathrm{LR}}|^2 \propto |\langle\psi \mid 1,-1\rangle|^2 = \frac{1}{4}(1-\cos\theta)^2$$

Lorentz Invariant form of Matrix Element

- Before concluding this discussion, note that the spin-averaged Matrix Element derived above is written in terms of the muon angle in the C.o.M. frame.

$$\begin{aligned}\left\langle |M_{fi}|^2 \right\rangle &= \frac{1}{4} \times \left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + \left| M_{LL}^2 \right| \right) \\ &= \frac{1}{4} e^4 \left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2 \right) \\ &= e^4 \left(1 + \cos^2\theta \right)\end{aligned}$$



- The matrix element is Lorentz Invariant (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz Invariant 4-vector scalar products.
- In the C.o.M. $p_1 = (E, 0, 0, E)$ $p_2 = (E, 0, 0, -E)$

$$p_3 = (E, E\sin\theta, 0, E\cos\theta) \quad \text{and} \quad p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

giving: $p_1 \cdot p_2 = 2E^2$; $p_1 \cdot p_3 = E^2(1-\cos\theta)$; $p_1 \cdot p_4 = E^2(1+\cos\theta)$

- Hence we can write

$$\left\langle |M_{fi}|^2 \right\rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \left(\frac{t^2 + u^2}{s^2} \right)$$

which is valid in any frame!

Tidying up a source of frustration

QED scattering processes seem to look 'simpler' in the $E \gg m$ limit, but ugly when $E \sim m$. [E.g. page 109 showed that certain helicity combinations disappear when $E \gg m$.]

This motivates the following question:

Is there a better way of categorising particles so that scattering looks simple for **all energies**? (Not just nice for $E \gg m$!)

There is a nice answer:

QED scattering at all energies looks nice in the so-called **Chiral Basis**.

- The next few slides will explore what Chirality is.
- Even though we will see that Chirality is completely different to Helicity, we will see that Chirality happens to coincide with Helicity in the limit $E \gg m$.

$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and the CHIRAL projector operators P_R and P_L

Example sheet Question 10 should persuade you that the matrix γ^5 defined thus:

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3} \tag{58}$$

has the following properties for **any** representation of gamma matrices:

$$(\gamma^5)^2 = 1, \tag{59}$$

$$\gamma^{5\dagger} = \gamma^5, \qquad \text{(i.e. } \gamma^5 \text{ is Hermitian, } \textbf{and so might be observable}\text{)}, \tag{60}$$

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5, \quad \text{(i.e. } \gamma^5 \text{ anticommuites with all other } \gamma\text{-matrices)} \tag{61}$$

and that therefore the operators

$$\boxed{P_R = \frac{1}{2}(1+\gamma^5)} \text{ and } \boxed{P_L = \frac{1}{2}(1-\gamma^5)} \tag{62}$$

are complementary projection operators
($P_R^2 = P_R$, $P_L^2 = P_L$ and $P_R + P_L = 1$) satisfying

$$P_R^\dagger = P_R \tag{63}$$

$$P_L^\dagger = P_L \tag{64}$$

$$P_R\gamma^\mu = \gamma^\mu P_L \qquad \text{and} \tag{65}$$

$$P_L\gamma^\mu = \gamma^\mu P_R. \tag{66}$$

Aside: in the Dirac Representation:

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$P_R = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} \qquad \text{and} \tag{67}$$

$$P_L = \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{68}$$

The complementary projection operators P_R and P_L allow us to write any spinor ψ as the sum of **RIGHT** and **LEFT CHIRAL** parts ψ_R and ψ_L.

Definition:

$$\psi_R \equiv P_R\psi, \qquad \psi_L \equiv P_L\psi, \tag{69}$$

$$\bar{\psi}_R \equiv \overline{P_R\psi} \quad \text{and} \quad \bar{\psi}_L \equiv \overline{P_L\psi}. \tag{70}$$

Check that the chiral parts of each spinor satisfy

$$\psi \equiv \psi_R + \psi_L, \quad \text{and} \tag{71}$$

$$\bar{\psi} \equiv \bar{\psi}_R + \bar{\psi}_L \tag{72}$$

and are eigenstates of γ^5 with opposite chirality:

$$\gamma^5\psi_R = +\psi_R \qquad \text{and} \qquad \gamma^5\psi_L = -\psi_L. \tag{73}$$

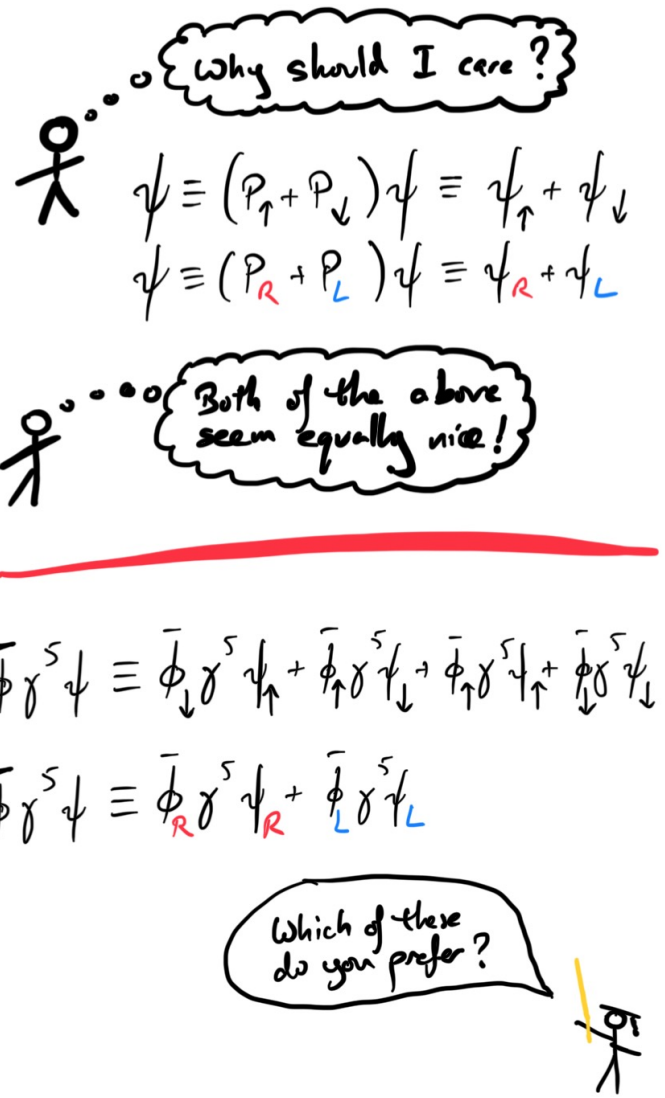
Example sheet Q10 asks you to prove that

$$\bar{\psi}_R \equiv \bar{\psi}P_L \tag{74}$$

$$\bar{\psi}_L \equiv \bar{\psi}P_R \tag{75}$$

and that therefore $\bar{\phi}_R\gamma^\mu\psi_L = \bar{\phi}_L\gamma^\mu\psi_R = 0$ and so

$$\boxed{\bar{\phi}\gamma^\mu\psi \equiv \bar{\phi}_R\gamma^\mu\psi_R + \bar{\phi}_L\gamma^\mu\psi_L}. \tag{76}$$



Link between HELICITY and CHIRALITY when $E \gg m$

- The helicity eigenstates for particles and anti-particles shown in (56) were:

$$u_\uparrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; u_\downarrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; v_\uparrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; v_\downarrow\Big|_{E\gg m} = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}.$$

- From P_R and P_L ((67), (68) and right) we note:

$$u_\uparrow\Big|_{E\gg m} = P_R u_\uparrow\Big|_{E\gg m}, \quad u_\downarrow\Big|_{E\gg m} = P_L u_\downarrow\Big|_{E\gg m},$$

$$v_\uparrow\Big|_{E\gg m} = P_L v_\uparrow\Big|_{E\gg m}, \quad v_\downarrow\Big|_{E\gg m} = P_R v_\downarrow\Big|_{E\gg m}.$$

Dirac Representation has:

$$P_R = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} \quad \text{and}$$

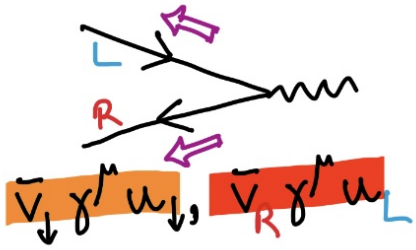
$$P_L = \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

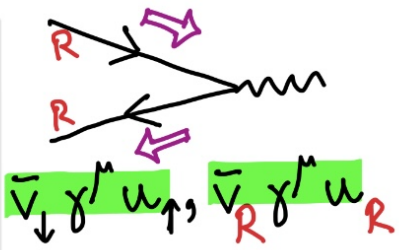
Summary: in the high energy limit $E \gg m$ ...

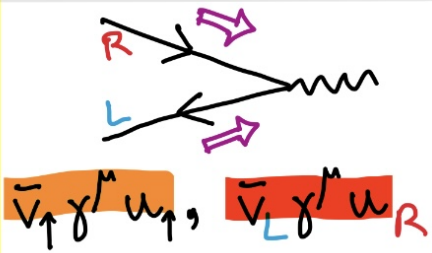
Right Helicity Particles and Left Helicity Anti-Particles are Right Chiral, and
Left Helicity Particles and Right Helicity Anti-Particles are Left Chiral!

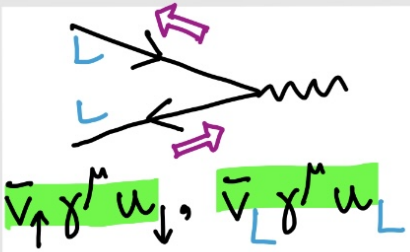
Favoured/Disfavoured/Forbidden QED Helicity and Chirality Combinations

Annihilation

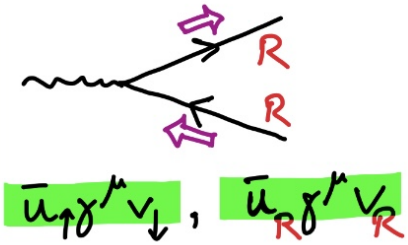


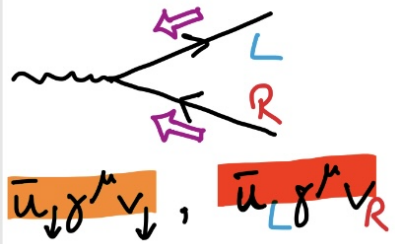


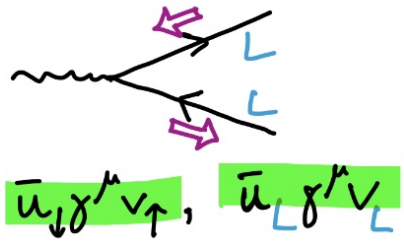


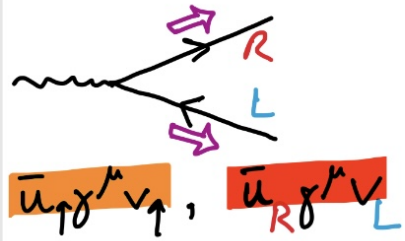


Pair Production

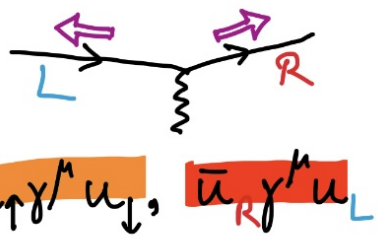


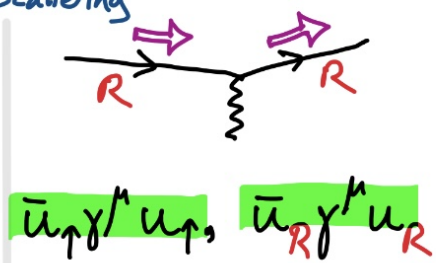


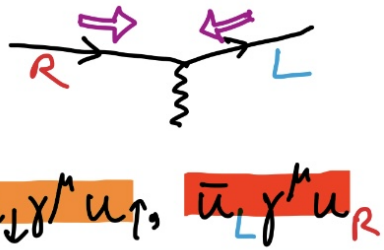


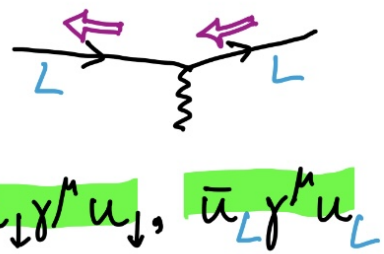


Particle Scattering

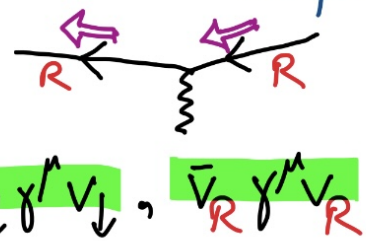


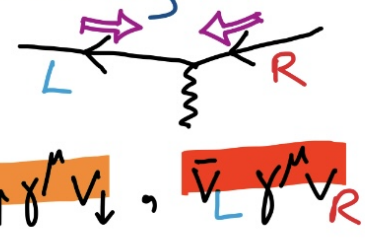


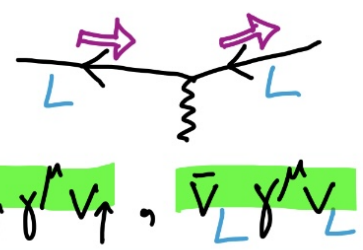


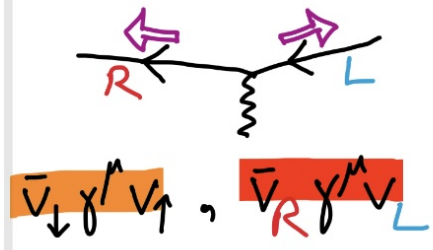


Antiparticle Scattering









Summary

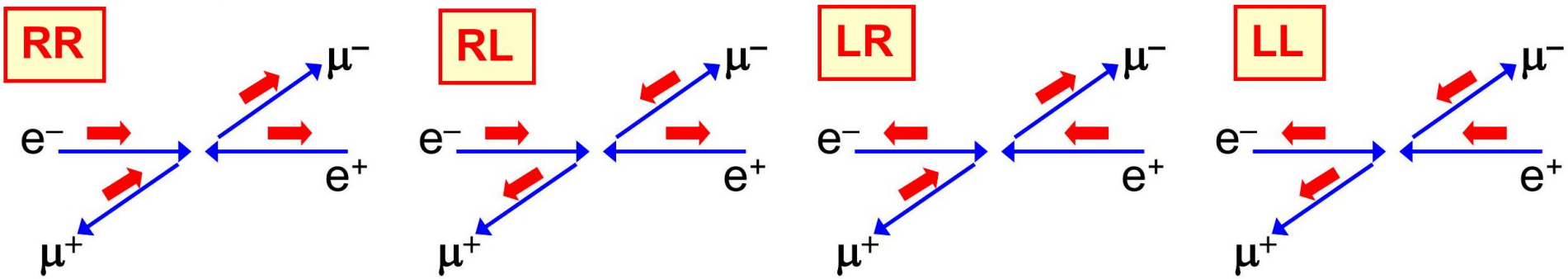
- In the centre-of-mass frame the $e^+e^- \to \mu^+\mu^-$ differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right).$$

  NOTE: neglected masses of the muons, i.e. assumed $E \gg m_\mu$.
- In QED only certain combinations of LEFT- and RIGHT-HANDED CHIRAL states give non-zero matrix elements.
- CHIRAL states defined by chiral projection operators:

$$P_R = \frac{1}{2}\left(1+\gamma^5\right); \quad P_L = \frac{1}{2}\left(1-\gamma^5\right).$$

- In limit $E \gg m$ the chiral eigenstates correspond to the HELICITY eigenstates and only certain HELICITY combinations give non-zero matrix elements:



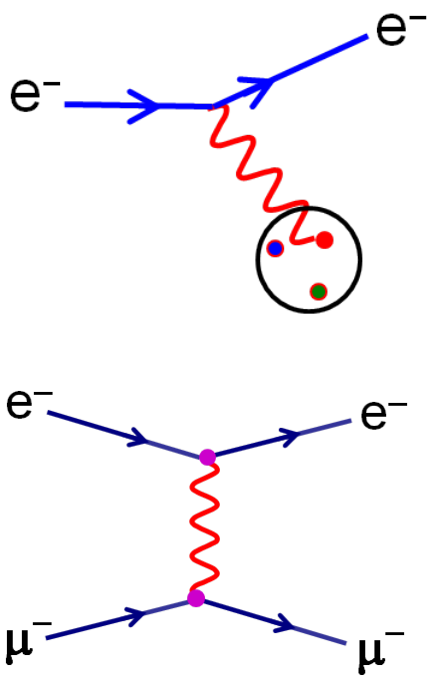
Prof. Alex Mitov, 2026



H5: Electron-Proton Elastic Scattering

Electron-Proton Scattering

- In this handout aiming towards a study of electron-proton scattering as a probe of the structure of the proton
- Two main topics:
  - $e^- p \to e^- p$ elastic scattering (this handout)
  - $e^- p \to e^- X$ deep inelastic scattering (Handout 6)
- But first consider scattering from a point-like particle e.g. $e^- \mu^- \to e^- \mu^-$ in the ultra-relativistic limit ($E \gg m$).



Two ways to proceed:

- perform QED calculation from scratch (example sheet Q11)

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) \right]$$

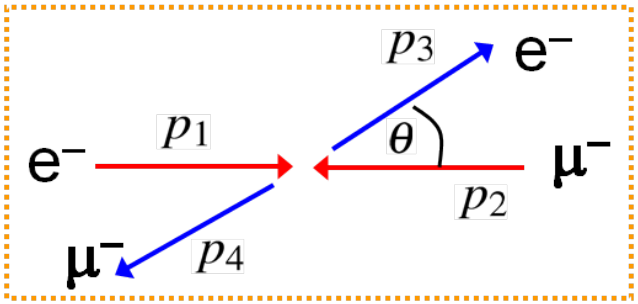
- or take results from $e^+ e^- \to \mu^+ \mu^-$ and use "Crossing Symmetry" to obtain the matrix element for $e^- \mu^- \to e^- \mu^-$ (Appendix IX).

Consequences of scattering being t-channel rather than s-channel

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2} \equiv 2e^4 \left(\frac{s^2 + u^2}{t^2} \right)$$

Work in the C.o.M.:

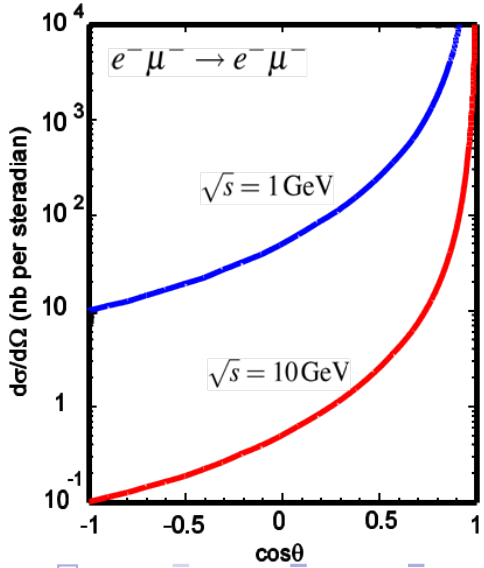
$$p_1 = (E, 0, 0, E)$$
$$p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



giving $p_1.p_2 = 2E^2$; $p_1.p_3 = E^2(1-\cos\theta)$; $p_1.p_4 = E^2(1+\cos\theta)$ so

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{E^4(1+\cos\theta)^2 + 4E^4}{E^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle |M_{fi}|^2 \rangle = \frac{e^4}{8\pi^2 s} \frac{\left[1 + \frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$

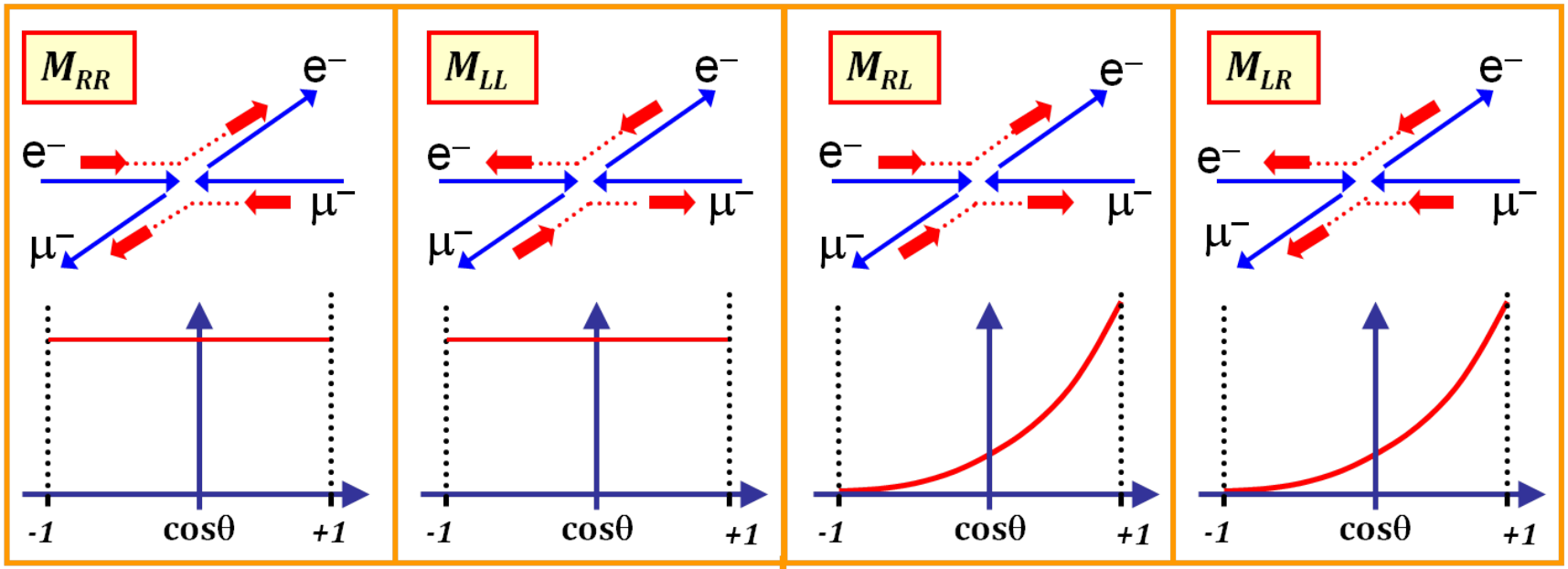


The denominator arises from the propagator $-ig_{\mu\nu}/q^2$ here $q^2 = (p_1 - p_3)^2 = E^2(1-\cos\theta)$. As $q^2 \to 0$ the cross section tends to infinity like $\frac{1}{q^4}$.

- What about the angular dependence of the numerator ?

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{8\pi^2 s} \frac{\left[1 + \frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$

- The factor $1 + \frac{1}{4}(1+\cos\theta)^2$ reflects helicity (really chiral) structure of QED
- Of the 16 possible helicity combinations only 4 are non-zero:



$S_z = 0$ → $\frac{d\sigma}{d\Omega} \propto 1$

$S_z = +1$ $\quad$ $S_z = -1$ → $\frac{d\sigma}{d\Omega} \propto \frac{1}{4}(1+\cos\theta)^2$

i.e. no preferred polar angle

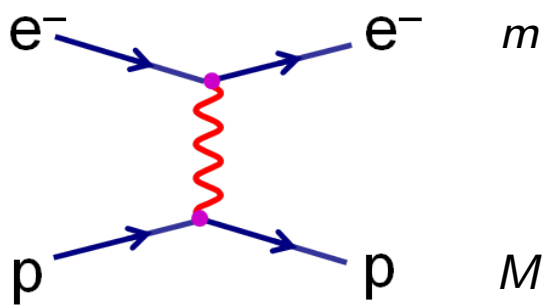
spin 1 rotation again

- The cross section calculated above is appropriate for the scattering of two spin-half Dirac (i.e. point-like) particles in the ultra-relativistic limit (where both electron and muon masses can be neglected). In this case:

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2}.$$

- We will use this again in the discussion of "Deep Inelastic Scattering" of electrons from the quarks within a proton (Handout 6).
- Before doing so we will consider the scattering of electrons from protons – the results of which will show us that the proton is not a fundamental "point-like" particle!

- When doing so we will not be able to use the relativistic limit (because the proton is too heavy!) and so we require the general expression for the matrix element (derived in the optional part of Question 11 in the examples sheet):

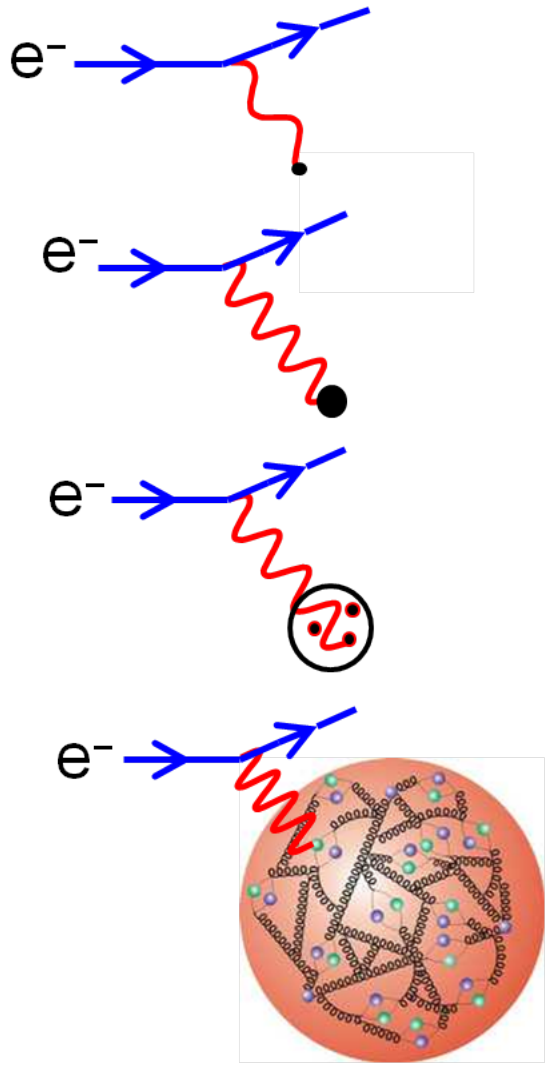


$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_1.p_4)m^2 + 2m^2M^2 \right] \quad (77)$$

Probing the Structure of the Proton

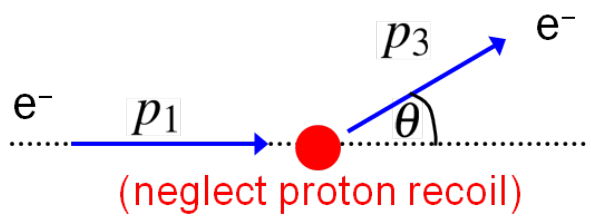
In $e^- p \to e^- p$ scattering the nature of the interaction of the virtual photon with the proton depends strongly on wavelength:

- At very low electron energies $\lambda \gg r_p$: the scattering is equivalent to that from a "point-like" spin-less object.
- At low electron energies $\lambda \sim r_p$: the scattering is equivalent to that from a extended charged object.
- At high electron energies $\lambda < r_p$: the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks.
- At very high electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons.



Rutherford Scattering Revisited

- Rutherford scattering is the low energy limit where the recoil of the proton can be neglected and the electron is non-relativistic.
- Start from RH and LH Helicity particle spinors



$$u_\uparrow = N \begin{pmatrix} c \\ e^{i\phi} s \\ \frac{p}{E+m} c \\ \frac{p}{E+m} e^{i\phi} s \end{pmatrix}; \quad u_\downarrow = N \begin{pmatrix} -s \\ e^{i\phi} c \\ \frac{p}{E+m} s \\ -\frac{p}{E+m} e^{i\phi} c \end{pmatrix}$$

with $N = \sqrt{E+m}$,
$s = \sin(\theta/2)$, and
$c = \cos(\theta/2)$.

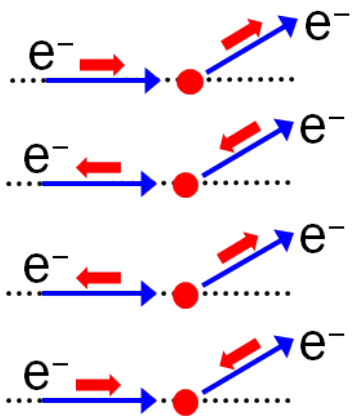
- In terms of $\alpha = \frac{|\vec{p}|}{E+m_e}$ $\left\{ \begin{array}{ll} \text{Non-relativistic limit:} & \alpha \to 0 \\ \text{Ultra-relativistic limit:} & \alpha \to 1 \end{array} \right\}$ we could write:

$$u_\uparrow = N \begin{pmatrix} c \\ e^{i\phi} s \\ \alpha c \\ \alpha e^{i\phi} s \end{pmatrix}; \quad u_\downarrow = N \begin{pmatrix} -s \\ e^{i\phi} c \\ \alpha s \\ -\alpha e^{i\phi} c \end{pmatrix}$$

and the possible initial and final state electron spinors are:

$$u_\uparrow(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix}; \; u_\downarrow(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix}; \; u_\uparrow(p_3) = N_e \begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix}; \; u_\downarrow(p_3) = N_e \begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}.$$

Consider all four possible electron currents: helicities $R \to R$, $L \to L$, $L \to R$, $R \to L$:



$$\overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right], \quad (78)$$

$$\overline{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right], \quad (79)$$

$$\overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right], \quad (80)$$

$$\overline{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right]. \quad (81)$$

- In the relativistic limit $E \gg m$ (i.e. $\alpha = 1$) the expressions in (80) and (81) are identically zero; only $R \to R$ and $L \to L$ combinations are non-zero.
- In the non-relativistic limit, $|\vec{p}| \ll E$, we have $\alpha = 0$ so:

$$\overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = +\overline{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = (2m_e)\left[c, 0, 0, 0\right],$$
$$\overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = -\overline{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = (2m_e)\left[s, 0, 0, 0\right].$$

All four electron helicity combinations have non-zero Matrix Element,

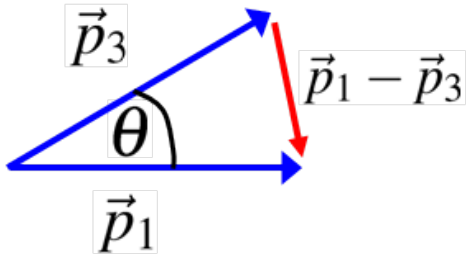
i.e. Helicity eigenstates $\neq$ Chirality eigenstates!

- The initial and final state proton spinors (assuming no recoil, so at rest in both places) are: $u_\uparrow(0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $u_\downarrow(0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ giving the proton currents:

$$j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p\,(1,0,0,0)$$
$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$$

The spin-averaged ME summing over the 8 allowed helicity

$$\langle|M_{fi}^2|\rangle = \frac{1}{4}\frac{e^4}{q^4}(16M_p^2m_e^2)(4c^2+4s^2) = \frac{16M_p^2m_e^2e^4}{q^4}$$



where $q^2 = (p_1 - p_3)^2 = (0, \vec{p_1} - \vec{p_3})^2 = -4|\vec{p}|^2\sin^2(\theta/2)$

$$\langle|M_{fi}^2|\rangle = \frac{M_p^2m_e^2e^4}{|\vec{p}|^4\sin^4(\theta/2)} \tag{82}$$

where $\vec{p} = \vec{p}_1 - \vec{p}_3$. (Note: in this limit all angular dependence is in the propagator!)

- The formula for the differential cross-section in the lab. frame was derived in Handout 1:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2|M_{fi}|^2 \tag{83}$$

- Here the electron is non-relativistic so $E \sim m_e \ll M_p$ and we can neglect E_1 in the denominator of equation (83)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2} |M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = \frac{p^2}{2m_e}$:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16 E_K^2 \sin^4 \frac{\theta}{2}}} \tag{84}$$

- This is the normal expression for the Rutherford cross section. It could have been derived by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton $V(\vec{r})$, **without any consideration of the interaction due to the intrinsic magnetic moments** of the electron or proton. From this we can conclude that **in this non-relativistic limit** only the interaction between the electric charges of the particles matters.

The Mott Scattering Cross Section

- For Rutherford scattering we were in the limit where the target recoil was neglected and the scattered particle was non-relativistic $E_K \ll m_e$.
- The limit where the target recoil is neglected and the scattered particle is relativistic (i.e. just neglect the electron mass) is called Mott Scattering .
- In this limit $(\alpha \to 1)$ the electron currents, equations (78) and (80) become:
  $\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E\,[c, s, -is, c]$ $\qquad$ $\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = E\,[0, 0, 0, 0]$

  Relativistic Electron $\Longrightarrow$ "helicity conserved"

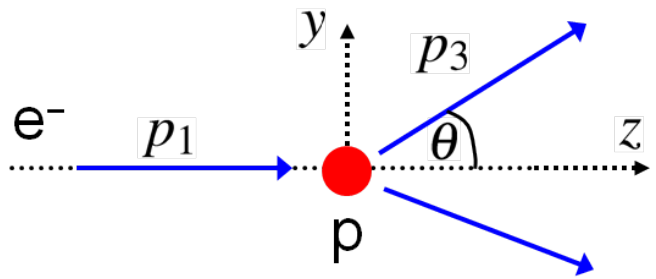
- It is then straightforward to obtain the result:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \underbrace{\frac{\alpha^2}{4E^2 \sin^4 \theta/2}}_{\substack{\text{Rutherford formula with} \\ E_K = E\ (E \gg m_e)}} \cdot \underbrace{\cos^2 \frac{\theta}{2}}_{\substack{\text{Overlap between initial/final-state} \\ \text{electron wave-function (spin-}\frac{1}{2}\text{)}}} \tag{85}$$

- NOTE: we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$ The interaction is ELECTRIC rather than magnetic (spin-spin) in nature.
- Still haven't taken into account the charge distribution of the proton ...

Point-like Electron-Proton Elastic Scattering with RECOIL

- So far have only considered the case we the proton does not recoil... For $E_1 \gg m_e$ the general case is:



$$p_1 = (E_1, 0, 0, E_1)$$
$$p_2 = (M, 0, 0, 0)$$
$$\boxed{p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)}$$
$$p_4 = (E_4, \vec{p}_4)$$

- From Eqn. (77) with $m = m_e = 0$ the matrix element for this process is:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 \right] \qquad (86)$$

- Experimentally observe scattered electron so eliminate p_4. The scalar products not involving p_4 are:

$$p_1.p_2 = E_1 M, \quad p_1.p_3 = E_1 E_3(1 - \cos\theta), \quad p_2.p_3 = E_3 M.$$

- Can eliminate p_4 using momentum conservation: $p_4 = p_1 + p_2 - p_3$

$$p_3.p_4 = p_3.p_1 + p_3.p_2 - \cancelto{0}{p_3.p_3} = E_1 E_3(1 - \cos\theta) + E_3 M$$
$$p_1.p_4 = \cancelto{0}{p_1.p_1} + p_1.p_2 - p_1.p_3 = E_1 M - E_1 E_3(1 - \cos\theta)$$

- Substituting these scalar products into equation (86) gives

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{8e^4}{(p_1-p_3)^4} ME_1E_3\left[(E_1-E_3)(1-\cos\theta)+M(1+\cos\theta)\right] \\ &= \frac{8e^4}{(p_1-p_3)^4} 2ME_1E_3\left[(E_1-E_3)\sin^2(\theta/2)+M\cos^2(\theta/2)\right] \qquad (87)\end{aligned}$$

- Now obtain expressions for $q^4=(q^2)^2=(p_1-p_3)^4$ and (E_1-E_3) neglecting mass of electron:

$$q^2=(p_1-p_3)^2=p_1^2+p_3^2-2p_1.p_3\approx -2E_1E_3(1-\cos\theta) \qquad (88)$$

$$=-4E_1E_3\sin^2\theta/2 \qquad (89)$$

NOTE: $q^2<0$ (space-like!). [See page 153 for proof outside of high energy limit.]

- To get (E_1-E_3) note that

$$q.p_2=(p_1-p_3).p_2=M(E_1-E_3)$$

and so it suffices to find $q\cdot p_2$. And $q\cdot p_2$ may be found from:

$$\begin{aligned}(q+p_2)^2 &= p_4^2 \quad\Longrightarrow \\ q^2+p_2^2+2q.p_2 &= p_4^2 \quad\Longrightarrow \\ q^2+M^2+2q.p_2 &= M^2 \quad\Longrightarrow \\ q.p_2 &= -q^2/2\end{aligned}$$

- ... hence the energy transferred to the proton, $E_1 - E_3$, is:

$$\boxed{E_1 - E_3 = -\frac{q^2}{2M}} \tag{90}$$

  Because q^2 is always negative $E_1 - E_3 > 0$ and the scattered electron is always lower in energy than the incoming electron
- Combining equations (86), (88) and (89):

$$\begin{aligned}\langle |M_{fi}|^2 \rangle &= \frac{8e^4}{16E_1^2 E_3^2 \sin^4\theta/2} 2ME_1E_3 \left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2\right] \\ &= \frac{M^2e^4}{E_1E_3\sin^4\theta/2}\left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right]\end{aligned}$$

- For $E \gg m_e$ we have (see Handout 1)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2, \qquad \text{or}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)}. \tag{91}$$

$(\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137})$

Interpretation

- So far have derived the differential cross-section for $e^- p \to e^- p$ elastic scattering assuming point-like Dirac spin-$\frac{1}{2}$ particles. How should we interpret the equation?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left(\cos^2 \theta/2 - \frac{q^2}{2M^2} \sin^2 \theta/2 \right)$$

- Compare with

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \cos^2 \frac{\theta}{2}$$

the important thing to note about the Mott cross-section is that it is equivalent to scattering of spin-$\frac{1}{2}$ electrons in a fixed electro-static potential. Here the term E_3/E_1 is due to the proton recoil.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left(\cos^2 \theta/2 - \frac{q^2}{2M^2} \sin^2 \theta/2 \right)$$

- the new term: $\propto \sin^2 \frac{\theta}{2} \Leftrightarrow$ Magnetic interaction : due to the spin-spin interaction

- The above differential cross-section depends on a single parameter. For an electron scattering angle θ , both q^2 and the energy, E_3 , are fixed by kinematics.
- Equating (88) and (90): $-2M(E_1 - E_3) = -2E_1E_3(1 - \cos\theta)$ so

$$\frac{E_3}{E_1} = \frac{M}{M + E_1(1 - \cos\theta)}.$$

- Substituting back into (88):

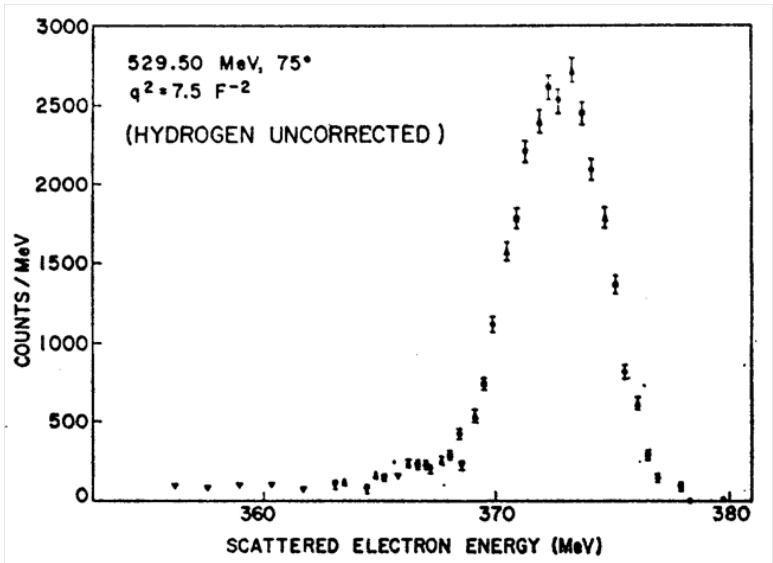
$$q^2 = -\frac{2ME_1^2(1 - \cos\theta)}{M + E_1(1 - \cos\theta)}$$

- e.g. for $e^- p \to e^- p$ with $E_{beam} = 529.5$ MeV and electron detector at $\theta = 75^\circ$ then for elastic scattering expect:

$$E_3 = \frac{ME_1}{M + E_1(1 - \cos\theta)} = \frac{938 \times 529}{938 + 529(1 - \cos 75^\circ)} = 373 \text{ MeV}.$$

- The energy identifies the scatter as elastic. Also know squared four-momentum transfer

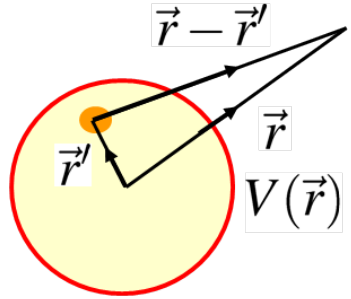
$$|q^2| = \frac{2 \times 938 \times 529^2(1 - \cos 75^\circ)}{938 + 529(1 - \cos 75^\circ)} = 294\,\text{MeV}^2.$$



(Plot taken from [4].)

Form Factors

Consider the scattering of an electron in the static potential due to an extended charge distribution. The potential at $\vec{r}$ from the center is given by:



$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r}) d^3\vec{r} = 1$$

- In first order perturbation theory, the matrix element is given by:

$$M_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3\cdot\vec{r}} V(\vec{r}) e^{i\vec{p}_1\cdot\vec{r}} d^3\vec{r} \qquad \text{(defining } \vec{q} = \vec{p}_1 - \vec{p}_3\text{)}$$

$$= \iint e^{i\vec{q}\cdot\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' d^3\vec{r} = \iint e^{i\vec{q}\cdot(\vec{r}-\vec{r}')} e^{i\vec{q}\cdot\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' d^3\vec{r}$$

- Fix $\vec{r}'$ and integrate over $d^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$:

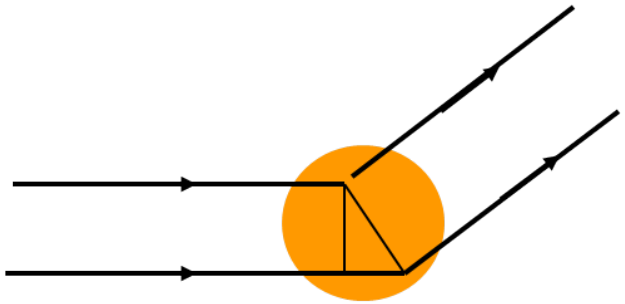
$$M_{fi} = \int e^{i\vec{q}\cdot\vec{R}} \frac{Q}{4\pi|\vec{R}|} d^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}\cdot\vec{r}'} d^3\vec{r}' = (M_{fi})_{\text{point}} F(\vec{q}^2)$$

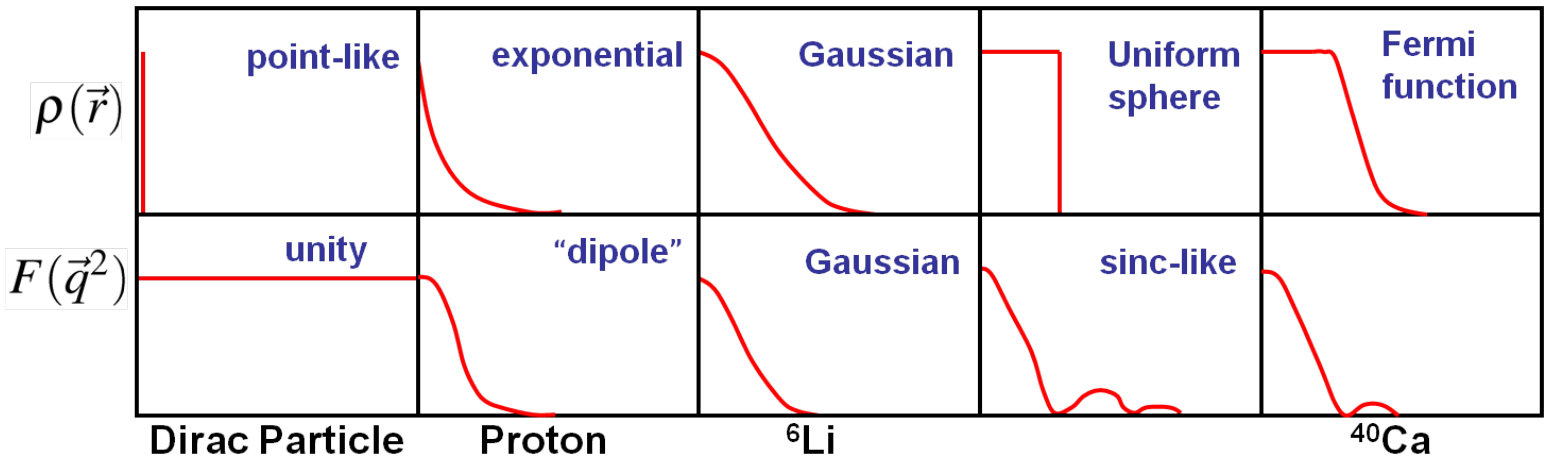
- The resulting matrix element is equivalent to the matrix element for scattering from a point source multiplied by the form factor:

$$\boxed{F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \rightarrow \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \cos^2 \frac{\theta}{2} |F(\vec{q}^2)|^2$$

- There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.
- The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space". If wavelength is long compared to size all waves in phase and $F(\vec{q}^2) = 1$.





- Note that the form factor for a point charge is unity.

Elastic Scattering from a Finite Size Proton

- In general the finite size of the proton can be accounted for by introducing two structure functions. One related to the charge distribution in the proton, $G_E(q^2)$ and the other related to the distribution of the magnetic moment of the proton, $G_M(q^2)$
- It can be shown that equation (91) generalizes to the ROSENBLUTH FORMULA.

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)}$$

with the Lorentz Invariant quantity:

$$\boxed{\tau = -\frac{q^2}{4M^2} > 0}.$$

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions. But

$$q^2 = (E_1 - E_3)^2 - \vec{q}^2$$

and from eq (90) obtain

$$-\vec{q}^2 = q^2 \left[1 - \left(\frac{q}{2M} \right)^2 \right]$$

so for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$.

- Hence in the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transforms of the charge and magnetic moment distributions:

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}\cdot\vec{r}}\rho(\vec{r})\, d^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}\cdot\vec{r}}\mu(\vec{r})\, d^3\vec{r}$$

- The derivation of the Rosenbluth formula assumes that the proton is a spin-half Dirac particle, i.e.

$$\vec{\mu} = \frac{e}{M}\vec{S}.$$

- However, the experimentally measured value of the proton magnetic moment is larger than expected for a point-like Dirac particle:

$$\vec{\mu} = 2.79\frac{e}{M}\vec{S}$$

- So for the proton we expect

$$G_E(0) = \int \rho(\vec{r})\, d^3\vec{r} = 1$$

$$G_M(0) = \int \mu(\vec{r})\, d^3\vec{r} = \mu_p = +2.79$$

Of course, the anomalous magnetic moment of the proton is already evidence that it is not point-like!

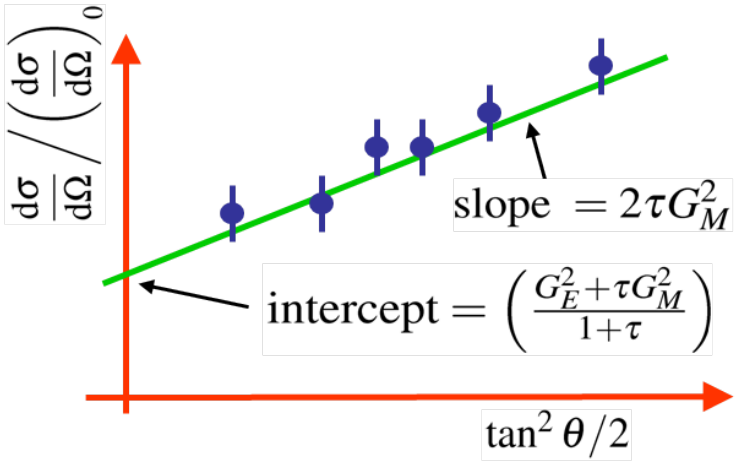
Measuring $G_E(q^2)$ and $G_M(q^2)$

- Express the Rosenbluth formula as :

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

where $\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$

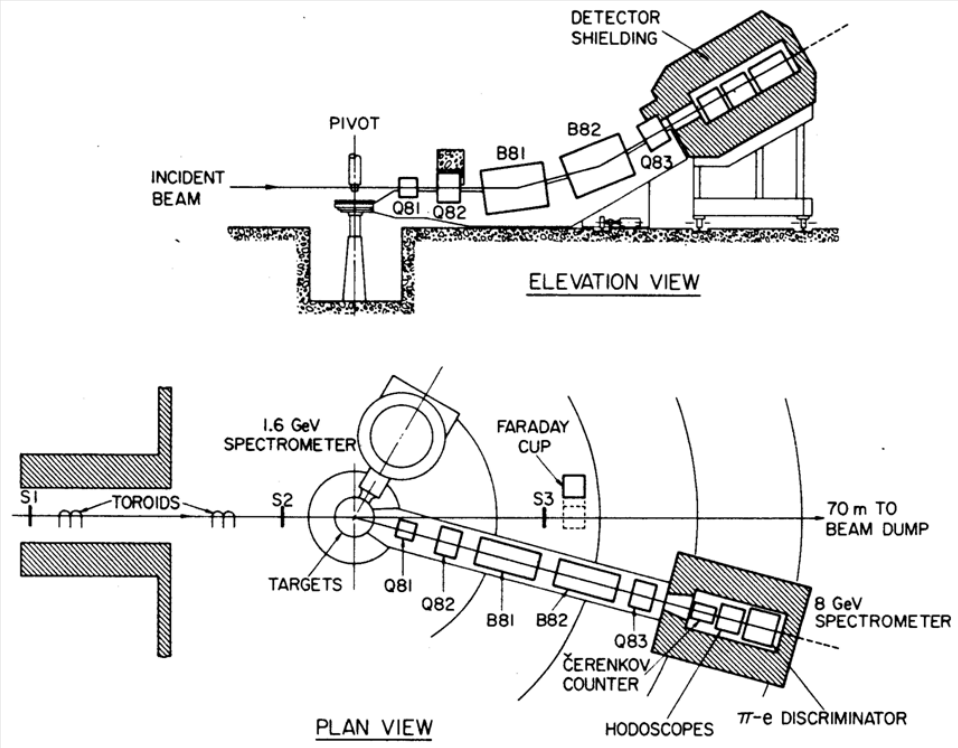
(which is the Mott cross-section including the proton recoil. It corresponds to scattering from a spin-0 proton).



- In general we are sensitive to both structure functions! These can be resolved from the angular dependence of the cross section at FIXED q^2.

Higher Energy Electron-Proton Scattering

- Use electron beam from SLAC LINAC: 5 GeV $< E_{\text{beam}} <$ 20 GeV
- Detect scattered electrons using the "8 GeV Spectrometer"

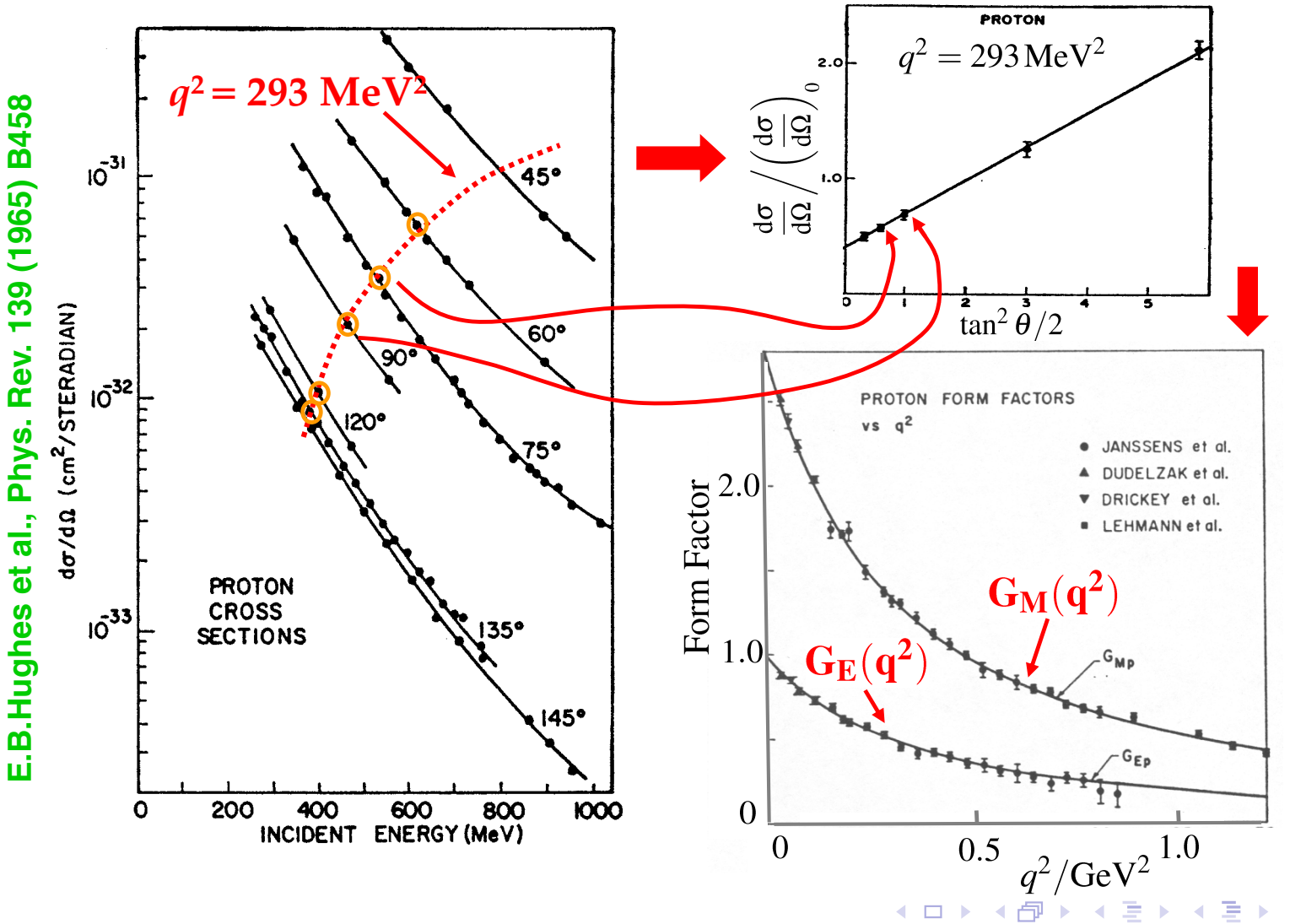




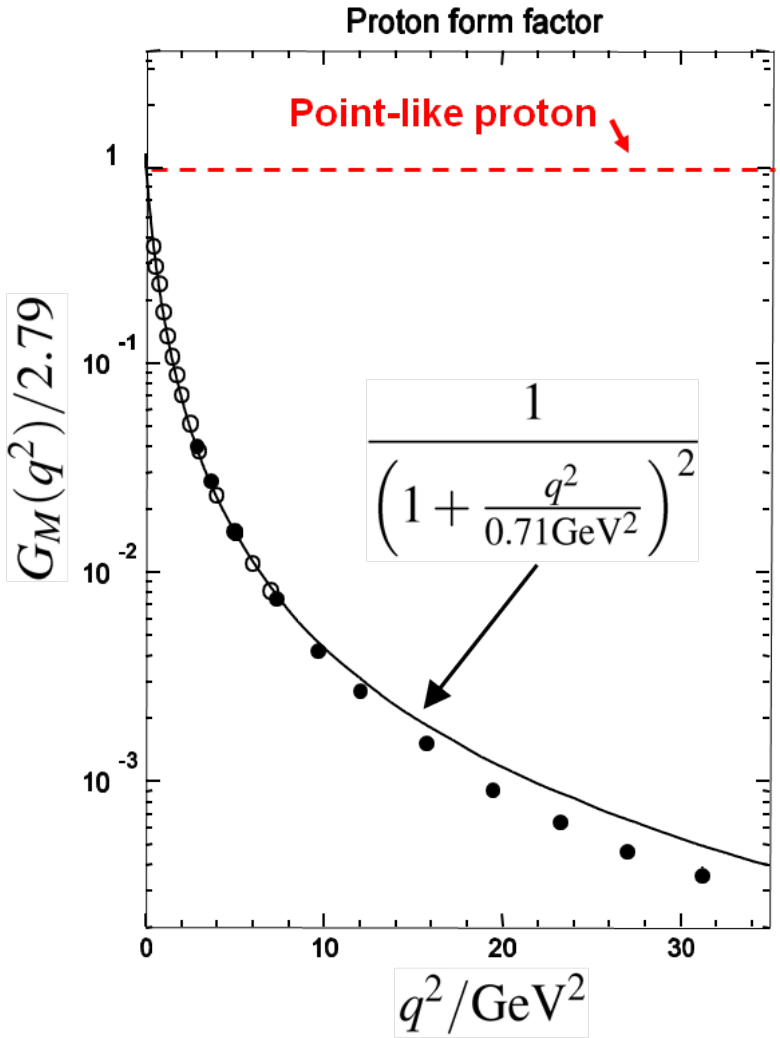
P.N.Kirk et al., Phys Rev D8 (1973) 63 [5]

High $q^2 \rightarrow$ Measure $G_M(q^2)$

- EXAMPLE: $e^- p \to e^- p$ at $E_{beam} = 529.5$ MeV Electron beam energies chosen to give certain values of q^2 Cross sections measured to 2-3%.
- NOTE Experimentally find $G_M(q_2) = 2.79 G_E(q_2)$, i.e. the electric and and magnetic form factors have same distribution.



High q^2 Results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671 A.F.Sill et al., Phys. Rev. D48 (1993) 29

- Form factor falls rapidly with q^2
  - Proton is not point-like
  - Good fit to the data with "dipole form":

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2}$$

- Taking FT find spatial charge and magnetic moment distribution

$$\boxed{\rho(r) \approx \rho_0 e^{-r/a}}$$

with $a \approx 0.24$ fm

- Corresponds to a rms charge radius $r_{rms} \approx 0.8$ fm.
- Although suggestive, does not imply proton is composite !
- Note: so far have only considered ELASTIC scattering; Inelastic scattering is the subject of next handout. (Now try Question 12!)

Summary: Elastic Scattering

- For elastic scattering of relativistic electrons from a point-like Dirac proton:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \frac{E_3}{E_1} \left(\cos^2(\theta/2) - \frac{q^2}{2M^2} \sin^2(\theta/2) \right)$$

Rutherford Proton recoil Electric/ Magnetic scattering Magnetic term due to spin.

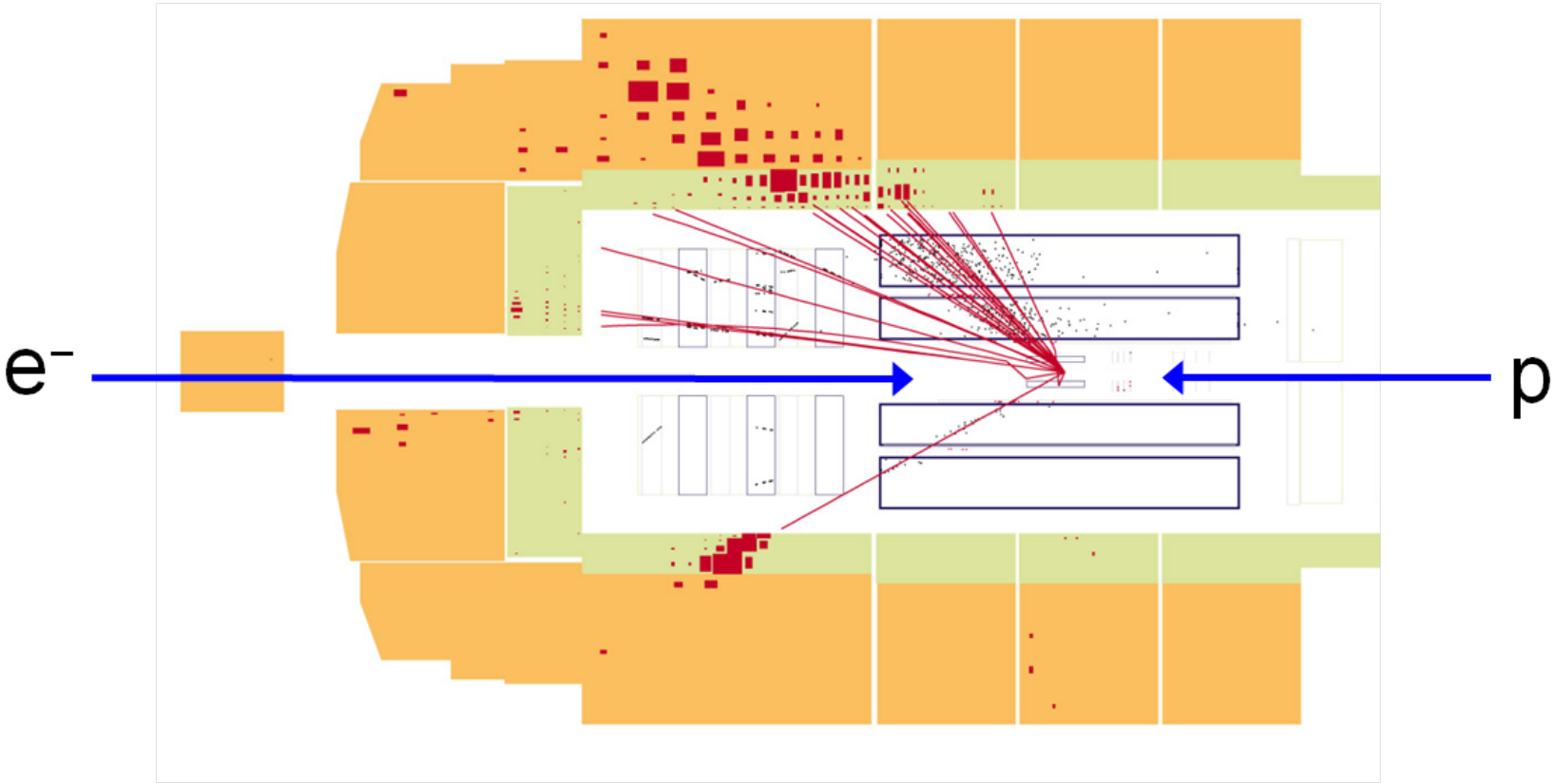
- For elastic scattering of relativistic electrons from an extended proton:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

(Rosenbluth Formula).

- Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of 0.8 fm.

Prof. Alex Mitov, 2026



H6: Deep Inelastic Scattering

Electron-Proton Elastic Scattering at High Q^2 $(= -q^2)$

- At high Q^2 the Rosenbluth expression for elastic scattering becomes

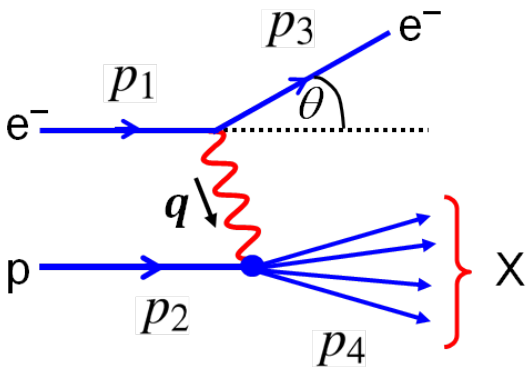
$$\left(\frac{d\sigma}{d\Omega}\right)_{elastic} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{q^2}{2M^2}G_M^2 \sin^2\frac{\theta}{2}\right) \quad \text{with} \quad \tau = -\frac{q^2}{4M^2} \gg 1$$

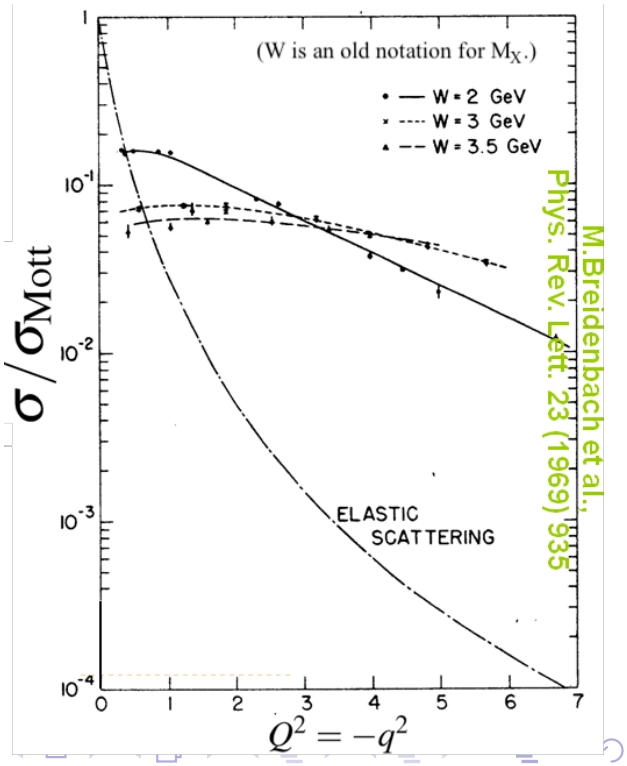
- From $e^- p$ elastic scattering, the proton magnetic form factor is

$$G_M(q^2) \approx \frac{1}{(1 + q^2/(0.71\,\mathrm{GeV}^2))^2} \quad \text{so} \quad G_M(q^2) \propto q^{-4} \text{at high } q^2$$

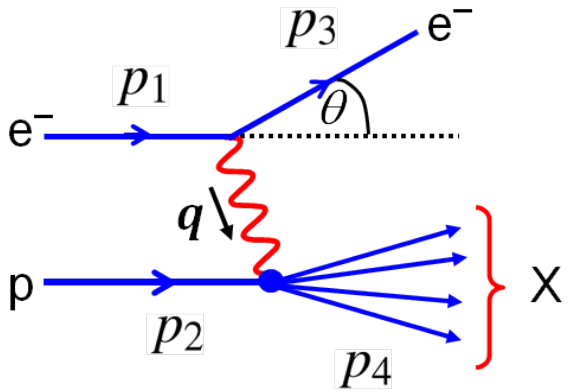
$$\Longrightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} \propto q^{-6}.$$

- Due to the finite proton size, elastic scattering at high Q^2 is unlikely and inelastic reactions where the proton breaks up dominate.





Kinematic Variables for Inelastic Scattering



- For inelastic scattering the mass M_X of the final state hadronic system X is larger than the proton mass, M, since X contains at least one baryon.
- Therefore: $M_X \geq M$ with equality for elastic and inequality for inelastic scattering.

- For inelastic scattering we will introduce four new kinematic variables:

$$Q^2 \equiv -q^2, \qquad x \equiv \frac{Q^2}{2p_2 \cdot q}, \qquad y \equiv \frac{p_2 \cdot q}{p_2 \cdot p_1} \qquad \text{and} \qquad \nu \equiv \frac{p_2 \cdot q}{M}.$$

('Bjorken x') ('Bjorken y')

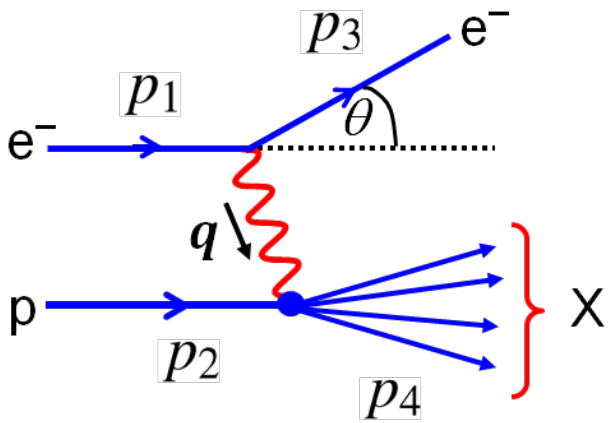
Properties of x and Q^2:

1. $Q^2 \geq 0$ (since $Q^2 = -(p_1 - p_3)^2 = 2p_1 \cdot p_3 - 2m_e^2 = 2(m_e E_3 - \vec{0} \cdot \vec{p}_3) - 2m_e^2$ [which used rest frame of p_1 as L.I.] so $Q^2 = 2m_e(E_3 - m_e) \geq 0$),
2. $2p_2 \cdot q \geq Q^2$ (with equality for elastic and inequality for inelastic scattering) (since: $2p_2 \cdot q = (p_2 + q)^2 - p_2^2 - q^2 = p_4^2 - p_2^2 + Q^2 = M_X^2 - M^2 + Q^2 \geq Q^2$),
3. and so $0 < x < 1$ for inelastic and $x = 1$ for elastic scattering.

(In many text books W is often used in place of M_X.)

Deep inelastic scattering variables in L.I. form and in lab-frame variables

- The angle, θ, and energy, E_3, of the scattered electron are all easy to measure in the lab frame, and so it is not uncommon to see the scattering variables Q^2, x, y and ν expressed in terms of lab-frame varibles. See below. (Derivations and interpretations are on the next page. M means mass of p_2.)



$$Q^2 = -q^2, \tag{92}$$

$$x = \frac{Q^2}{2p_2 \cdot q}, \tag{93}$$

$$y = \frac{p_2 \cdot q}{p_2 \cdot p_1}, \tag{94}$$

$$\nu = \frac{p_2 \cdot q}{M}. \tag{95}$$

$$Q^2 = 4E_1 E_3 \sin^2 \theta/2, \tag{96}$$

$$x = \frac{Q^2}{2M(E_1 - E_3)}, \tag{97}$$

$$y = 1 - \frac{E_3}{E_1}, \tag{98}$$

$$\nu = E_1 - E_3. \tag{99}$$

Properties of Bjorken y:

1. In the lab frame $p_1 = (E_1, 0, 0, E_1)$, $p_2 = (M, 0, 0, 0)$ and $q = (E_1 - E_3, \vec{p}_1 - \vec{p}_3)$ so

$$y \equiv \frac{p_2 \cdot q}{p_2 \cdot p_1} = \frac{M(E_1 - E_3)}{ME_1} = 1 - \frac{E_3}{E_1}$$

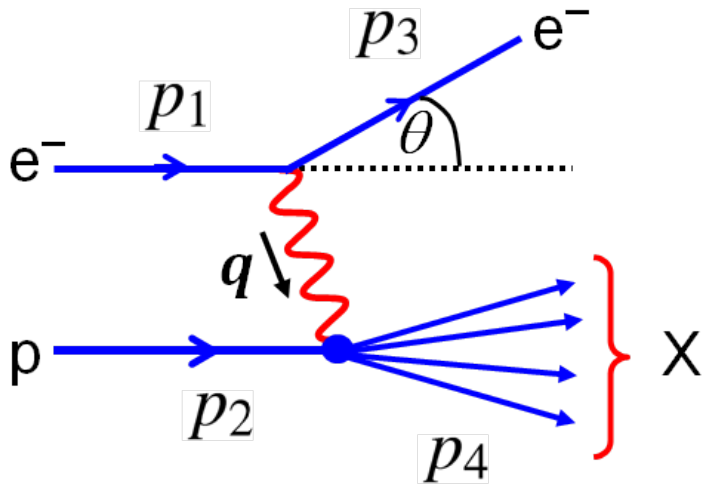
   and so y is the fractional energy loss of the incoming particle in the lab frame.
2. $0 < y < 1$ follows from the last property.

Properties of ν:

1. In the Lab Frame:

$$\nu \equiv \frac{p_2.q}{M} = \frac{M(E_1 - E_3)}{M} = E_1 - E_3$$

   so ν is the energy lost by the incoming particle in the lab frame.



Relationships between Kinematic Variables for inelastic collisions

- We saw in lecture 1 that elastic collisions **need only two variables** to describe their kinematic differences, as all four external particles have fixed masses.
  (These two variables are usually chosen to be Mandelstam s together with one of t or u.)
- However, in inelastic collisions the proton breaks up, and so we need an extra variable to describe the variability in the mass of p_4:
  inelastic collisions **require three variables** to describe their kinematic differences.

Thus, at fixed s, at most two of

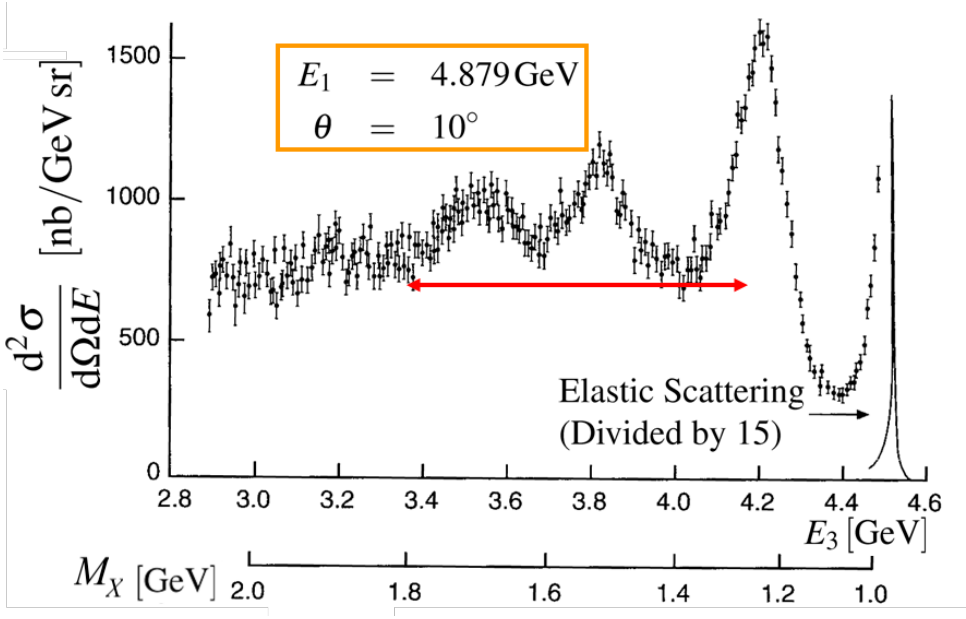
$$Q^2 \equiv -q^2, \quad x \equiv \frac{Q^2}{2p_2.q}, \quad y \equiv \frac{p_2.q}{p_2.p_1} \quad \text{and} \quad \nu \equiv \frac{p_2.q}{M}$$

are independent! There must exist many relationships among them. E.g.:

- A simple division of the above definitions shows that
$$x = \frac{Q^2}{2M\nu}.$$
- From $s = (p_1 + p_2)^2 = m_e^2 + M^2 + 2p_1 \cdot p_2 \approx 2p_1 \cdot p_2 + M^2$ we can show (when m_e is negligible) that:
$$y = \frac{2M}{s - M^2}\nu.$$
- Putting the last two relations together:
$$xy = \frac{Q^2}{s - M^2} \quad \text{and} \quad Q^2 = (s - M^2)xy.$$

Elastic vs Inelastic vs Deep Inelastic Scattering

Example: Scattering of 4.879 GeV electrons from protons at rest



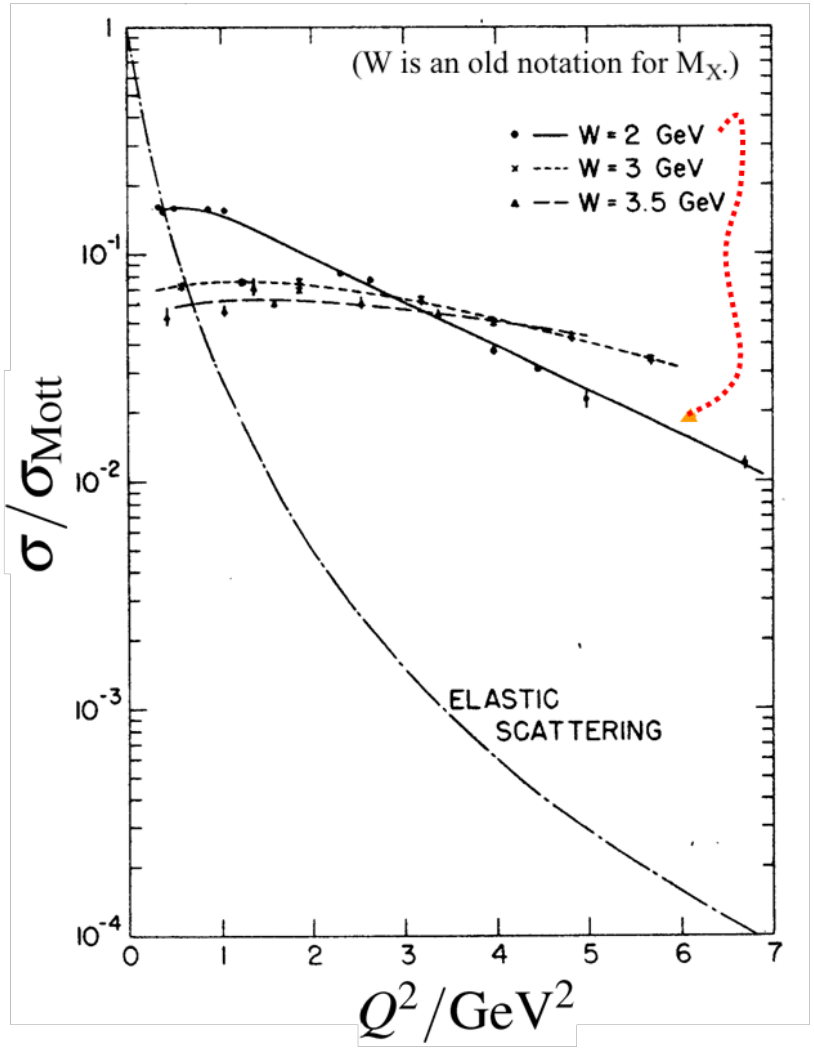
- Place detector at 10° to beam and measure the energies of scattered e^-.
- Kinematics fully determined from the electron energy and angle !
- For this energy and angle the invariant mass of the final state hadronic system M_X is about 1 GeV.
  (Try and show this!)

- **Elastic Scattering**: proton remains intact ($M_X = M$).
- **Inelastic Scattering** (red arrow on plot) produces 'excited states' of proton, e.g. $\Delta^+(1232)$, $M_X = M_\Delta$.
- **Deep Inelastic Scattering** is where the proton breaks up completely resulting in a many particle final state (far L.H.S. of plot where $M_X \gg M$).

Deep Inelastic Scattering = Large M_X = Total proton breakup.

Inelastic Cross Sections

M.Breidenbach et al., Phys. Rev. Lett. 23 (1969) 935



- Repeat experiments at different angles/beam energies and determine Q^2 dependence of elastic and inelastic cross-sections.
- Elastic scattering falls of rapidly with due to the proton not being point-like (i.e. form factors).
- Inelastic scattering cross sections only weakly dependent on Q^2.
- Deep Inelastic scattering cross sections almost independent of Q^2.
  i.e. "Form factor" $\rightarrow 1$.

There appears to be Scattering from point-like objects within the proton!

Elastic → Inelastic Scattering

Elastic scattering (Handout 5). Recall:

- Only one independent variable desscribes final state. In Lab. frame express differential cross section in terms of the electron scattering angle (Rosenbluth formula)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right) \qquad (\text{with } \tau = \frac{Q^2}{4M^2}).$$

  Note: here the energy of the scattered electron E_3 is completely determined by the angle θ.

- We could express the differential cross section in terms of a single variable Q^2. (Q14 on examples sheet.)

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2 + \tau G_M^2}{(1+\tau)}\left(1 - y - \frac{M^2y^2}{Q^2}\right) + \frac{1}{2}y^2 G_M^2\right]$$

  which can be written as

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[f_2(Q^2)\left(1 - y - \frac{M^2y^2}{Q^2}\right) + \frac{1}{2}y^2 f_1(Q^2)\right].$$

Inelastic scattering:

- For Deep Inelastic Scattering (DIS) have two independent variables for final sate. Therefore need a double differential cross section for DIS ... as shown on next page.

Deep Inelastic Scattering

- It can be shown that the most general Lorentz Invariant expression for $e^- p \to e^- X$ inelastic scattering (via a single exchanged photon is):

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \qquad \boxed{\text{INELASTIC}} \tag{100}$$

c.f. $$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2y^2}{Q^2}\right) f_2(Q^2) + \frac{1}{2}y^2 f_1(Q^2)\right] \qquad \boxed{\text{ELASTIC}}$$

We will soon see how this connects to the quark model of the proton.

- NOTE: The **form factors** $f_1(Q^2)$ and $f_2(Q^2)$ have been replaced by the **structure functions** $F_1(x, Q^2)$ and $F_2(x, Q^2)$ which are functions of x and Q^2: they cannot be interpreted as the Fourier transforms of the charge and magnetic moment distributions. We shall soon see that they describe the **momentum distribution** of the quarks within the proton.
- In the limit of high energy (or more correctly $Q^2 \gg M^2y^2$) equation (100) becomes:

$$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right]}. \tag{101}$$

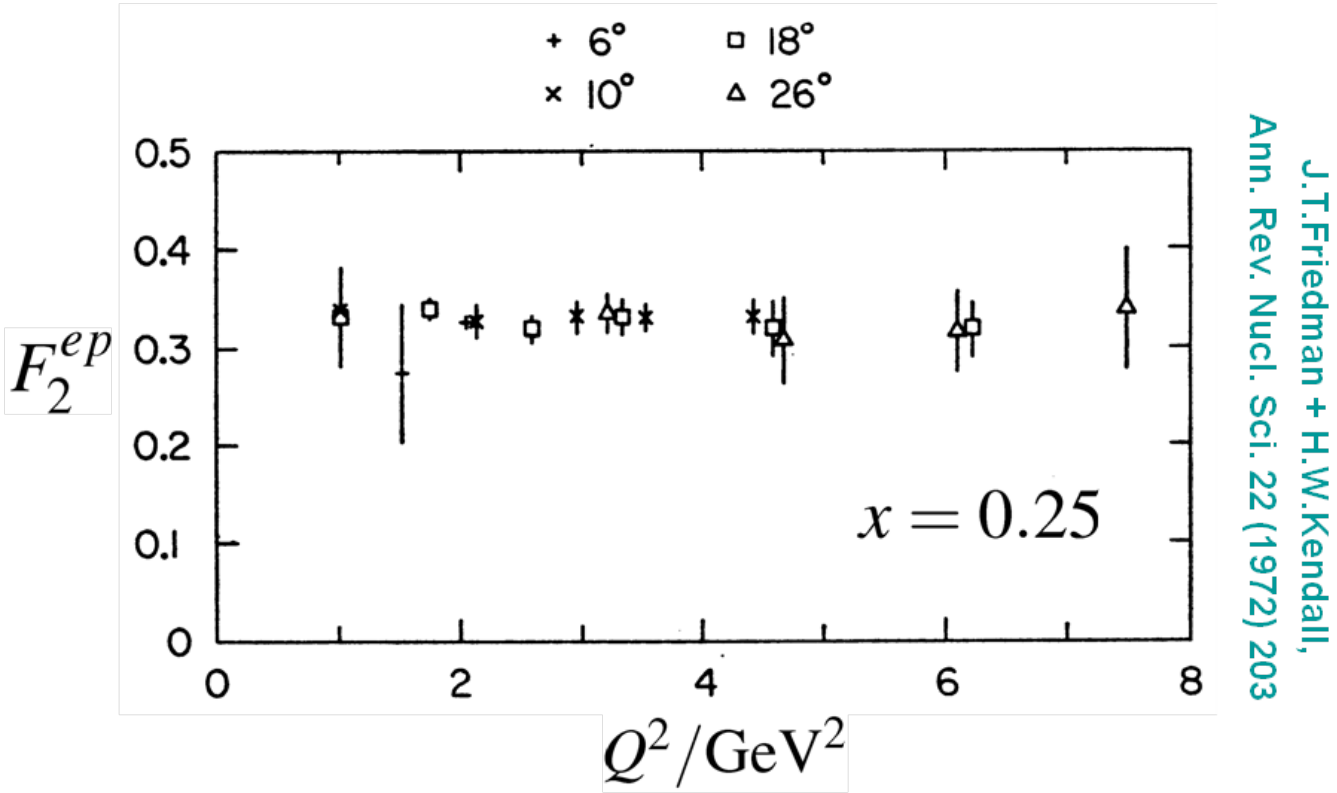
- In the Lab. frame, Equation (101) becomes: (see examples sheet Q14)

$$\frac{d^2\sigma}{dE_3\, d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \left[\frac{1}{\nu} F_2(x, Q^2) \cos^2\left(\frac{\theta}{2}\right) + \frac{2}{M} F_1(x, Q^2) \sin^2\left(\frac{\theta}{2}\right) \right] \quad (102)$$

Electromagnetic Structure Function ; Pure Magnetic Structure Function

Measuring the Structure Functions

- To determine $F_1(x, Q^2)$ and $F_2(x, Q^2)$ for a given x and Q^2 need measurements of the differential cross section at several different scattering angles and incoming electron beam energies (see Q14 on examples sheet)
  Example: electron-proton scattering F_2 vs. Q^2 at fixed x:



- Experimentally it is observed that both F_1 and F_2 are (almost) independent of Q^2!

Bjorken Scaling and the Callan-Gross Relation

- The near (see later) independence of the structure functions on Q^2 is known as **Bjorken Scaling**, i.e.

$$F_1(x, Q^2) \to F_1(x) \quad \text{and} \quad F_2(x, Q^2) \to F_2(x).$$

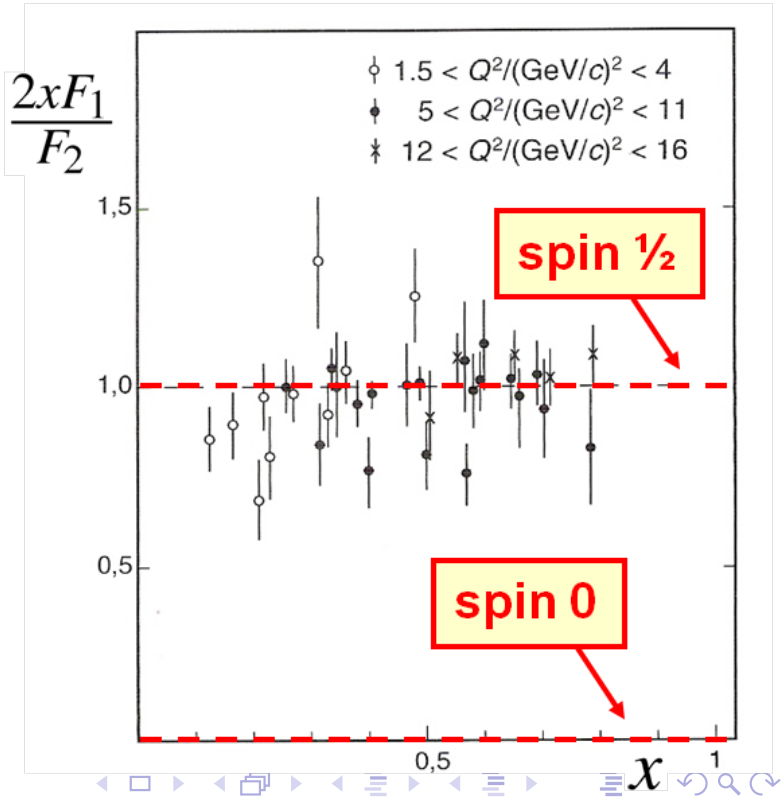
- It is **strongly suggestive of scattering from point-like constituents** within the proton

- It is also observed that $F_1(x)$ and $F_2(x)$ are not independent but satisfy the **Callan-Gross relation**

$$F_2(x) = 2xF_1(x).$$

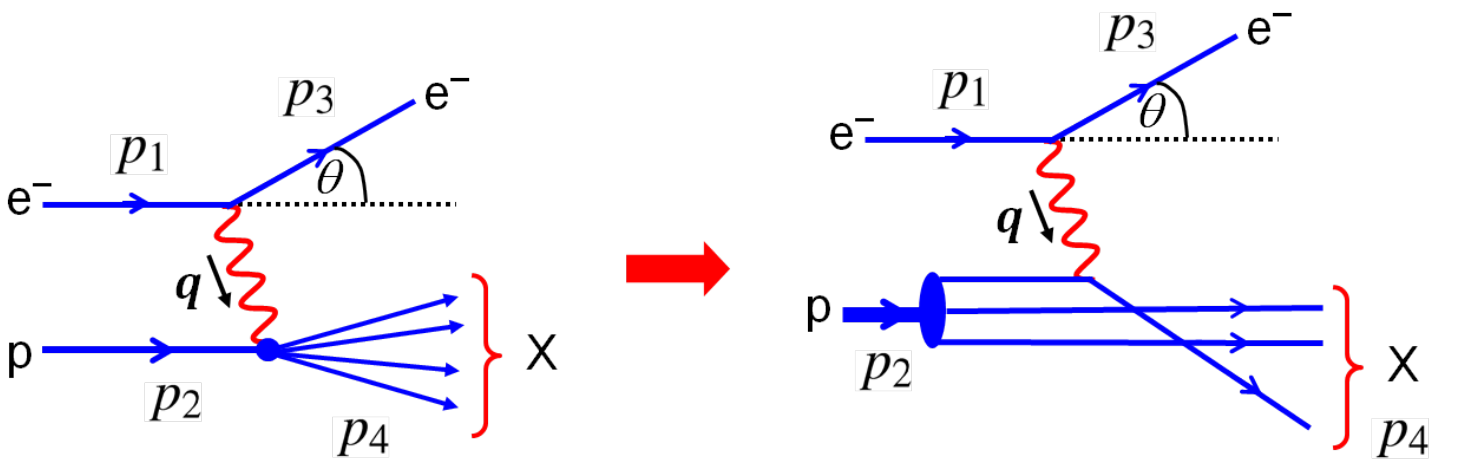
- As we shall soon see this is exactly what is expected for scattering from spin-half quarks.

- **Note:** if quarks were spin zero particles we would expect the purely magnetic structure function to be zero, i.e. $F_1(x) = 0$.



The Quark-Parton Model

- Before quarks and gluons were generally accepted Feynman proposed that the proton was made up of point-like constituents "partons".
- Both **Bjorken Scaling** and the **Callan-Gross relationship** can be explained by assuming that Deep Inelastic Scattering is dominated by the scattering of a single virtual photon from point-like spin-half constituents of the proton.



Scattering from a proton with structure functions → **Scattering from a point-like quark within the proton**

- How do these two pictures of the interaction relate to each other?

- In the parton model the basic interaction is ELASTIC scattering from a "quasi-free" spin-$\frac{1}{2}$ quark in the proton, i.e. treat the quark as a free particle!
- The parton model is most easily formulated in a frame where the proton has very high energy, often referred to as the "infinite momentum frame", where we can neglect the proton mass, and $p_2 = (E_2, 0, 0, E_2)$.
- In this frame can also neglect the mass of the quark and any momentum transverse to the direction of the proton.
- Let the quark carry a fraction ξ of the proton's four-momentum. $(\xi^2 p_2^2 = m_q^2 \approx 0)$



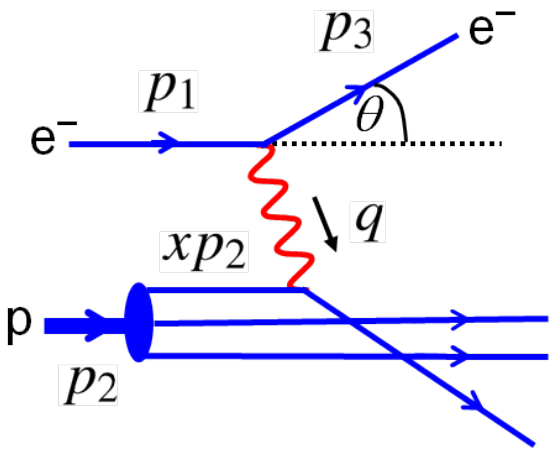
- After the interaction the struck quark's four-momentum is $\xi p_2 + q$ and it is still a quark, therefore: $m_q^2 = (\xi p_2 + q)^2 = (\xi p_2)^2 + 2\xi p_2 \cdot q + q^2 = m_q^2 + 2\xi p_2 \cdot q + q^2$ and so comparing the first and last expressions we see that $\xi = -q^2/(2p_2 \cdot q)$ and thus

$$\xi = \frac{Q^2}{2p_2.q} = x \qquad \ldots$$

therefore Bjorken x can be identified as the fraction of the proton momentum carried by the struck quark in the infinite momentum frame.

- In terms of the proton momentum

$$s = (p_1 + p_2)^2 \simeq 2p_1.p_2 \quad y = \frac{p_2.q}{p_2.p_1} \quad x = \frac{Q^2}{2p_2.q}$$



- But for the underlying quark interaction:

$$s^q = (p_1 + xp_2)^2 = 2xp_1.p_2 = xs$$

$$y_q = \frac{p_q.q}{p_q.p_1} = \frac{xp_2.q}{xp_2.p_1} = y \qquad \text{and} \qquad x_q = 1.$$

(i.e. quark coll. is elastic: quark does not break up).

- Q11 on the examples sheet (leaning on results from Handout 4) derives the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit. Using that result for $e^-q \to e^-q$ assuming quark charge is e_q (e.g. $e_u = +2/3$) gives:

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2 e_q^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s_q}\right)^2\right]$$

- Using $-q^2 = Q^2 = (s_q - m^2)x_q y_q$ from page 156 gives $\frac{q^2}{s_q} = -x_q y_q = -1.y = -y$ so

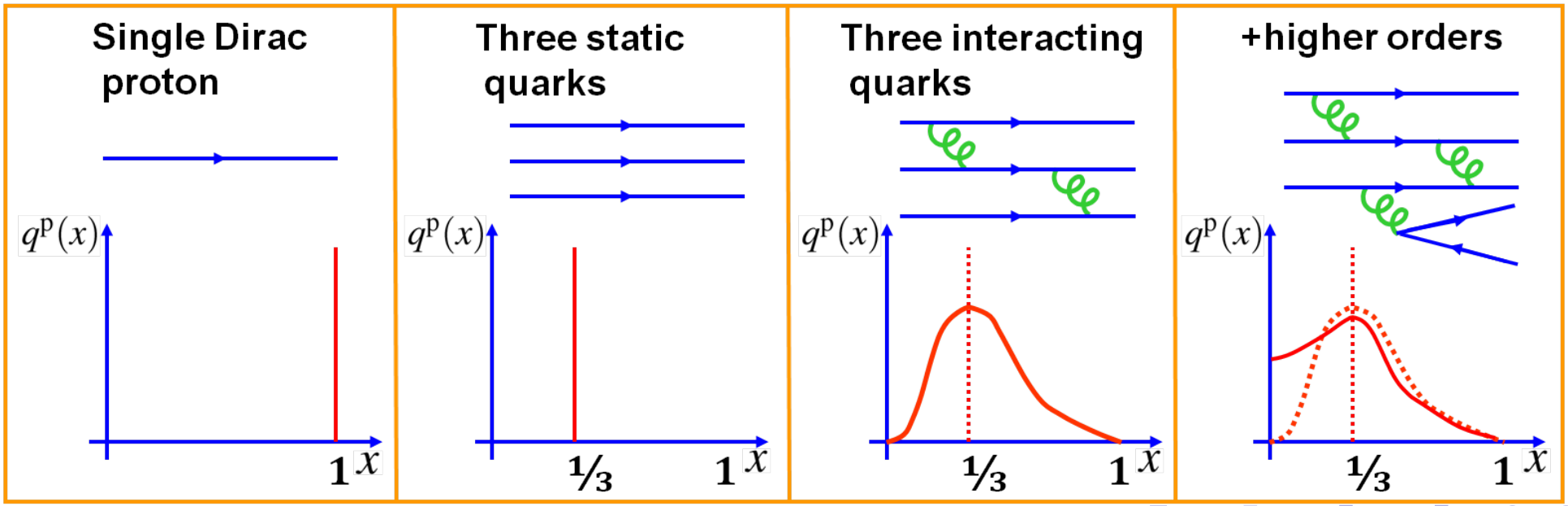
$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1 + (1-y)^2\right]$$

(where the last two expressions assume the massless quark limit $m = 0$).

Parton Distribution Functions

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2 e_q^2}{Q^4}\left[(1-y)+\frac{y^2}{2}\right] \tag{103}$$

- This is the expression for the differential cross-section for elastic $e^- q$ scattering from a quark carrying a fraction x of the proton momentum.
- Now need to account for distribution of quark momenta within proton
- Introduce **parton distribution functions** such that $q^{\mathrm{p}}(x)dx$ is **the number of quarks of type q within a proton p with momenta between $x \to x + dx$.**
- Possible forms of the parton distribution functions:



- The cross section for scattering from a particular quark type within the proton, which in the range $x \to x + \mathrm{d}x$, is

$$\frac{d^2\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right] \cdot e_q^2 q^{\mathrm{p}}(x)\mathrm{d}x$$

- Summing over all types of quarks within the proton gives the expression for the electron-proton scattering cross section

$$\frac{d^2\sigma^{\mathrm{ep}}}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]\sum_q e_q^2 q^{\mathrm{p}}(x) \tag{104}$$

- Compare with the electron-proton scattering cross section in terms of structure functions (equation (101)):

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \tag{105}$$

- By comparing (104) and (105), obtain the parton model prediction for the structure functions in the general Lorentz invariant form for the differential cross section

$$\boxed{F_2^{\mathrm{ep}}(x,Q^2) = 2xF_1^{\mathrm{ep}}(x,Q^2) = x\sum_q e_q^2 q^{\mathrm{p}}(x)}$$

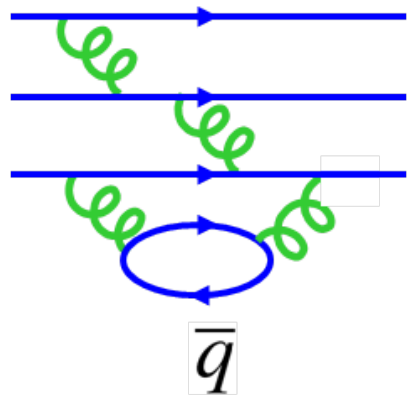
Can relate measured structure functions to the underlying quark distributions.

Predictions of The Parton Model

The parton model predicts:

- **Bjorken Scaling** $F_1^{\mathrm{ep}}(x, Q^2) \to F_1^{\mathrm{ep}}(x)$ and $F_2^{\mathrm{ep}}(x, Q^2) \to F_2^{\mathrm{ep}}(x)$. This is due to scattering from point-like particles within the proton
- **Callan-Gross Relation** $F_2^{\mathrm{ep}}(x) = x \sum_q e_q^2 q^{\mathrm{p}}(x)$. This is due to scattering from spin half Dirac particles where the magnetic moment is directly related to the charge; hence the 'electro-magnetic' and 'pure magnetic' terms are fixed with respect to each other.
- At present parton distributions cannot be calculated from QCD.
  (Can't use perturbation theory due to large coupling constant.)
- Measurements of the structure functions enable us to determine the parton distribution functions !
- For electron-proton scattering we have:

$$F_2^{\mathrm{ep}}(x) = x \sum_q e_q^2 q^{\mathrm{p}}(x)$$



- Due to higher orders, the proton contains not only up and down quarks but also anti-up and anti-down quarks
  (will neglect the small contributions from heavier quarks).

- For electron-proton scattering have:

$$F_2^{\rm ep}(x) = x\sum_q e_q^2 q^{\rm p}(x) = x\left(\frac{4}{9}u^{\rm p}(x) + \frac{1}{9}d^{\rm p}(x) + \frac{4}{9}\overline{u}^{\rm p}(x) + \frac{1}{9}\overline{d}^{\rm p}(x)\right)$$

- For electron-neutron scattering have:

$$F_2^{\rm en}(x) = x\sum_q e_q^2 q^{\rm n}(x) = x\left(\frac{4}{9}u^{\rm n}(x) + \frac{1}{9}d^{\rm n}(x) + \frac{4}{9}\overline{u}^{\rm n}(x) + \frac{1}{9}\overline{d}^{\rm n}(x)\right)$$

- Now assume "isospin symmetry", i.e. that the neutron (ddu) is the same as a proton (uud) with up and down quarks interchanged, i.e.

$$d^{\rm n}(x) = u^{\rm p}(x) \qquad \text{and} \qquad u^{\rm n}(x) = d^{\rm p}(x)$$

and then define the neutron distributions functions in terms of those of the proton

$u(x) \equiv u^{\rm p}(x) = d^{\rm n}(x); \qquad d(x) \equiv d^{\rm p}(x) = u^{\rm n}(x)$
$\overline{u}(x) \equiv \overline{u}^{\rm p}(x) = \overline{d}^{\rm n}(x); \qquad \overline{d}(x) \equiv \overline{d}^{\rm p}(x) = \overline{u}^{\rm n}(x)$

giving:
$$F_2^{\rm ep}(x) = 2xF_1^{\rm ep}(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right) \quad (106)$$

$$F_2^{\rm en}(x) = 2xF_1^{\rm en}(x) = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\overline{d}(x) + \frac{1}{9}\overline{u}(x)\right). \quad (107)$$

- Integrating (106) and (107):

$$\int_0^1 F_2^{\mathrm{en}}(x)dx = \int_0^1 x\left(\frac{4}{9}[d(x)+\overline{d}(x)]+\frac{1}{9}[u(x)+\overline{u}(x)]\right)dx = \frac{4}{9}f_d+\frac{1}{9}f_u \qquad \text{and}$$

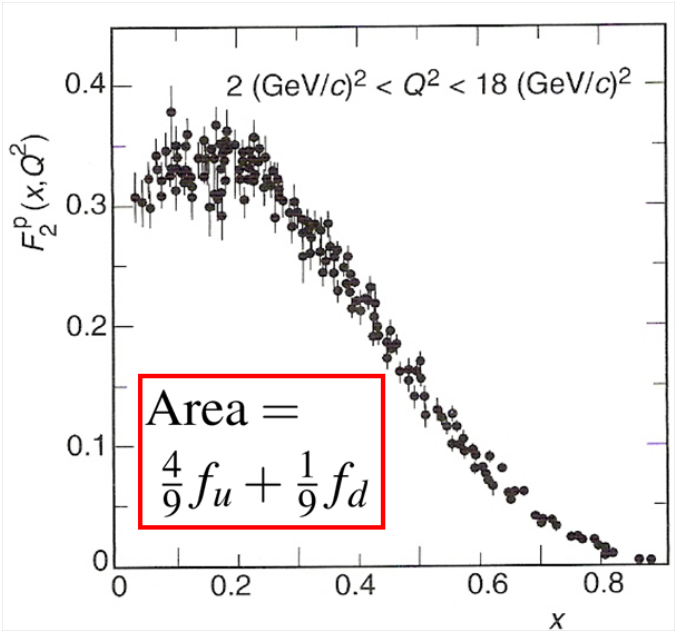
$$\int_0^1 F_2^{\mathrm{ep}}(x)dx = \int_0^1 x\left(\frac{4}{9}[u(x)+\overline{u}(x)]+\frac{1}{9}[d(x)+\overline{d}(x)]\right)dx = \frac{4}{9}f_u+\frac{1}{9}f_d \qquad \text{where}$$

$$f_u = \int_0^1 [xu(x)+x\overline{u}(x)]dx$$

is the fraction of the proton momentum carried by the up and anti-up quarks.

Experimentally:

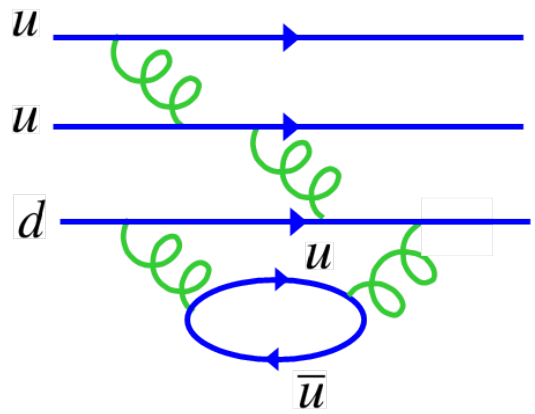
$$\begin{array}{l}\int F_2^{\mathrm{ep}}(x)dx \approx 0.18 \\ \int F_2^{\mathrm{en}}(x)dx \approx 0.12\end{array} \implies \boxed{f_u \approx 0.36 \text{ and } f_d \approx 0.18}$$



- In the proton, as expected, the up quarks carry twice the momentum of the down quarks.
- The quarks carry just over 50% of the total proton momentum. The rest is carried by gluons (which being neutral doesn't contribute to electron-nucleon scattering).

Valence and Sea Quarks

- The proton is complex! its parton distribution functions include contributions from the "valence" quarks and the virtual quarks produced by gluons: the "sea".



- Resolving into valence and sea contributions:

$$u(x) = u_{\mathrm{V}}(x) + u_{\mathrm{S}}(x) \qquad \overline{d}(x) = \overline{d}_{\mathrm{S}}(x)$$
$$\overline{u}(x) = \overline{u}_{\mathrm{S}}(x) \qquad d(x) = d_{\mathrm{V}}(x) + d_{\mathrm{S}}(x)$$

- Proton contains two valence up quarks and one valence down quark so would expect:

$$\int_0^1 u_{\mathrm{V}}(x)dx = 2 \quad \text{and} \quad \int_0^1 d_{\mathrm{V}}(x)dx = 1$$

... but we have no a priori expectation for the total number of sea quarks!

- Sea quarks arise from gluon quark/anti-quark pair production and with $m_u = m_d$ it is reasonable to expect

$$u_{\mathrm{S}}(x) = d_{\mathrm{S}}(x) = \overline{u}_{\mathrm{S}}(x) = \overline{d}_{\mathrm{S}}(x) = S(x).$$

- With these relations (106) and (107) become:

$$F_2^{\mathrm{ep}}(x) = x\left(\frac{4}{9}u_{\mathrm{V}}(x) + \frac{1}{9}d_{\mathrm{V}}(x) + \frac{10}{9}S(x)\right) \;\&\; F_2^{\mathrm{en}}(x) = x\left(\frac{4}{9}d_{\mathrm{V}}(x) + \frac{1}{9}u_{\mathrm{V}}(x) + \frac{10}{9}S(x)\right)$$

... giving the ratio

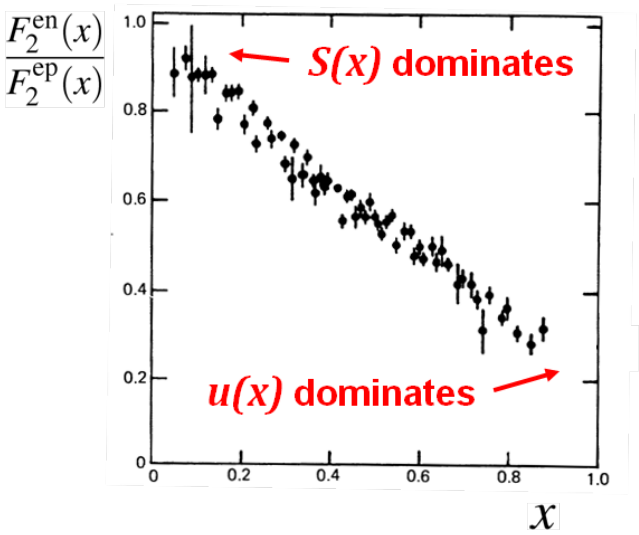
$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} = \frac{4d_V(x) + u_V(x) + 10S(x)}{4u_V(x) + d_V(x) + 10S(x)}.$$

- The sea component arises from processes such as $g \to \overline{u}u$. Due to the $1/q^2$ dependence of the gluon propagator, much more likely to produce low energy gluons. Expect the sea to comprise of low energy q and $\overline{q}$.

- Therefore at low x expect the sea to dominate:

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \to 1 \quad \text{as} \quad x \to 0.$$

Experiment agrees!



- At high x expect the sea contribution to be small

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} = \frac{4d_V(x) + u_V(x) + 10S(x)}{4u_V(x) + d_V(x) + 10S(x)}$$

Note: $u_V = 2d_V$ would give ratio $2/3$ as $x \to 1$

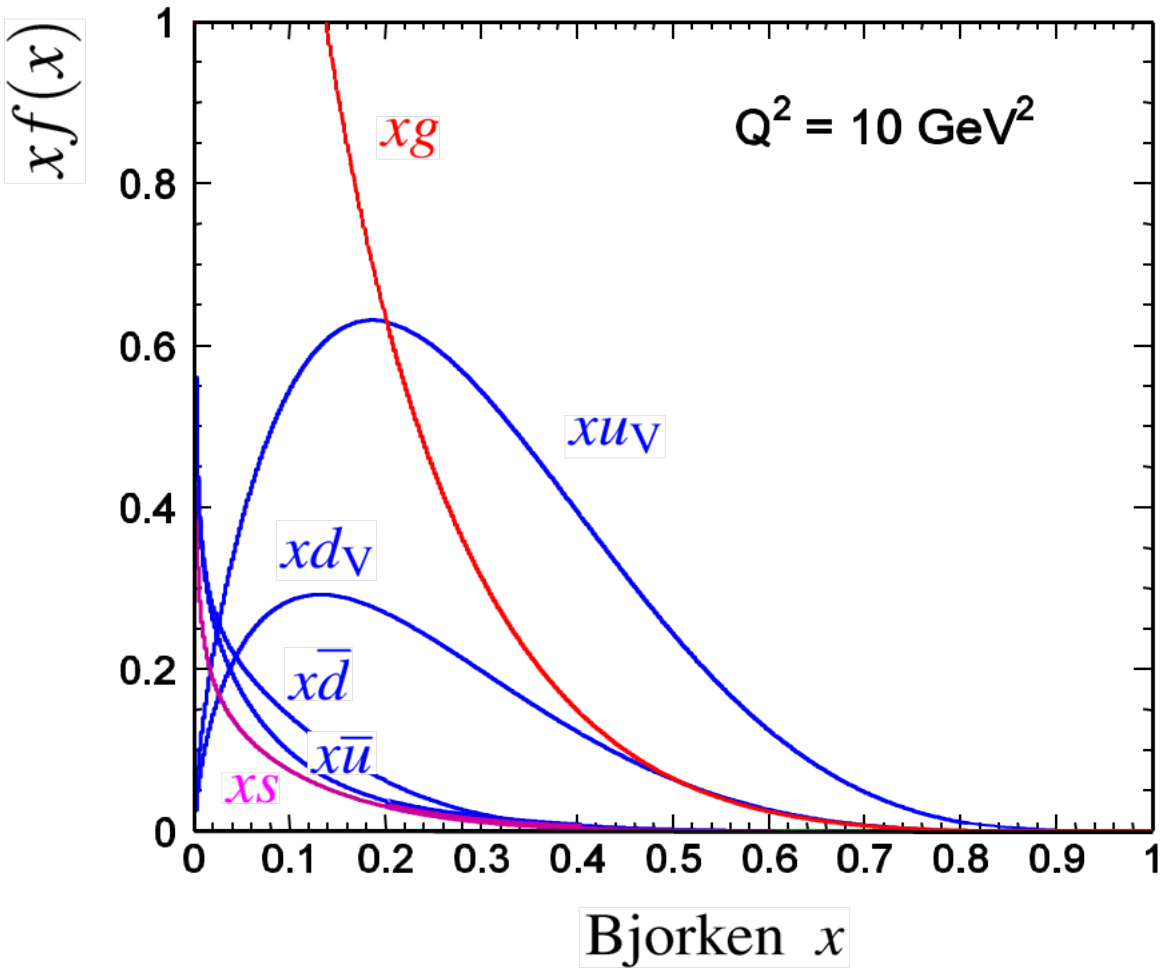
Experiment disagrees:

$$F_2^{en}(x)/F_2^{ep}(x) \to 1/4 \quad \text{i.e.} \quad d(x)/u(x) \to 0 \quad \text{as} \quad x \to 1.$$

This behaviour is not understood!

Parton Distribution Functions

- Ultimately the parton distribution functions are obtained from a fit to all experimental data including neutrino scattering (see Handout 10)
- Hadron-hadron collisions give information on gluon pdf, $g(x)$.



Note:

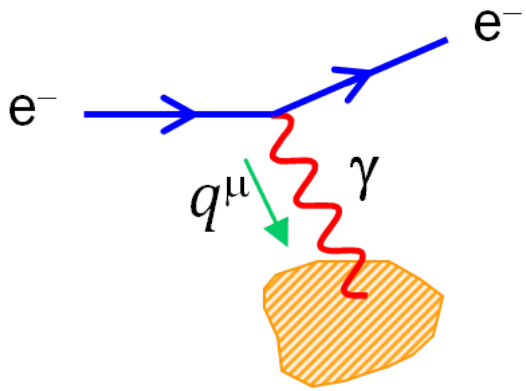
- Apart from at large x, $u_V(x) \approx 2d_V(x)$.
- For $x < 0.2$ gluons dominate.
- In fits to data assume $u_s(x) = \bar{u}(x)$
- $\bar{d}(x) > \bar{u}(x)$ not understood — exclusion principle?
- Small strange quark component $s(x)$.

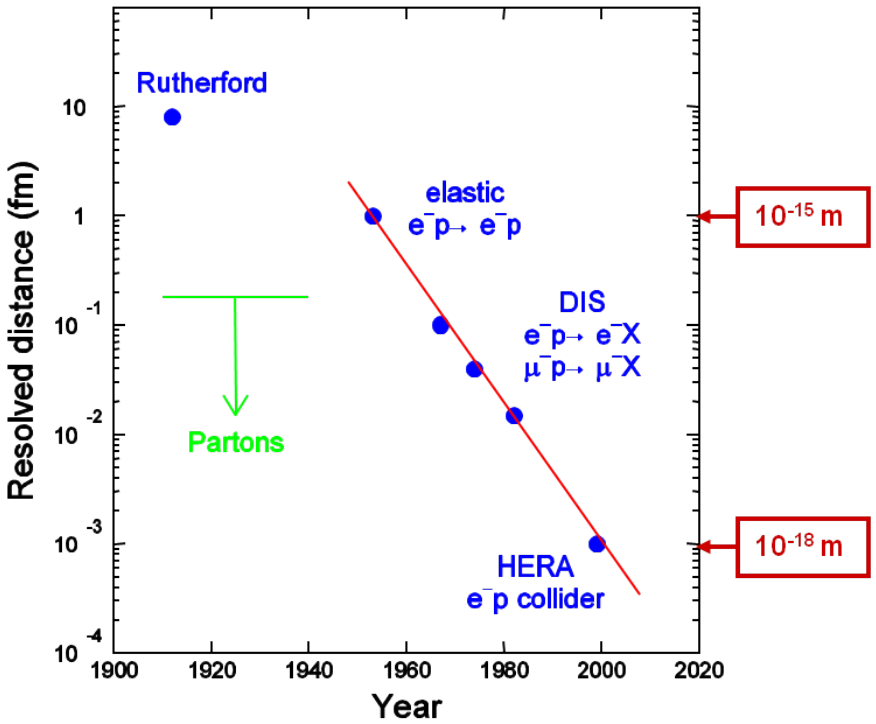
(Now try Question 13)

Quark substructure would be revealed as scaling violations high Q^2

- In last 40 years, experiments have probed the proton with virtual photons of ever increasing energy
- Non-point like nature of the scattering becomes apparent when $\lambda_\gamma \sim$ size of scattering centre

$$\lambda_\gamma = \frac{h}{|\vec{q}|} \sim O\left(\frac{1}{|\vec{q}|/\mathrm{GeV}}\right) \text{ fm.}$$





- Scattering from point-like quarks gives rise to Bjorken scaling: no Q^2 cross section dependence
- IF quarks were not point-like, at high Q^2 (when the wavelength of the virtual photon i$\sim$ size of quark) would observe rapid decrease in cross section with increasing Q^2.
- To search for quark sub-structure want to go to highest Q^2 ... see HERA collider (next slide).

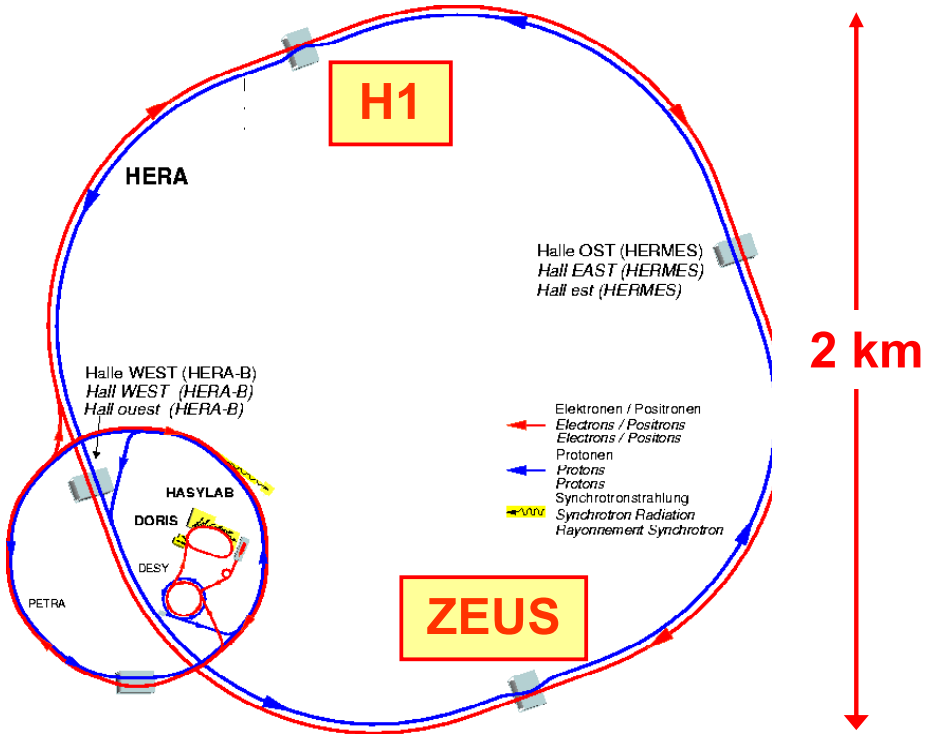
The HERA $e^{\pm}$-proton Collider : 1991-2007

★ **DESY (Deutsches Elektronen-Synchroton) Laboratory, Hamburg, Germany**

$e^{\pm}$ **27.5 GeV** → ← **820 GeV** p

$\sqrt{s} = 300$ GeV



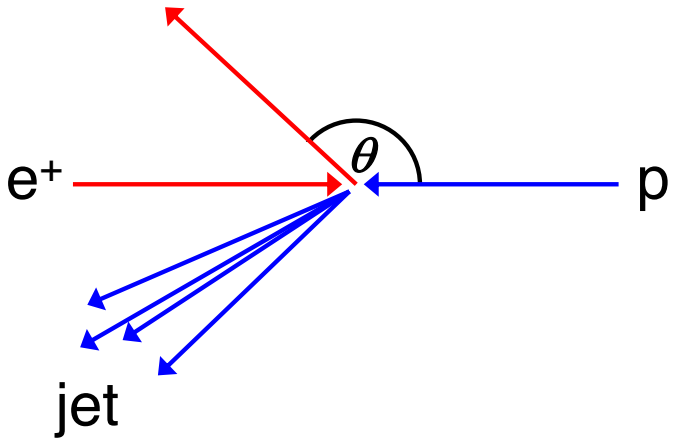


★ **Two large experiments : H1 and ZEUS**

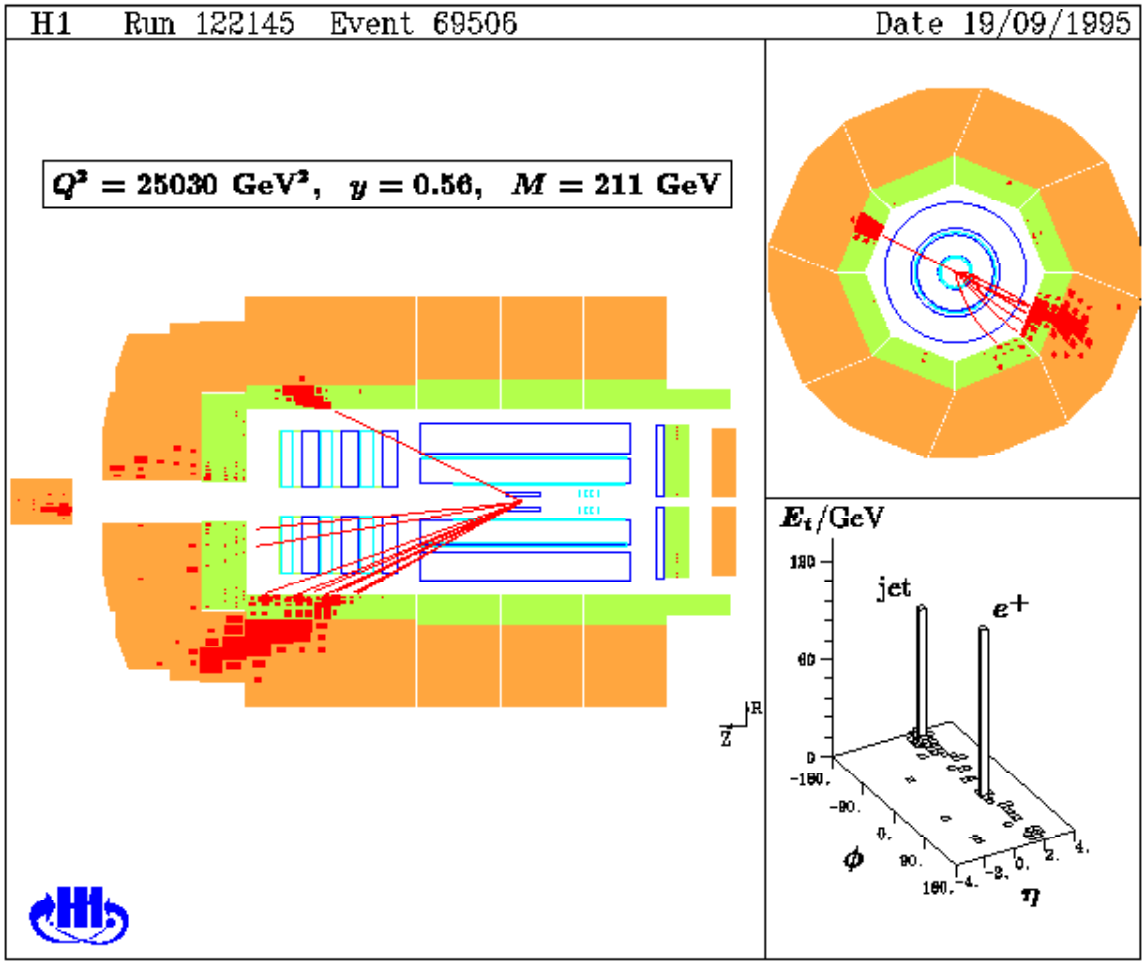
★ **Probe proton at very high Q^2 and very low x**

Example of a High Q^2 Event in the H1 Experiment at HERA

- **Event kinematics determined from electron angle and energy**



- **Also measure hadronic system (although not as precisely) – gives some redundancy**



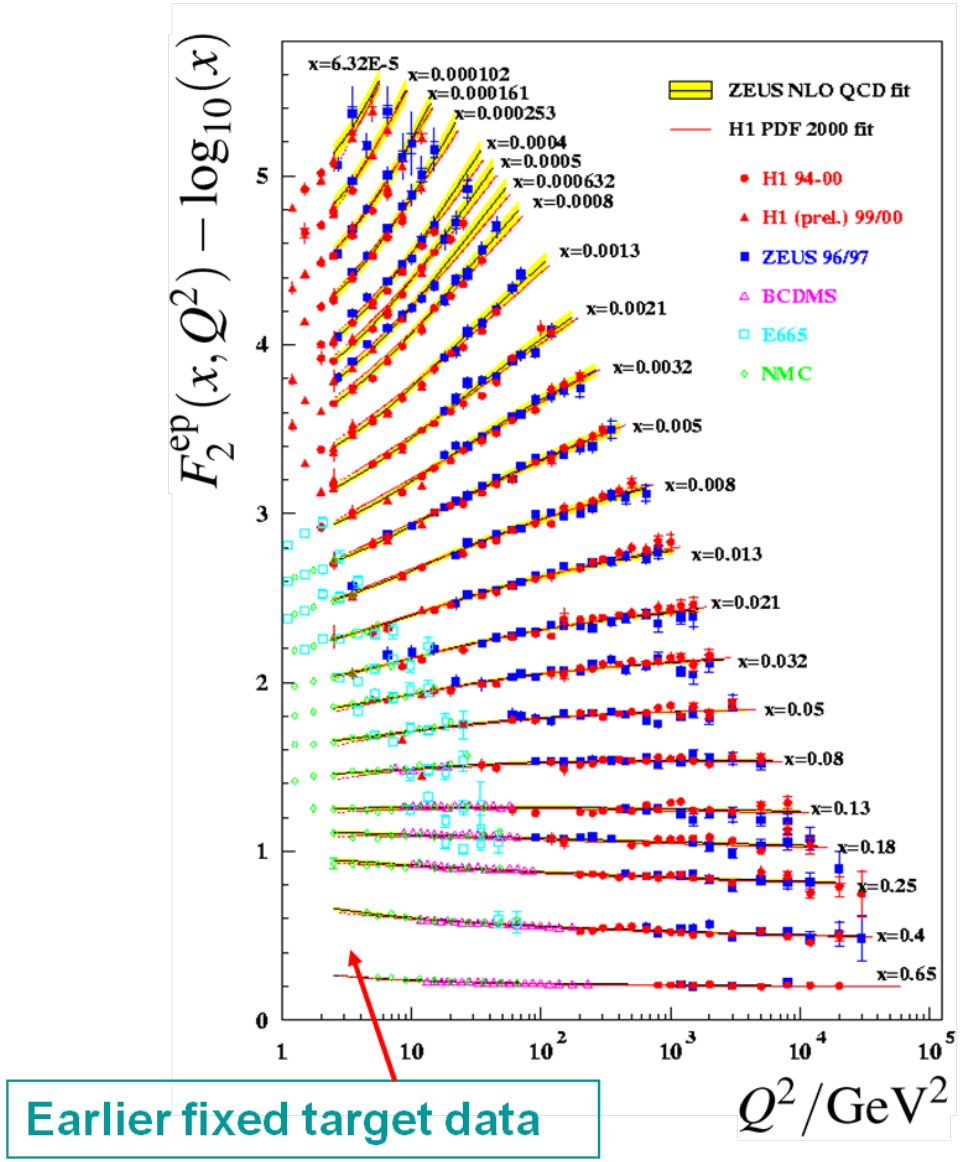
$F_2(x, Q^2)$ Results

- No evidence of rapid decrease of cross section at highest Q^2

$$\Longrightarrow \quad \boxed{R_{\text{quark}} < 10^{-18}\,\text{m}}.$$

- For $x > 0.05$, only weak dependence of F_2 on Q^2 : consistent with the expectation from the quark-parton model.
- Observe clear scaling violations, particularly at low x:

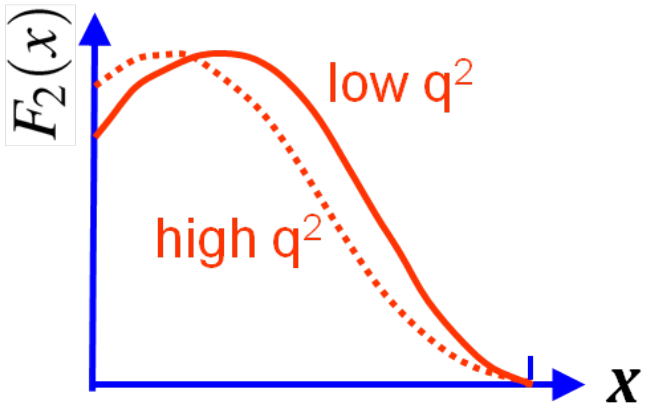
$$F_2(x, Q^2) \neq F_2(x).$$

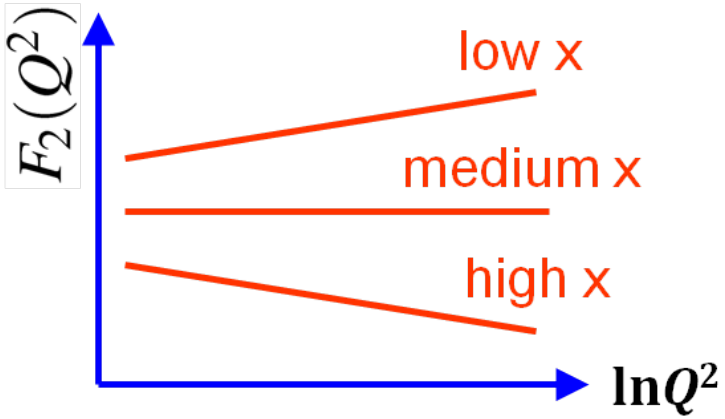


Origin of Scaling Violations at small x

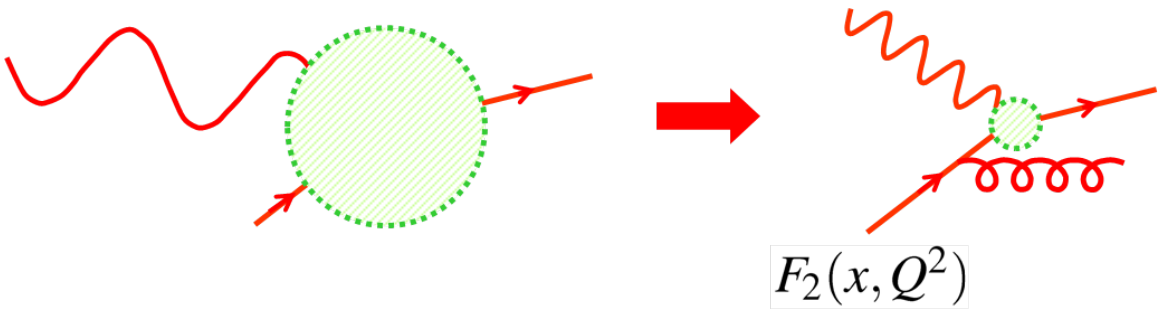
Not examinable

- Observe "small" deviations from exact Bjorken scaling $F_2(x) \to F_2(x, Q^2)$
- At high Q^2 observe more low x quarks.





- "Explanation": at high Q^2 (shorter wave-length) resolve finer structure: i.e. reveal quark is sharing momentum with gluons. At higher Q^2 expect to "see" more low x quarks.
- QCD cannot predict the x dependence of $F_2(x, Q^2)$



Not examinable

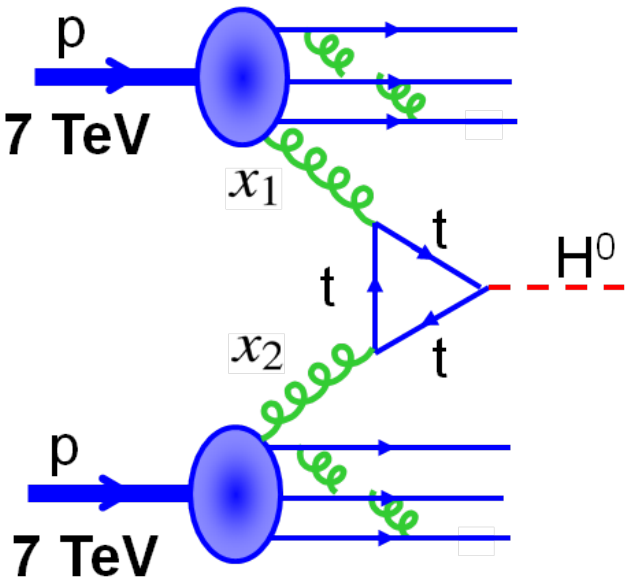
But QCD can predict the Q^2 dependence of $F_2(x, Q^2)$.

Proton-Proton Collisions at the LHC

- Measurements of structure functions not only provide a powerful test of QCD, the parton distribution functions are essential for the calculation of cross sections at pp and pp collider.

Example: Higgs production at the Large Hadron Collider LHC (2009-)

- The LHC collides up to $7\,TeV$ protons with $7\,TeV$ protons, however underlying collisions are between partons.



- Higgs production the LHC dominated by "gluon-gluon fusion"
- Cross section depends on gluon PDFs

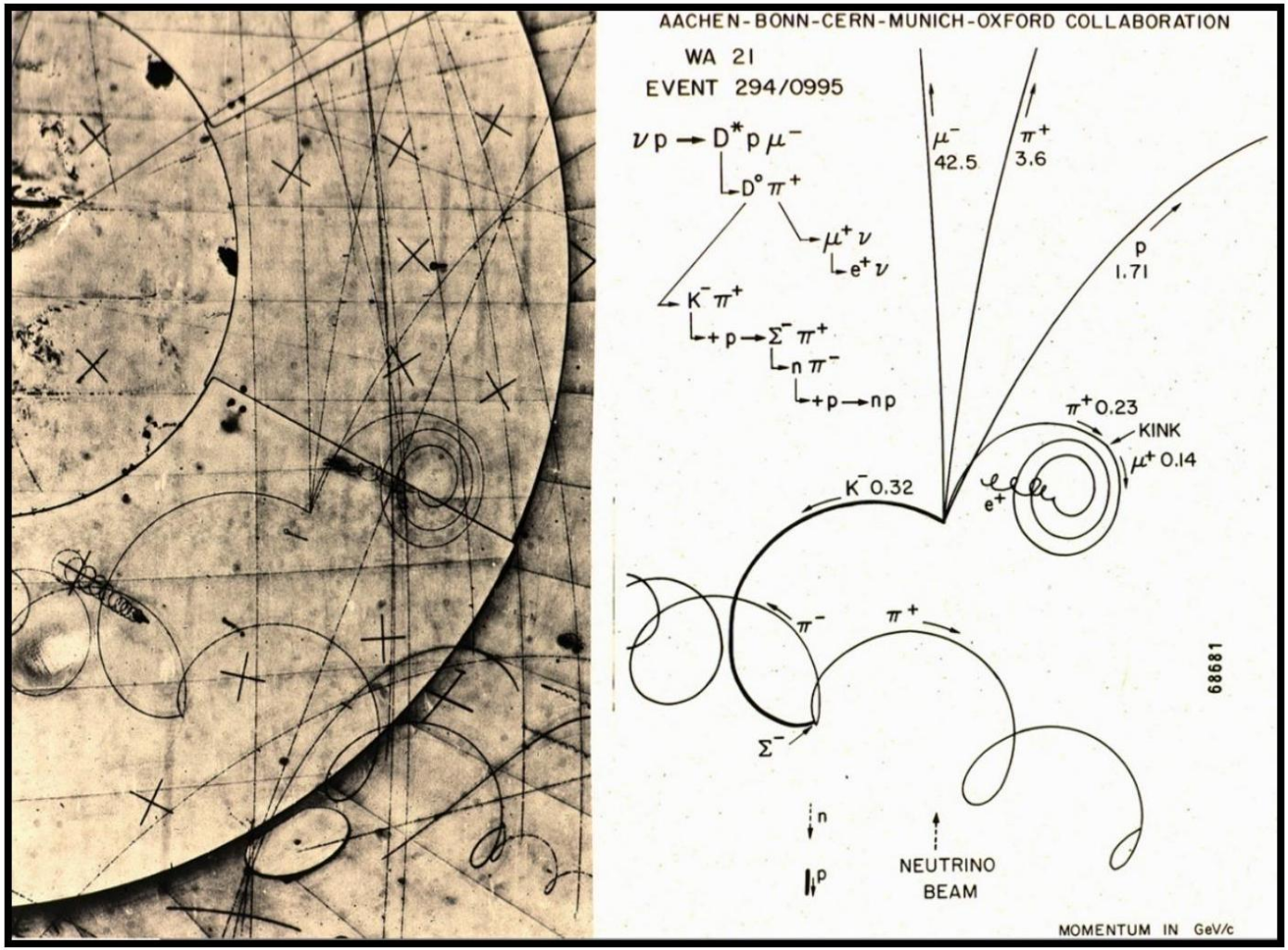
$$\sigma(pp \to HX) \sim \int_0^1 \int_0^1 g(x_1)g(x_2)\sigma(gg \to H)\, dx_1\, dx_2.$$

- Uncertainty in gluon PDFs lead to a $\pm 5\%$ uncertainty in Higgs production cross section.
- Prior to HERA data uncertainty was $\pm 25\%$.

Summary

- At very high electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons.
- Deep Inelastic Scattering = Elastic scattering from the quasi-free constituent quarks.
  - **Bjorken Scaling:** $F_1(x, Q^2) \to F_1(x)$ (point-like scattering).
  - **Callan-Gross:** $F_2(x) = 2xF_1(x)$ (scattering from spin-$\frac{1}{2}$).
- Describe scattering in terms of parton distribution functions $u(x), d(x), \ldots$ which describe momentum distribution inside a nucleon.
- The proton is much more complex than just *uud* — sea of anti-quarks/gluons.
- Quarks carry only 50% of the protons momentum — the rest is due to low energy gluons.
- We will come back to this topic when we discuss neutrino scattering ...

Prof. Alex Mitov, 2026



H7: Symmetries and the Quark Model

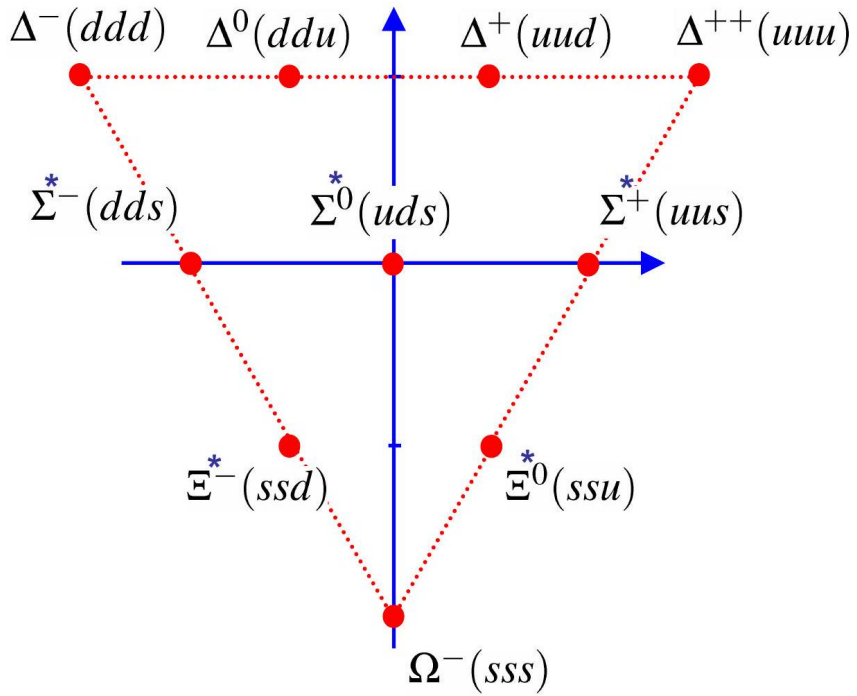
Symmetries in this handout

Symmetries play a central role in particle physics; one aim of particle physics is to discover the fundamental symmetries of our universe.

In this handout will apply the idea of symmetry to the quark model with the aim of:

- deriving hadron wave-functions,
- providing an introduction to the more abstract ideas of colour and QCD (Handout 8), and
- ultimately explaining why hadrons only exist as $\bar{q}q$ (mesons) qqq (baryons) or $\bar{q}\bar{q}\bar{q}$ (anti-baryons) and combinations thereof (tetra quarks $q\bar{q}q\bar{q}$, pentaquarks $q\bar{q}qqq$, etc.).

En route we will see some early usage of $SU(2)$ and $SU(3)$ symmetry groups which play a role both hear and later on (e.g. see Handout 13).



Symmetries and Conservation Laws

- Suppose physics is invariant under the transformation

$$\psi \to \psi' = \hat{U}\psi \qquad \text{e.g. rotation of the coordinate axes}$$

- If U is a symmetry which preserves state normalisations then for all $|\ \psi\ \rangle$ we require:

$$\langle \psi \mid \psi \rangle = \langle \psi' \mid \psi' \rangle = \langle \hat{U}\psi \mid \hat{U}\psi \rangle = \left\langle \psi \middle| \hat{U}^\dagger \hat{U} \middle| \psi \right\rangle$$

$$\implies \boxed{\hat{U}^\dagger \hat{U} = 1} \qquad \text{i.e. } \hat{U} \text{ has to be unitary.} \tag{108}$$

- For physical predictions to be unchanged by the symmetry transformation, we also require (for all $|\ \psi\ \rangle$) that

$$\langle \psi | \hat{H} | \psi \rangle = \left\langle \psi' | \hat{H} | \psi' \right\rangle .$$

Since $\left\langle \psi' | \hat{H} | \psi' \right\rangle = \left\langle \psi \middle| \hat{U}^\dagger \hat{H} \hat{U} \middle| \psi \right\rangle$ we need $\hat{U}^\dagger \hat{H} \hat{U} = \hat{H}$ which (using (108)) says:

$$\boxed{[\hat{H}, \hat{U}] = 0} \qquad \text{i.e. } \hat{U} \text{ commutes with the Hamiltonian.}$$

- If the symmetry can be small (i.e. almost the identity) then an infinitesimal transformation could be wrtten (in terms of $\varepsilon \ll 1$) as:

$$\hat{U} = 1 + i\varepsilon \hat{G}.$$

$\hat{G}$ is called the generator of the transformation.

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^\dagger = (1 + i\varepsilon\hat{G})\left(1 - i\varepsilon\hat{G}^\dagger\right) = 1 + i\varepsilon\left(\hat{G} - \hat{G}^\dagger\right) + O\left(\varepsilon^2\right)$$

neglecting terms in ε^2 $\quad UU^\dagger = 1 \quad \Rightarrow \quad \hat{G} = \hat{G}^\dagger$ i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity G !

- Furthermore, $\quad [\hat{H}, \hat{U}] = 0 \quad \Rightarrow [\hat{H}, 1 + i\varepsilon\hat{G}] = 0 \quad \Rightarrow [\hat{H}, \hat{G}] = 0.$
But from Ehrenfest Theorem in QM:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H}, \hat{G}]\rangle = 0$$

i.e. G is a conserved quantity.

$$\boxed{\text{Symmetry} \Longleftrightarrow \text{Conservation Law}}$$

- Each such symmetry of nature therefore has an observable conserved quantity.
Example: Infinitesimal spatial translation $x \to x + \varepsilon$ i.e. expect physics to be invariant under $\psi(x) \to \psi' = \psi(x + \varepsilon)$:

$$\psi'(x) = \psi(x + \varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1 + \varepsilon\frac{\partial}{\partial x}\right)\psi(x) = \left(1 + i\varepsilon(-i\frac{\partial}{\partial x})\right)\psi(x)$$

but

$$\hat{p}_x = -i\frac{\partial}{\partial x} \Rightarrow \psi'(x) = (1 + i\varepsilon\hat{p}_x)\,\psi(x).$$

The generator of the symmetry transformation is $\hat{p}_x$ and so p_x is conserved.

- Translational invariance of physics implies momentum conservation!

- In general the symmetry operation may depend on more than one parameter:

$$\hat{U} = 1 + i\vec{\varepsilon} \cdot \vec{G}$$

For example for an infinitesimal 3D linear translation: $\vec{r} \longrightarrow \vec{r} + \vec{\varepsilon}$ with $\vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ we have

$$\hat{U} = 1 + i\vec{\varepsilon} \cdot \vec{p}.$$

- So far have only considered an infinitesimal transformation. Fortunately, finite transformations can be expressed as a series of infinitesimal transformations:

$$\hat{U}(\vec{\varepsilon}) = \lim_{n\to\infty} \left(1 + i\frac{\vec{\varepsilon}}{n} \cdot \vec{G}\right)^n = e^{i\vec{\varepsilon}\cdot\vec{G}}.$$

- Example: Finite spatial translation in 1D: $x \to x + x_0$ with $\hat{U}(x_0) = e^{ix_0\hat{p}_x}$

$$\begin{aligned}
\psi'(x) = \psi(x + x_0) = \hat{U}\psi(x) = \exp\left(x_0\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \quad \left(p_x = -i\frac{\partial}{\partial x}\right) \\
= \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x) \\
= \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots
\end{aligned}$$

confirming that one obtains the expected Taylor expansion for a translated field.

Symmetries in Particle Physics : Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}.$$

- To reflect this symmetry, Heisenberg (1932) proposed that:

> If you could "switch off" the electric charge of the proton there would be no way to distinguish between a proton and neutron.

- More specifically he proposed that the neutron and proton should be considered as two states of a single entity; the nucleon:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- Symmetry ended up being called isospin since maths is very similar that of spin-up/spin-down states of spin-$\frac{1}{2}$ particles.
- Expect physics to be invariant under continuous (ahem) 'rotations' in isospin space just as the axis against which ordinary spin is measured can be continuously varied.
- The neutron and proton form an isospin doublet with total isospin $I = \frac{1}{2}$ and third component $I_3 = \pm\frac{1}{2}$.

Flavour Symmetry of the Strong Interaction

We can extend this idea to the quarks: we can assume that assume the strong interaction treats all quark flavours equally (which it does!).

- Because $m_u \approx m_d$ the strong interaction possesses an approximate flavour symmetry i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and vice versa.
- Choose the basis
$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
- Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under "rotations" in an abstract isospin space:
$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}.$$

In general a 2×2 complex matrix has 4 complex (i.e. 8 real) degrees of freedom. However, when such a matrix is unitary it has only 4 real degrees of freedom as $\hat{U}^\dagger \hat{U} = 1$ imposes 4 real constraints (since no matter what $\hat{U}$ is, $\hat{U}^\dagger \hat{U}$ is always Hermitian).

$$\implies \text{4 independent d.o.f. in these matrices.}$$

- In the language of group theory, the matrices form the $U(2)$ group.

Generators for $SU(2)$-flavour a.k.a. 'isospin'

- One of the four available degrees of freedom (d.o.f.) corresponds to multiplying by a phase factor. This is not a **flavour transformation** and so is not relevant.

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

- The remaining three d.o.f. parameterise the $SU(2)$ group of 'special' (i.e. $\det U = 1$) unitary matrices. For infinitessimal transformations, place these d.o.f. in ε_i and write:

$$\hat{U}(\vec{\varepsilon}) = 1 + i \sum_{i=1}^{3} \varepsilon_i \hat{G}_i$$

where the three $\hat{G}_i$ are called the **generators** of the symmetry. $[\vec{G} \equiv (\hat{G}_1, \hat{G}_2, \hat{G}_3)]$

- **EXERCISE:** check that $\boxed{(\det U = 1) \iff (\mathrm{Tr}(\hat{G}_i) = 0)}$ for infinitessimal tfms.
- **EXERCISE:** (re)check that $\boxed{(\hat{U}^\dagger \hat{U} = 1) \iff (\hat{G}_i^\dagger = \hat{G}_i)}$ for infinitessimal tfms.
- The Pauli Matrices are three linearly independent traceless and Hermitian matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- The three generators $\hat{G}_i$ of $SU(2)$-flavour (a.k.a. isospin) symmetry are traditionally denoted $\hat{T}_i$ and are defined by $\boxed{\hat{T}_i = \frac{1}{2}\sigma_i}$ making the general tfm.: $\boxed{U(\vec{\varepsilon}) = e^{i\vec{\varepsilon}\cdot\vec{T}}}$.

(The $\frac{1}{2}$ does not stop these $\hat{T}_i$ being traceless, Hermitian and linearly independent!)

Aside: a check that isospin has claimed properties

Not examinable

For any $SU(2)$-flavour (i.e. isospin) transformation $U(\varepsilon)$ we have:

$$\begin{aligned}\hat{U}(\varepsilon) &= 1 + \frac{1}{2} i\vec{\varepsilon}\cdot\vec{\sigma} + O(\varepsilon^2) \\ &= 1 + \frac{i}{2}\left(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3\right) + O(\varepsilon^2) \\ &= \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i\left(\varepsilon_1 - i\varepsilon_2\right) \\ \frac{1}{2}i\left(\varepsilon_1 + i\varepsilon_2\right) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix} + O(\varepsilon^2).\end{aligned}$$

$\hat{U}(\vec{\varepsilon})$ is evidently unitary (at least for infinitessimal transformations) because:

$$\begin{aligned}U^\dagger U &= \begin{pmatrix} 1 - \frac{1}{2}i\varepsilon_3 & -\frac{1}{2}i\left(\varepsilon_1 - i\varepsilon_2\right) \\ -\frac{1}{2}i\left(\varepsilon_1 + i\varepsilon_2\right) & 1 + \frac{1}{2}i\varepsilon_3 \end{pmatrix} \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i\left(\varepsilon_1 - i\varepsilon_2\right) \\ \frac{1}{2}i\left(\varepsilon_1 + i\varepsilon_2\right) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix} + O(\varepsilon^2) \\ &= I + O(\varepsilon^2) \qquad \text{(multiply the terms above to see why!)}.\end{aligned}$$

$\hat{U}(\vec{\varepsilon})$ also has determinant 1 (at least for infinitessimal transformations) because:

$$\begin{aligned}\det U &= \left| \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i\left(\varepsilon_1 - i\varepsilon_2\right) \\ \frac{1}{2}i\left(\varepsilon_1 + i\varepsilon_2\right) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix} + O(\varepsilon^2) \right| \\ &= (1 + \frac{1}{2}i\varepsilon_3)(1 - \frac{1}{2}i\varepsilon_3) - O(\varepsilon^2) \\ &= 1 + O(\varepsilon^2).\end{aligned}$$

Not examinable

Mathematical similarities between Isopin and Spin

- Isospin generators have exactly the same properties as those of spin:

$$[T_1, T_2] = iT_3, \quad [T_2, T_3] = iT_1, . \quad [T_3, T_1] = iT_2$$
$$\left[T^2, T_3\right] = 0, \quad T^2 = T_1^2 + T_2^2 + T_3^2.$$

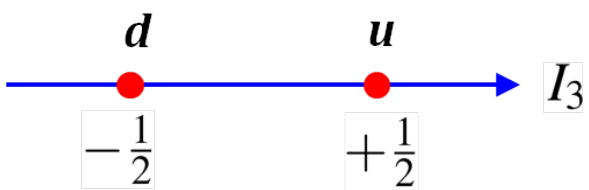
As in the case of spin, have three non-commuting Hermitian operators, T_1, T_2, T_3, so even though all three correspond to observables, we can't measure them simultaneously.

- So label states in terms of total isospin I and the third component of isospin I_3. These eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s, m\rangle \rightarrow |I, I_3\rangle$:

$$T^2 |I, I_3\rangle = I(I+1) |I, I_3\rangle \quad T_3 |I, I_3\rangle = I_3 |I, I_3\rangle$$

- In terms of isospin:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left|\frac{1}{2}, +\frac{1}{2}\right\rangle, \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle.$$



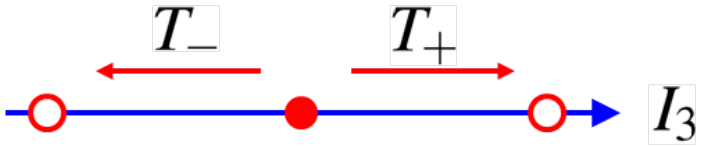
$$\boxed{I = \frac{1}{2}, I_3 = \pm\frac{1}{2}}$$

- In general $I_3 = \frac{1}{2}(N_u - N_d)$.

Can define isospin ladder operators - analogous to spin ladder operators

$T_- \equiv T_1 - iT_2$

$u \to d$



$T_+ \equiv T_1 + iT_2$

$d \to u$

$$T_+ |I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}\, |I, I_3+1\rangle$$
$$T_- |I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}\, |I, I_3-1\rangle$$

These ops step up/down in I_3 until reach end of multiplet: $T_+|I,+I\rangle = 0 \quad T_-|I,-I\rangle = 0.$

$$\boxed{T_+ u = 0, \quad T_+ d = u, \quad T_- u = d, \quad T_- d = 0}$$

What is isospin of a sytemt of two quarks?

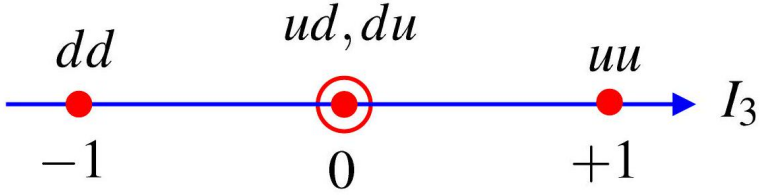
- Combining/adding isospin works same as combining/adding angular momentum:

$$\left|I^{(1)}, I_3^{(1)}\right\rangle \left|I^{(2)}, I_3^{(2)}\right\rangle \to |I, I_3\rangle\,.$$

- I_3 additive : $\quad I_3 = I_3^{(1)} + I_3^{(2)}$,
- I in integer steps from $\left|I^{(1)} - I^{(2)}\right|$ to $\left|I^{(1)} + I^{(2)}\right|$.
- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum.

Combining Quarks

Isospin starts to become useful in defining states of more than one quark.
Two quarks states could use a $\{uu, ud, du, dd\}$ basis:



This is not a good basis, though, becayse two of these states are not eigenstates of total isospin. Here (and on next slide) we will derive a better basis whose elements have well defined I.

- We can immediately identify the extremes since I_3 is additive:

$$uu \equiv \left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = |1,+1\rangle \quad dd \equiv \left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle = |1,-1\rangle.$$

- To obtain the $|1,0\rangle$ state use ladder operators

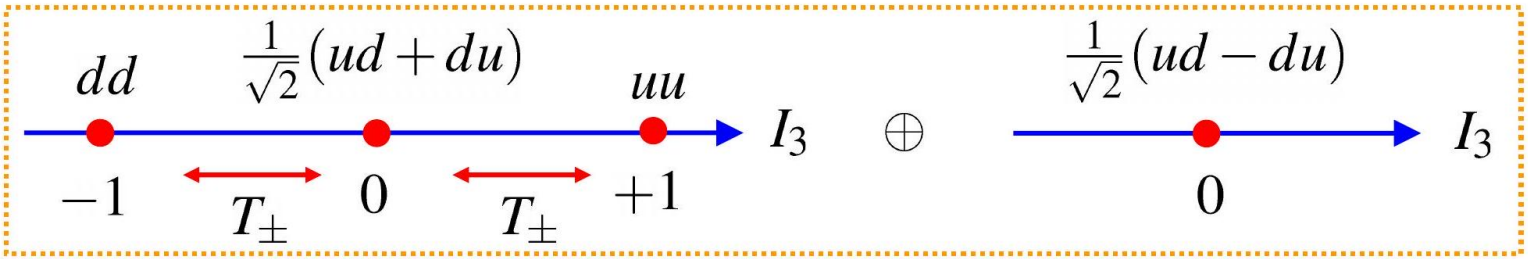
$$T_-|1,+1\rangle = \sqrt{2}|1,0\rangle = T_-(uu) = ud + du$$

$$\Longrightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}(ud + du)$$

- The final (fourth) basis state, $|0,0\rangle$, can be found from orthogonality with $|1,0\rangle$

$$\Longrightarrow \quad |0,0\rangle = \frac{1}{\sqrt{2}}(ud - du).$$

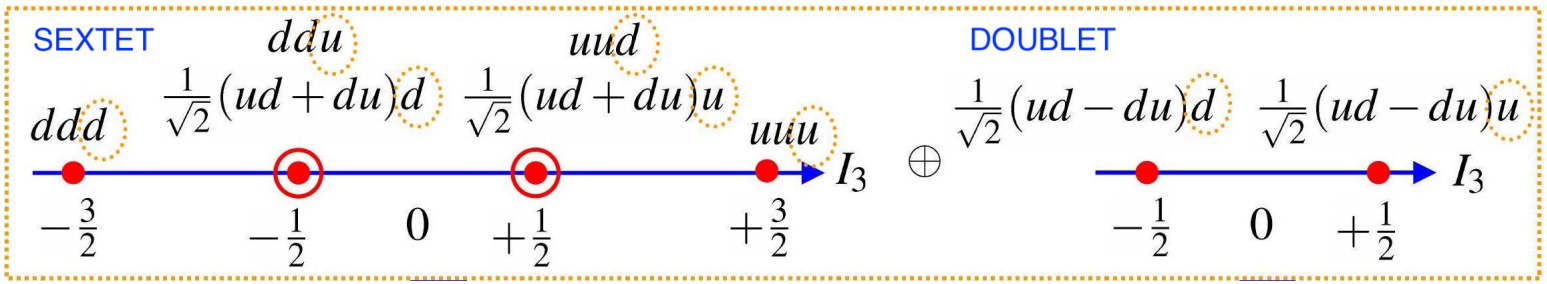
- From four possible combinations of isospin doublets obtain a triplet of isospin 1 states and a singlet isospin 0 state $2 \otimes 2 = 3 \oplus 1$



- Can move around within multiplets using ladder operators.
- As anticipated $I_3 = \frac{1}{2}(N_u - N_d)$.
- States with different total isospin are physically different — the isospin 1 triplet is symmetric under interchange of quarks 1 and 2 whereas singlet is anti-symmetric.

Now add an additional up or down quark:

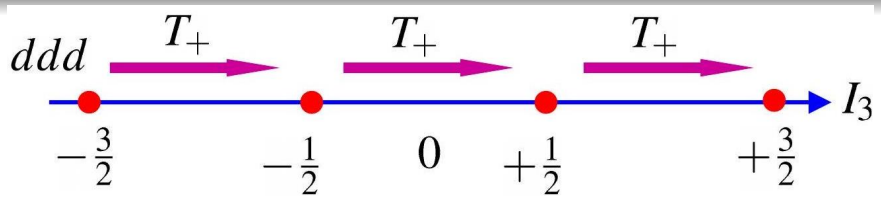
From each of the above 4 states get two new isospin states with $I_3' = I_3 \pm \frac{1}{2}$:



- As for two quark case, the **extremal** states of a given multiplet have well defined isospin, but we must use ladder operators and orthogonality find the states which shares (or have different) properties to the extremal states. E.g. to obtain the $I = \frac{3}{2}$ states, step up from *ddd* (or down from *uuu*) within the SEXTET. (See next page!)

Four states that are symmetric under exchange of first two quarks

Derive the $I = \frac{3}{2}$ states from $ddd \equiv \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$:



$$\boxed{\left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = ddd}$$

$$\Longrightarrow \quad T_+ \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = T_+(ddd) = (T_+ d)\, dd + d\,(T_+ d)\, d + dd\,(T_+ d)$$

$$\Longrightarrow \quad \sqrt{3}\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = udd + dud + ddu$$

$$\Longrightarrow \quad \boxed{\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(udd + dud + ddu)}$$

$$\Longrightarrow \quad T_+ \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}\, T_+(udd + dud + ddu)$$

$$\Longrightarrow \quad 2\left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(uud + udu + uud + duu + udu + duu)$$

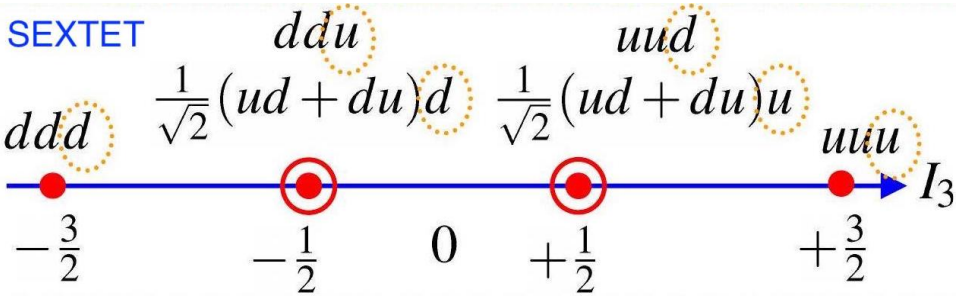
$$\Longrightarrow \quad \boxed{\left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(uud + udu + duu)}$$

$$\Longrightarrow \quad T_+ \left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}\, T_+(uud + udu + duu)$$

$$\Longrightarrow \quad \sqrt{3}\left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(\ uuu\ +\ uuu\ +\ uuu\ )$$

$$\Longrightarrow \quad \boxed{\left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle = \ uuu\ }.$$

Two more states that are symmetric under exchange of first two quarks



- The SEXTET (above) contained six states.
  All were symmetric under exchange of the first two quarks, $q_1 \leftrightarrow q_2$.
- On last page we found four states with the same $q_1 \leftrightarrow q_2$ symmetry (but with well defined isopin). There must therefore be two more states which are orthogonal to the four just found, and which are symmetric under $q_1 \leftrightarrow q_2$ but which have well defined isospin.
- The two missing orthogonal states hidden within the SEXTET are:

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -\tfrac{1}{\sqrt{6}}(2ddu - udd - dud), \qquad \text{and}$$
$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle = +\tfrac{1}{\sqrt{6}}(2uud - udu - duu).$$

[EXERCISE: Check that these states are mapped to each other by $T_\pm$, that they have $I = 1/2$, and that they are orthogonal to each other and to the other states.]

Flavour basis for states containing three quarks

We still have the two DOUBLET states from page 194 – which are (conveniently) already an iso-doublet. Thus, we have managed to replace the 8-dimensional basis $\{uuu, uud, udu, udd, duu, dud, ddu, ddd\}$ with one formed from **one iso-quadruplet** and **two iso-doublets**:

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

Each multiplet has its own distinct symmetry:

$$\left.\begin{aligned} \left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle &= uuu \\ \left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(uud + udu + duu) \\ \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(ddu + dud + udd) \\ \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle &= ddd \end{aligned}\right\} S; \qquad S = \boxed{\text{A quadruplet of states which are Symmetric under the interchange of any two quarks.}}$$

$$\left.\begin{aligned} \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle &= +\tfrac{1}{\sqrt{6}}(2uud - udu - duu) \\ \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle &= -\tfrac{1}{\sqrt{6}}(2ddu - udd - dud) \end{aligned}\right\} M_S; \qquad M_S = \boxed{\text{Mixed, Symmetric under interchange of quarks } 1 \leftrightarrow 2.}$$

$$\left.\begin{aligned} \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(udu - duu) \\ \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(udd - dud) \end{aligned}\right\} M_A; \qquad M_A = \boxed{\text{Mixed, Antisymmetric under interchange of quarks } 1 \leftrightarrow 2.}$$

The states in M_S and M_A have no definite symmetry under interchange of the third quark with either of the others. This will change when we add spin later.

Spin basis for states containing three spin-half particles

Since the maths for spin is the same as the maths for isospin, we can replace the 8-dimensional basis $\{\uparrow\uparrow\uparrow, \uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, \uparrow\downarrow\downarrow, \downarrow\uparrow\uparrow, \downarrow\uparrow\downarrow, \downarrow\downarrow\uparrow, \downarrow\downarrow\downarrow\}$ with one formed from **one iso-quadruplet** and **two iso-doublets**:

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

Each multiplet has its own distinct symmetry:

$$\left.\begin{aligned}
\left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle &= \uparrow\uparrow\uparrow \\
\left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\
\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow) \\
\left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle &= \downarrow\downarrow\downarrow
\end{aligned}\right\} S; \qquad S = \boxed{\text{A quadruplet of states which are Symmetric under the interchange of any two quarks.}}$$

$$\left.\begin{aligned}
\left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle &= +\tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\
\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle &= -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)
\end{aligned}\right\} M_S; \qquad M_S = \boxed{\text{Mixed, Symmetric under interchange of quarks } 1 \leftrightarrow 2.}$$

$$\left.\begin{aligned}
\left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\
\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)
\end{aligned}\right\} M_A; \qquad M_A = \boxed{\text{Mixed, Antisymmetric under interchange of quarks } 1 \leftrightarrow 2.}$$

As was the case for the flavour basis, the states in M_S and M_A have no definite symmetry under interchange of the third quark with either of the others.

Baryon Wave-functions

Quarks are fermions so the total wave-function must be anti-symmetric under the interchange of any two quarks.

- The total wave-function can be factorised into:

$$\psi = \phi_{\text{flavour}}\, \chi_{\text{spin}}\, \xi_{\text{colour}}\, \eta_{\text{space}}$$

- ξ_{colour} is anti-symmetric as it is a bound qqq states (see Handout 8).
- η_{space} is symmetric (for us) as we will only consider the lowest mass, ground state baryons ($L = 0$). [Symmetry is $(-1)^L$ in general.]

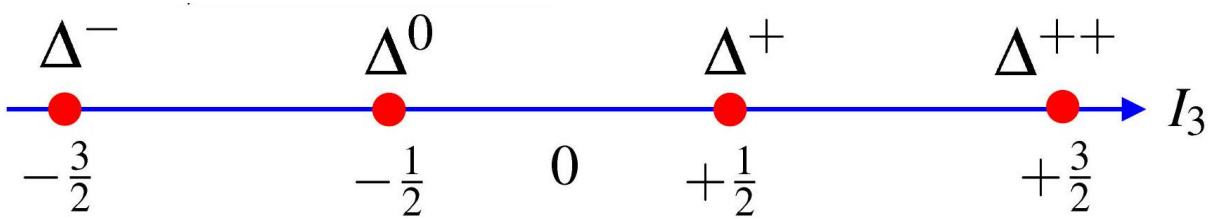
Thus:

$\phi_{flavour}\chi_{spin}$ must be symmetric under the interchange of any two quarks

- Two ways to form a totally symmetric wave-function from spin and isospin states:

(1) combine totally symmetric spin and isospin wave-functions $\phi(S)\chi(S)$

$ddd \quad \frac{1}{\sqrt{3}}(ddu + dud + udd) \quad \frac{1}{\sqrt{3}}(uud + udu + duu) \quad uuu$



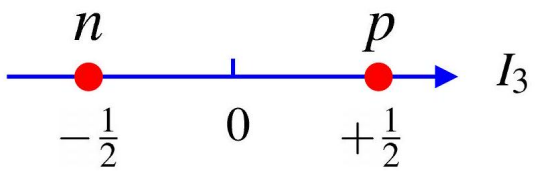
Spin 3/2
Isospin 3/2

(2) combine mixed symmetry spin and mixed symmetry isospin states:

- Both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
- Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \ldots$
- However, it is not difficult to show that the (normalised) linear combination:

$$\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)$$

is totally symmetric (i.e. symmetric under $q_1 \leftrightarrow q_2; q_1 \leftrightarrow q_3; q_2 \leftrightarrow q_3$).



Spin 1/2
Isospin 1/2

- The spin-up proton wave-function is therefore:

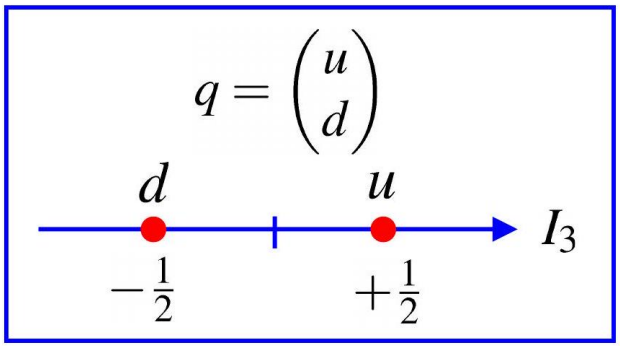
$$|p\uparrow\rangle = \tfrac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \tfrac{1}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

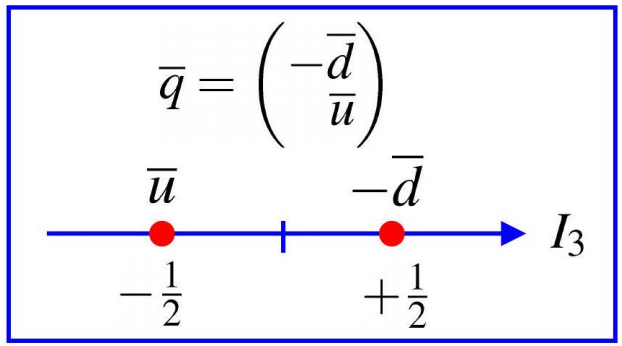
$$\Longrightarrow |p\uparrow\rangle = \left\{ \begin{array}{l} \frac{1}{\sqrt{18}}(\quad 2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow \\ \qquad +2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow \\ \qquad +2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\downarrow \quad) \end{array} \right\}.$$

NOTE: it is not always necessary to use the fully symmetrised proton wave-function. E.g. the first three terms are sufficient for calculating the proton magnetic moment.

Anti-quarks and Mesons (u and d)

⋆ The **u**, **d** quarks and $\bar{\mathbf{u}}, \bar{\mathbf{d}}$ anti-quarks are represented as isospin doublets





$$\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\bar{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- The ordering and the minus sign in the anti-quark doublet ensures that anti-quarks and quarks transform in the same way (see Appendix X). This is necessary if we want physical predictions to be invariant under $u \leftrightarrow d; \bar{u} \leftrightarrow \bar{d}$
- For anti-quarks the ladder operators introduce an extra minus sign not seen for quarks. E.g.:

$$T_+ \bar{u} = T_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\bar{d}$$

The effect of the ladder operators on all isospin states are:

anti-quarks : $T_+ \bar{u} = -\bar{d}$, $T_+ \bar{d} = 0$, $T_- \bar{u} = 0$, $T_- \bar{d} = -\bar{u}$, and

quarks : $T_+ d = +u$, $T_+ u = 0$, $T_- d = 0$, $T_- u = +d$.

Light ud Mesons

- Can now construct meson states from combinations of up/down quarks



- Consider the $q\bar{q}$ combinations in terms of isospin

$$|1,+1\rangle = \left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle \overline{\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle} = -u\bar{d}$$
$$|1,-1\rangle = \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \overline{\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle} = \quad d\bar{u}$$

(The bar indicates this is the isospin representation of an anti-quark.)

- To obtain the $I_3 = 0$ states use ladder operators and orthogonality:

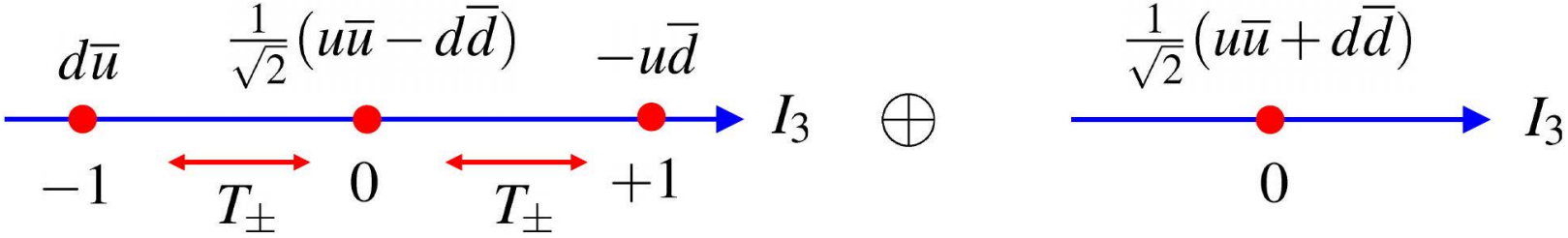
$$T_-|1,+1\rangle = T_-[-u\bar{d}]$$
$$\implies \quad \sqrt{2}|1,0\rangle = -T_-[u]\bar{d} - uT_-[\bar{d}]$$
$$= -d\bar{d} + u\bar{u}$$
$$\implies \quad \boxed{|1,0\rangle = \tfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})}.$$

Orthogonality gives: $\boxed{|0,0\rangle = \tfrac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})}$.

- To summarise:



decomposes into a triplet of $I = 1$ states and a singlet $I = 0$ state:



- You will see this written as $2 \otimes \bar{2} = 3 \oplus 1$
  with 2 being the quark doublet and $\bar{2}$ being the anti-quark doublet.
- To show the state obtained from orthogonality with $|1, 0\rangle$ is a singlet use ladder operators

$$T_+|0,0\rangle = T_+ \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) = \frac{1}{\sqrt{2}}(-u\bar{d} + u\bar{d}) = 0.$$

  Silimiarly

$$T_-|0,0\rangle = 0.$$

- A singlet state is a 'dead end' from the point of view of ladder operators.

$SU(3)$-flavour: the extension to the strange quark

- Since $m_s > m_u, m_d$ have only an approximate symmetry. Nonetheless, m_s is not 'very' different from m_u, m_d, and so the strong interaction acts as if its states were approximately symmetric under $u \leftrightarrow d \leftrightarrow s$.
- The assumed *uds* flavour symmetry can timesbe expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3×3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$ so the $U(3)$ group is $18 - 9 =$ 9-dimesional.
- As before, one d.o.f. simply allows for matrices in $U(3)$ to be multiplied by a complex phase and is of no interest in the context of flavour symmetry.
- The remaining 8 d.o.f. control variation between matrices with the 'special' property $\det U = 1$. The group which these d.o.f. parameterise is therefore called $SU(3)$.
- The eight traceless and Hermitian generators G_i for $SU(3)$-flavour are denoted: $\boxed{\vec{G} = \frac{1}{2}\vec{\lambda}}$ making a general element of the group: $\boxed{\hat{U}(\vec{\varepsilon}) = e^{i\vec{\varepsilon}\cdot\vec{G}} = e^{i\frac{1}{2}\vec{\varepsilon}\cdot\vec{\lambda}}}$.

- In SU(3) flavour, the three quark states are represented by:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- In SU(3) uds flavour symmetry contains SU(2) ud flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0\ 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0\ 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0\ 0 & 0 \end{pmatrix}$$

i.e. $\boxed{\mathbf{u} \leftrightarrow \mathbf{d}} \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- The third component of isospin is now written $I_3 = \frac{1}{2}\lambda_3$ (instead of $\frac{1}{2}\sigma_3$) thus

$$I_3 u = +\tfrac{1}{2}u, \quad I_3 d = -\tfrac{1}{2}d, \quad \text{and} \quad I_3 s = 0.$$

- I_3 counts the number of up quarks minus number of down quarks in a state.
- As before, ladder operators $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$ achieve $\boxed{d \longleftarrow T_\pm \longrightarrow u}$.

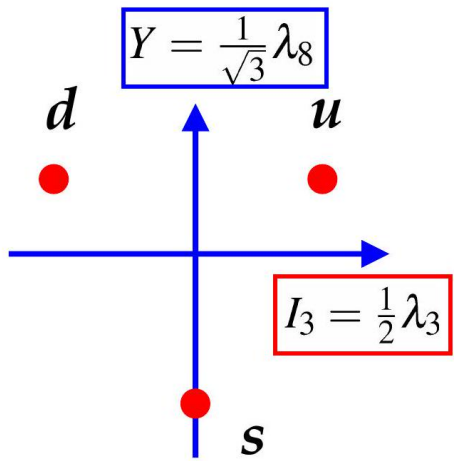
- Now consider the matrices corresponding to the **u** $\leftrightarrow$ **s** and **d** $\leftrightarrow$ **s**

u $\leftrightarrow$ **s**	$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
d $\leftrightarrow$ **s**	$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- Hence in addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have two other traceless diagonal matrices.
- However, these three diagonal matrices are not independent, so define the eighth matrix, λ_8, as the following linear combination:

$$\lambda_8 = \frac{1}{\sqrt{3}} \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which specifies the 'vertical position' in the 2D plane.

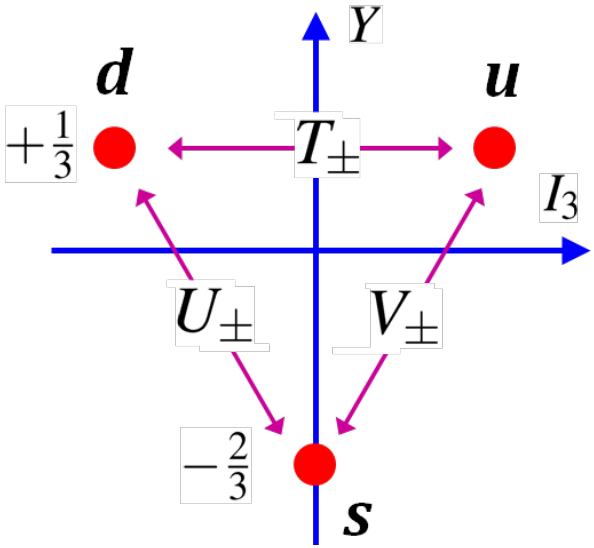


Only need two axes (quantum numbers) to specify a state in the 2D plane: (I_3, Y), i.e. third component of isospin and hypercharge, Y.

There are now six ladder operators (built from Gell-Mann matrices λ_i) which step between the states:

$$T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$$

$$I_3 = \tfrac{1}{2}\lambda_3$$
$$Y = \tfrac{1}{\sqrt{3}}\lambda_8$$



u ↔ d

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

u ↔ s

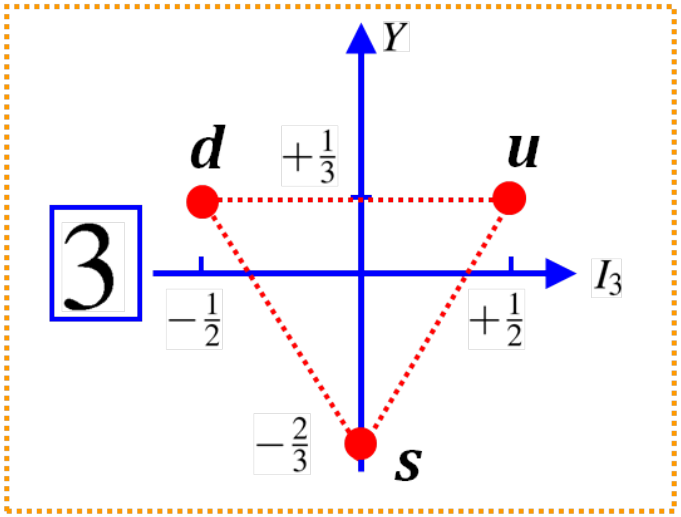
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

d ↔ s

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Quarks and anti-quarks in SU(3) Flavour

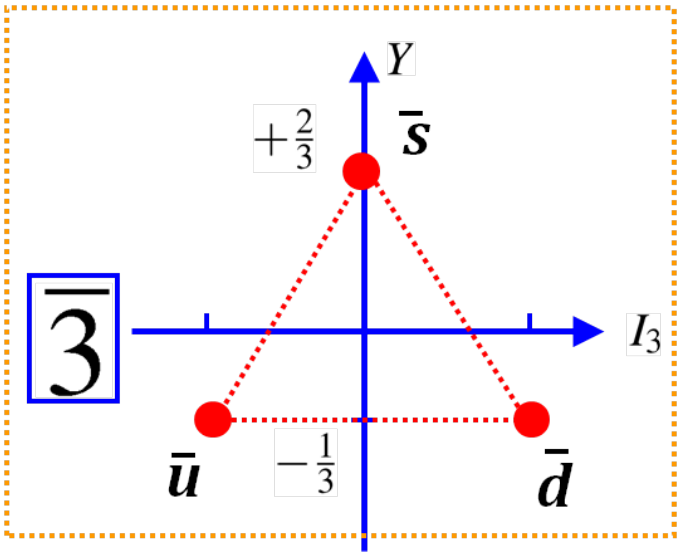


Quarks

$$I_3 u = +\tfrac{1}{2} u; \qquad I_3 d = -\tfrac{1}{2} d; \qquad I_3 s = 0$$

$$Yu = +\tfrac{1}{3} u; \qquad Yd = +\tfrac{1}{3} d; \qquad Ys = -\tfrac{2}{3} s$$

- The anti-quarks have opposite **SU**(3) flavour quantum numbers



Anti-Quarks

$$I_3 \bar{u} = -\tfrac{1}{2} \bar{u}; \qquad I_3 \bar{d} = +\tfrac{1}{2} \bar{d}; \qquad I_3 \bar{s} = 0$$

$$Y\bar{u} = -\tfrac{1}{3} \bar{u}; \qquad Y\bar{d} = -\tfrac{1}{3} \bar{d}; \qquad Y\bar{s} = +\tfrac{2}{3} \bar{s}$$

$SU(3)$ Ladder Operators

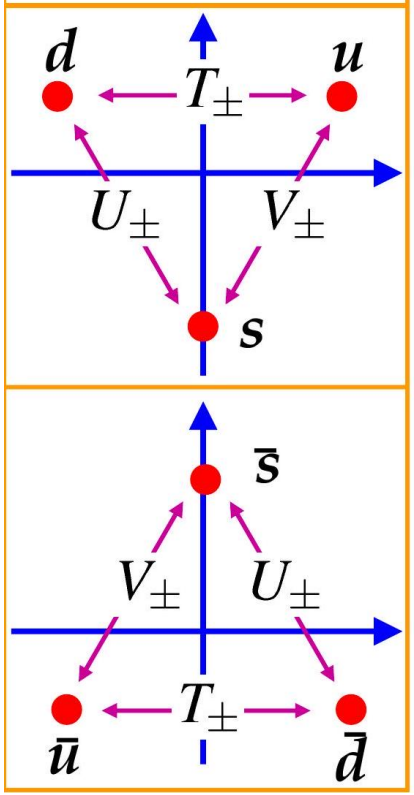
The uds $SU(3)$-flavour symmetry contains ud, us and ds $SU(2)$-flavour symmetries.

Consider the $u \leftrightarrow s$ symmetry 'V-spin' which has the associated $s \to u$ ladder operator

$$V_+ = \tfrac{1}{2}(\lambda_4 + i\lambda_5) = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \tfrac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so $$V_+ s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = +u.$$

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



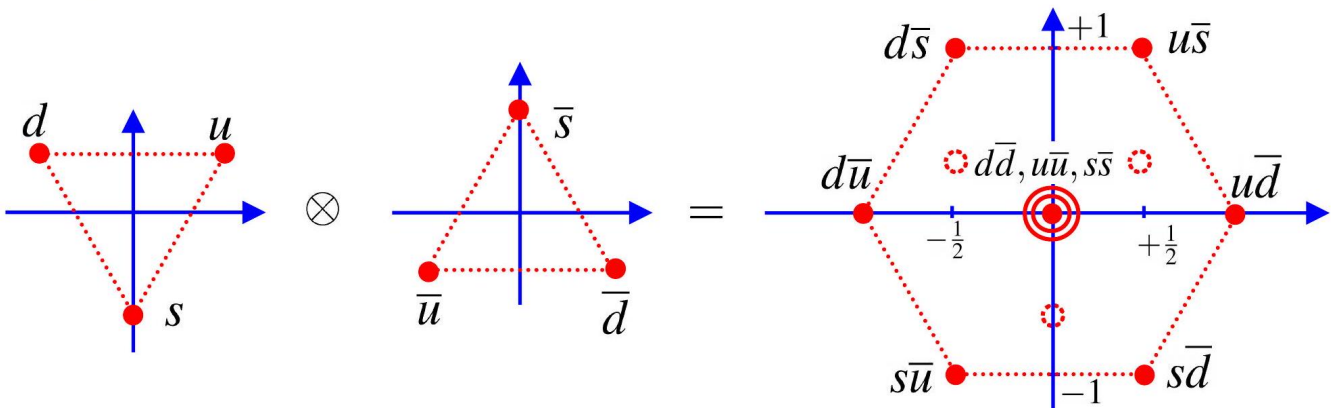
- The actions of all six ladder operators are:

$T_+ d = u;$	$T_- u = d;$	$T_+ \bar{u} = -\bar{d};$	$T_- \bar{d} = -\bar{u}$
$V_+ s = u;$	$V_- u = s;$	$V_+ \bar{u} = -\bar{s};$	$V_- \bar{s} = -\bar{u}$
$U_+ s = d;$	$U_- d = s;$	$U_+ \bar{d} = -\bar{s};$	$U_- \bar{s} = -\bar{d}$

and all other unlisted actions give zero.

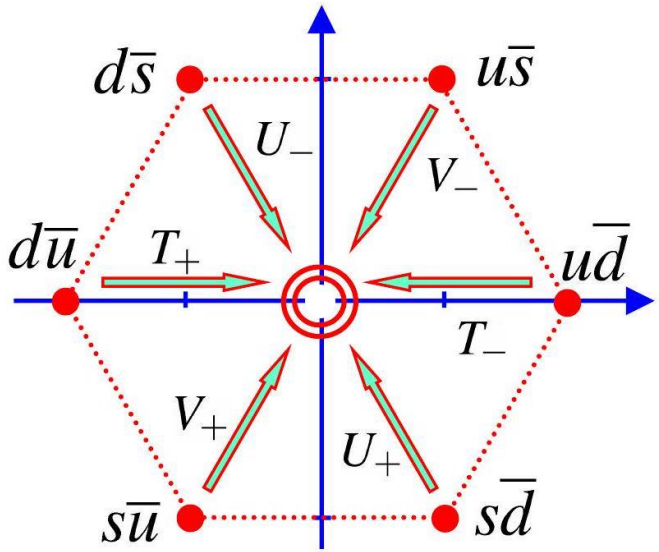
Light (uds) Mesons

- Use ladder operators to construct uds mesons from the nine possible $q\bar{q}$ states



- The central states with $Y = I_3 = 0$ can be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways:

$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \quad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$$
$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \quad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle$$
$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \quad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle$$



- Only two of these six states are linearly independent,
- but there are three original states with $Y = I_3 = 0$.
- Therefore one state is not part of this first multiplet, which is therefore an octet. The ninth state cannot be reached with ladder ops as it is orthogonal.

- First form two linearly independent orthogonal states from:

$$|u\bar{u}\rangle - |d\bar{d}\rangle \quad |u\bar{u}\rangle - |s\bar{s}\rangle \quad |d\bar{d}\rangle - |s\bar{s}\rangle.$$

- If the SU(3) flavour symmetry were exact, the choice of states wouldn't matter. However, $m_S > m_{u,d}$ and the symmetry is only approximate.
- Experimentally observe three light mesons with m 140 MeV: π^+, π^0, π^-
- Identify one central octet state (the π^0) with the isospin triplet derived previously:

$$\boxed{\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})}$$

- The second central octet state can be obtained as a linear combination of the other two states which is orthogonal to the π^0:

$$\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |s\bar{s}\rangle)$$

with orthonormality: $\langle\psi_1 \mid \psi_2\rangle = 0; \quad \langle\psi_2 \mid \psi_2\rangle = 1$

$$\implies \boxed{\psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})}$$

- The remaining central state (**which is not part of the octet!**) is then whatever is orthogonal to ψ_1 and ψ_2:

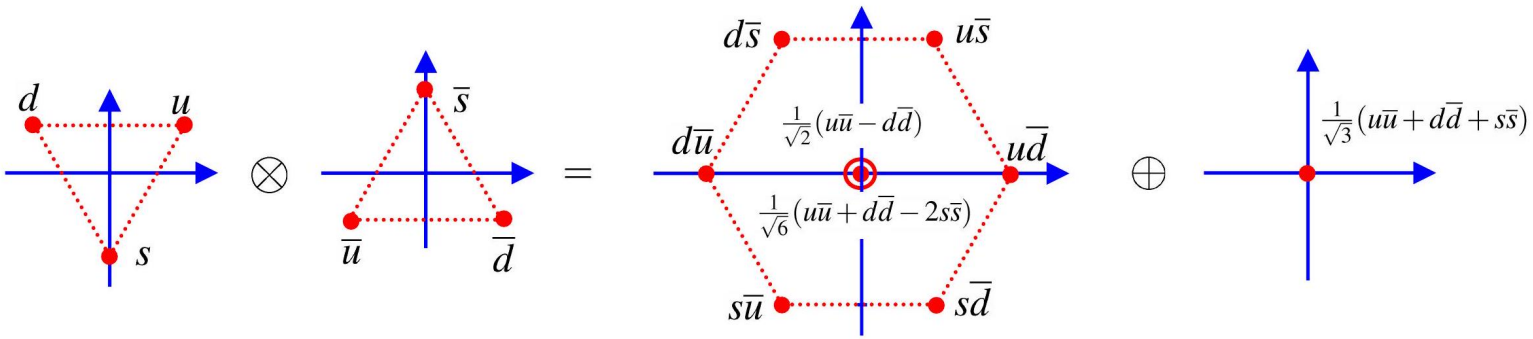
$$\implies \boxed{\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})} \text{ is the SINGLET!}$$

- It is easy to check that

$$T_+\psi_3 = T_-\psi_3 = U_+\psi_3 = U_-\psi_3 = V_+\psi_3 = V_-\psi_3 = 0$$

thereby confirming that $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ is a 'flavourless' SINGLET.

- Therefore the combination of a quark and anti-quark yields nine states which breakdown into an OCTET and a SINGLET:



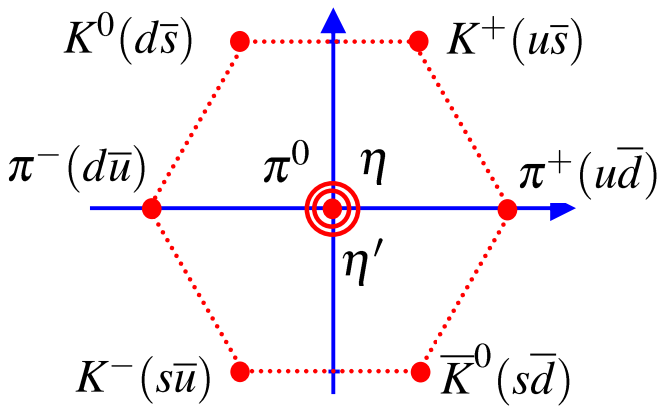
- In the language of group theory: $3 \otimes \bar{3} = 8 \oplus 1$
- Compare with combination of two spin-half particles $2 \otimes 2 = 3 \oplus 1$

$$\text{spin-1 TRIPLET: states: } |1,-1\rangle, |1,0\rangle, |1,+1\rangle,$$
$$\text{spin-0 SINGLET: state: } |0,0\rangle.$$

- These spin triplet states are connected by ladder operators just as the meson uds octet states are connected by $SU(3)$ flavour ladder operators.
- The singlet state carries no angular momentum — in this sense the $SU(3)$ flavour singlet is flavourless.

Meson summary

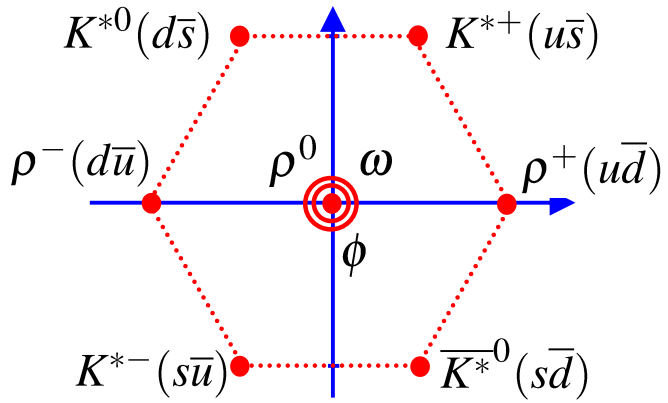
PSEUDOSCALAR MESONS (L=0, S=0, J=0, P= –1)



- Because SU(3) flavour is only approximate the physical states with $I_3 = 0,\ Y = 0$ can be mixtures of the octet and singlet states. Empirically find:

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \leftarrow \text{singlet}$$

VECTOR MESONS (L=0, S=1, J=1, P= –1)



- For the vector mesons the physical states are found to be approximately "ideally mixed":

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

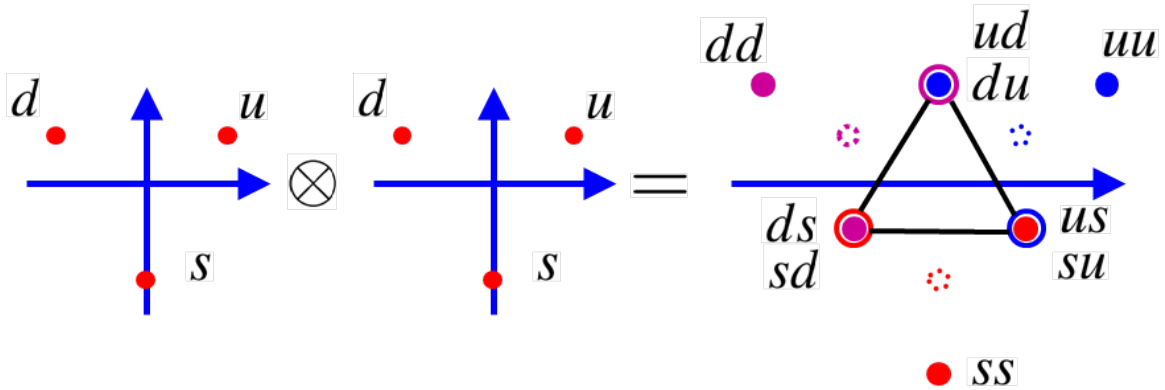
MASSES

$\pi^\pm : 140\,\text{MeV}$	$\pi^0 : 135\,\text{MeV}$
$K^\pm : 494\,\text{MeV}$	$K^0/\overline{K}^0 : 498\,\text{MeV}$
$\eta : 549\,\text{MeV}$	$\eta' : 958\,\text{MeV}$

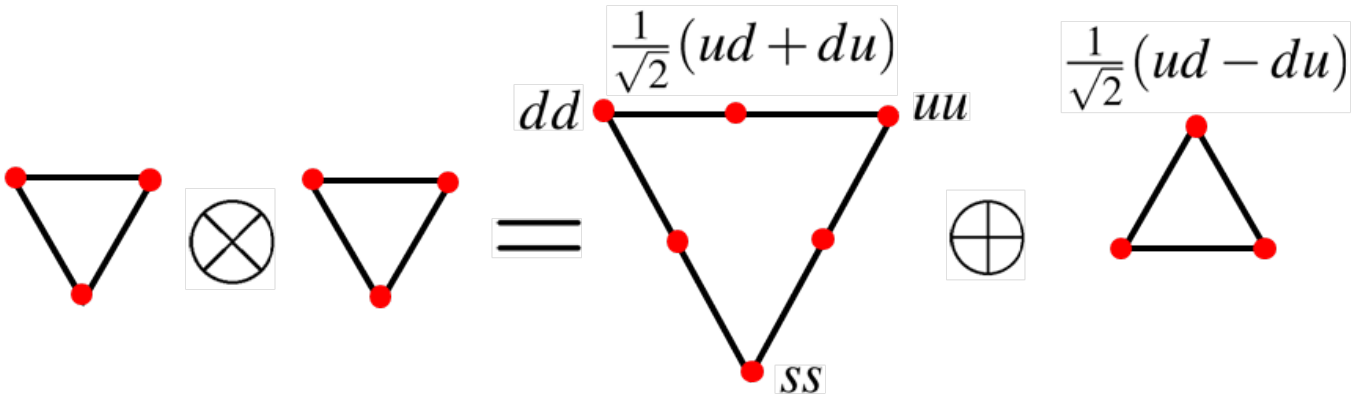
$\rho^\pm : 770\,\text{MeV}$	$\rho^0 : 770\,\text{MeV}$
$K^{*\pm} : 892\,\text{MeV}$	$K^{*0}/\overline{K^*}^0 : 896\,\text{MeV}$
$\omega : 782\,\text{MeV}$	$\phi : 1020\,\text{MeV}$

Combining *uds* quarks to form Baryons

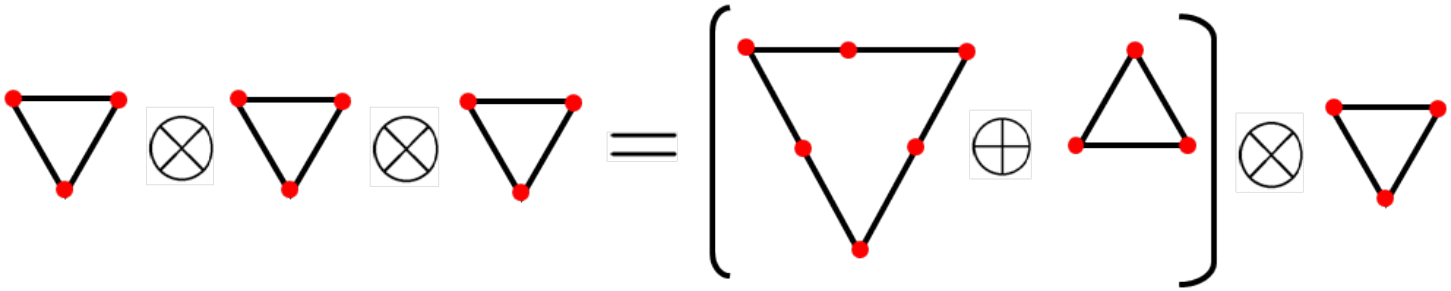
- Have already seen that constructing Baryon states is a fairly tedious process when we derived the proton wave-function. Concentrate on multiplet structure rather than deriving all the wave-functions.
- Everything we do here is relevant to the treatment of colour
- First combine two quarks:



- Yields a SYMMETRIC SEXTET and ANTI-SYMMETRIC TRIPLET:
  $3 \otimes 3 = 6 \oplus \overline{3}$:

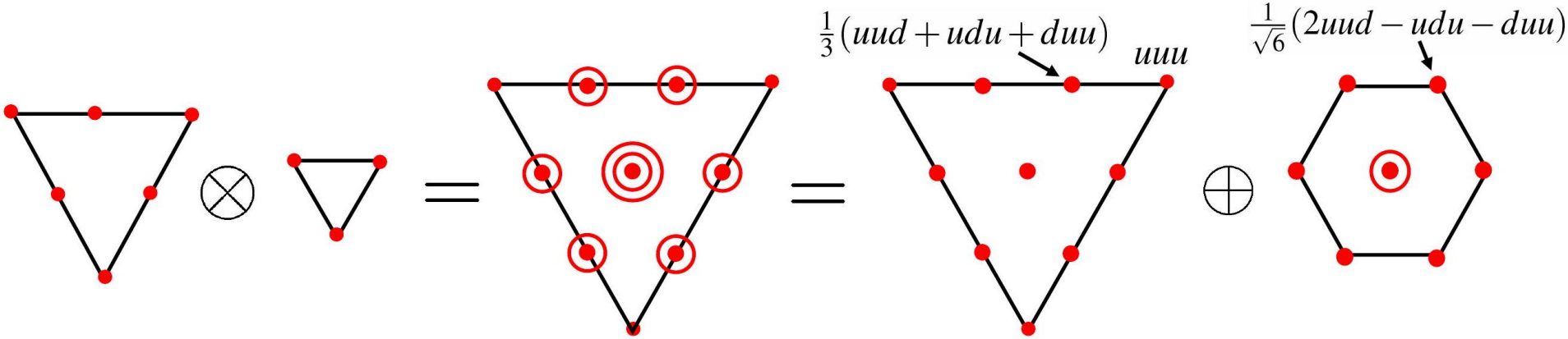


- Now add the third quark:



- Best considered in two parts, building on the sextet and triplet. Again concentrate on the multiplet structure (for the wave-functions refer to the discussion of proton wave-function).

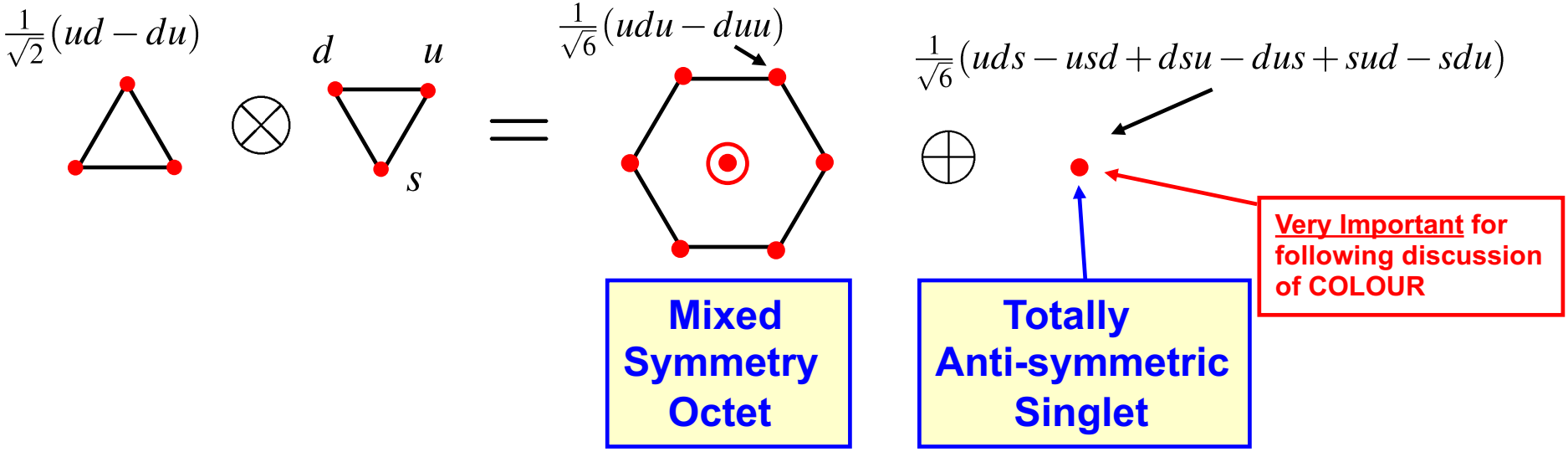
(1) Building on the sextet: $6 \otimes 3 = 10 \oplus 8$:



Symmetric Decuplet

Mixed Symmetry Octet

(2) Building on the triplet: $\bar{3} \otimes 3 = 8 \oplus 1$:



- We also saw $\bar{3} \otimes 3 = 8 \oplus 1$ when we combind quarks with antiquarks to get mesons.
- Can verify the wave-function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$ is a singlet by using ladder operators, e.g.

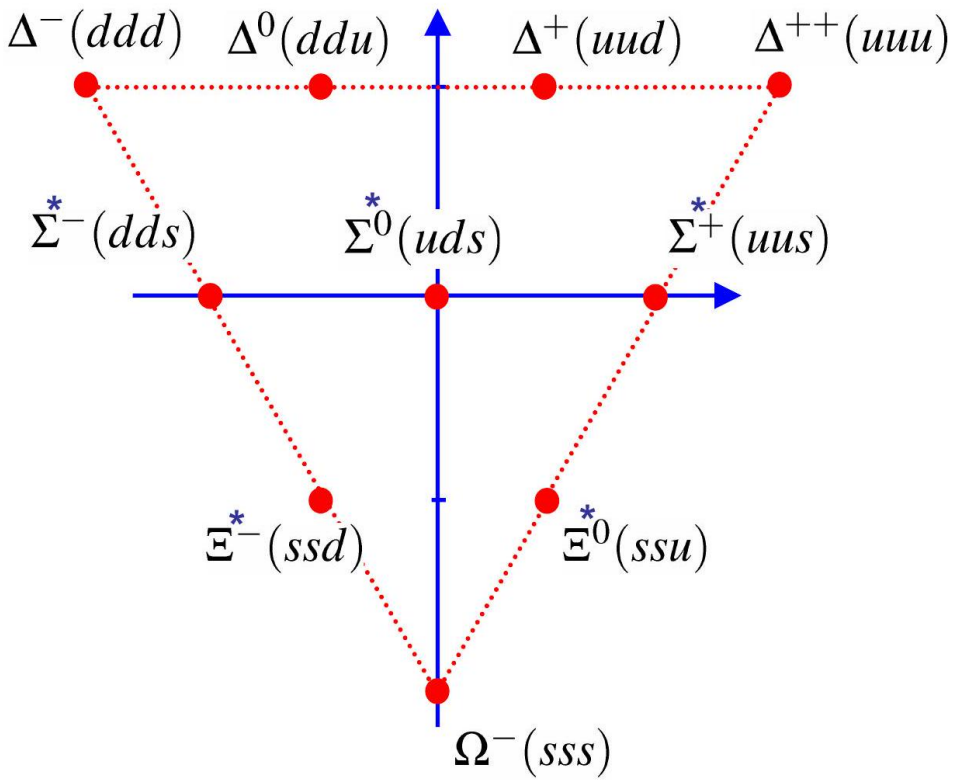
$$T_+ \psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus - usu + usu - uus + suu - suu) = 0.$$

- In summary, the combination of three uds quarks decomposes into

$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1.$$

Baryon Decuplet ($L = 0, S = 3/2, J = 3/2, P = +1$)

- The spin-$\frac{3}{2}$ decuplet is formed from symmetric flavour and symmetric spin wave-functions $\phi_{\text{flavour}}(S)\chi_{\text{spin}}(S)$:



Mass in MeV

$\Delta(1232)$

$\Sigma(1318)$

$\Xi(1384)$

$\Omega(1672)$

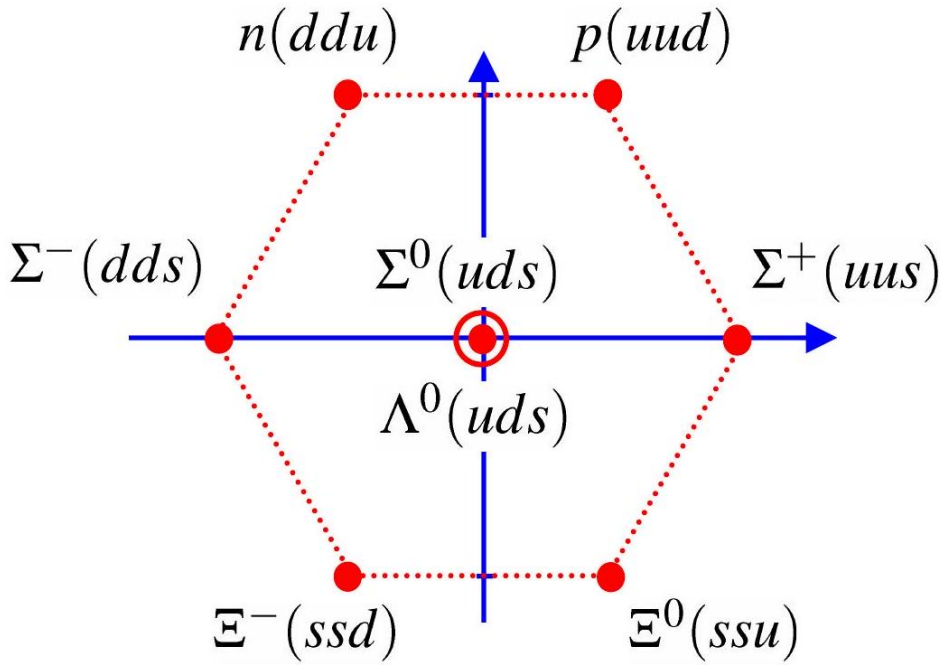
If $SU(3)$-flavour were an exact symmetry then all the masses would be the same. They are not, so $SU(3)$-flavour is a broken symmetry.

Baryon Octet ($L = 0, S = 1/2, J = 1/2, P = +1$)

- The spin $1/2$ octet is formed from mixed symmetry flavour and mixed symmetry spin wave-functions

$$\alpha\phi(M_S)\chi(M_S) + \beta\phi(M_A)\chi(M_A).$$

- Adapt previous discussion concerning proton to obtain wave-functions for all these:

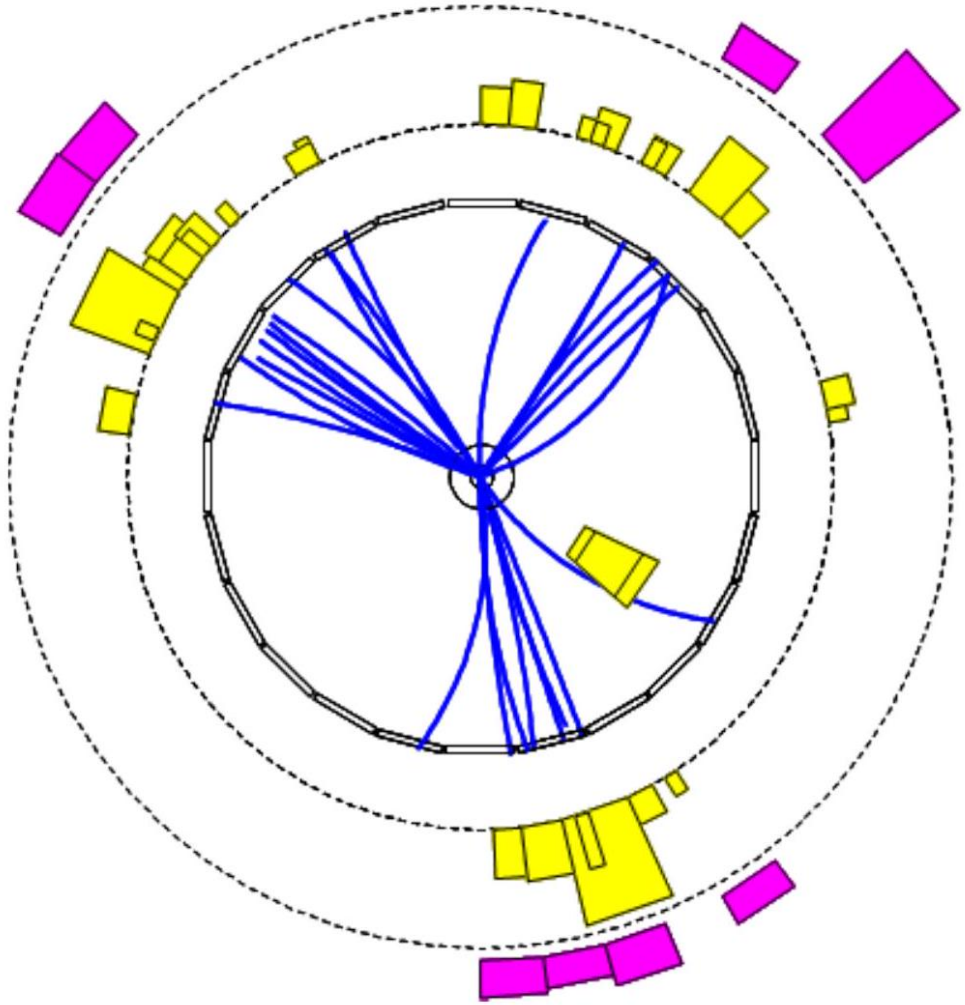


Cannot form a totally symmetric wave-function based on the anti-symmetric flavour singlet as there no totally anti-symmetric spin wave-function for three quarks.

Summary

- We considered $SU(2)$ *ud* and $SU(3)$ *uds* flavour symmetries.
- Although these flavour symmetries are only approximate, they can still be used to explain observed multiplet structure for mesons/baryons.
- $SU(3)$ flavour symmetry results, e.g. predicted wave-functions, should be treated with a pinch of salt as $m_s \neq m_{u/d}$.
- We introduced the idea of singlet states being 'spinless' and/or 'flavourless'
- In the next handout apply these ideas to colour and QCD ...

Prof. Alex Mitov, 2026



H8: Quantum Chromodynamics

The Local Gauge Principle

(see Appendices XI, XII, XV and XVI for more non-examinable details)

- All the interactions between fermions and spin-1 bosons in the SM are specified by the principle of LOCAL GAUGE INVARIANCE.
- To arrive at QED, we **require** physics to be invariant under the local phase transformation of particle wave-functions:

$$\psi \to \psi' = e^{iq\chi(x)}\psi.$$

- Note that the change of phase depends on the space-time coordinate: $\chi(t, \vec{x})$.

Under this transformation the Dirac Equation transforms as

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0} \longrightarrow \boxed{i\gamma^\mu \left(\partial_\mu + iq\partial_\mu \chi\right)\psi - m\psi = 0}.$$

The above is is bad news! We want everything physical (and thus the Dirac equation too) to be invariant under local gauge transformations.

- We are FORCED to introduce a massless gauge boson, A_μ, and the Dirac equation has to be modified to include this new field:

$$\boxed{i\gamma^\mu \left(\partial_\mu + iqA_\mu\right)\psi - m\psi = 0}.$$

- The modified Dirac equation is invariant under local phase transformations if:

$$\boxed{A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi}.$$

The Local Gauge Group ($U(1)$ for QED) specifies the Gauge Boson Interactions and thus the Feynman Rules

Thus the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (in this case the photon) once the gauge group is chosen.

For example, is from the iqA_μ term (which was created to keep everything gauge invariant) that the $ie\gamma^\mu$ vertex factor of QED's Feynman rules can be derived.

The local phase transformation of QED is a unitary $U(1)$ transformation:

$$\psi \to \psi' = \hat{U}\psi \quad \text{i.e.} \quad \psi \to \psi' = e^{iq\chi(x)}\psi \quad \text{with} \quad U^\dagger U = 1$$

We will now extend this idea ...

From QED ($U(1)$) to QCD ($SU(3)$-colour)

- Suppose there is another fundamental symmetry of the universe, say 'invariance under SU(3) local phase transformations'
- i.e. require invariance under $\boxed{\psi \to \psi' = e^{ig\vec{\lambda}\cdot\vec{\theta}(x)}\psi}$ where $\vec{\lambda}$ are the eight 3×3 Gell-Mann matrices introduced in Handout 7, and where $\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time.
- **Unavoidably**, the wave function is now a vector in colour space: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$.

The QCD Lagrangian is created by requiring invariance under local $SU(3)$ gauge tfm.

From that Lagrangian, the Feynman Rules and all other properties may be derived:

- the interaction vertex is: $-\frac{1}{2} i g_s \lambda^a \gamma^\mu$,
- there are 8 massless gauge bosons – the gluons – one for each λ, and
- there are 3 and 4 gluon vertices but no others. (The details are beyond the level of this course. See Gauge Field Theory course in Lent!)

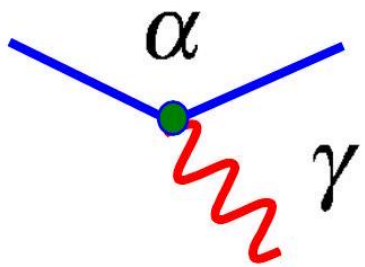
ASIDE: Our $SU(3)$-colour gauge tfm rotates states in colour space about and angle and an axis which is different at every space-time point. Why might this be desirable in a theory with multiple observers?

Colour in QCD

The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with three conserved 'colour' charges.

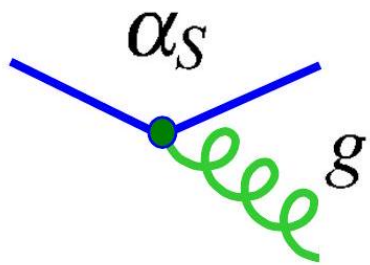
In QED:

- the electron carries one unit of charge $-e$,
- the anti-electron carries one unit of anti-charge $+e$,
- the force is mediated by a massless "gauge boson" – the photon.



In QCD:

- quarks carry colour charge: r, g, b,
- anti-quarks carry anti-charge: $\bar{r}, \bar{g}, \bar{b}$,
- The force is mediated by massless gluons.



The strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b; r \leftrightarrow g; b \longleftrightarrow g.$$

$SU(3)$-colour symmetry is exact

This $SU(3)$-colour symmstry is an exact symmetry, unlike the approximate *uds* $SU(3)$-flavour symmetry discussed previously.

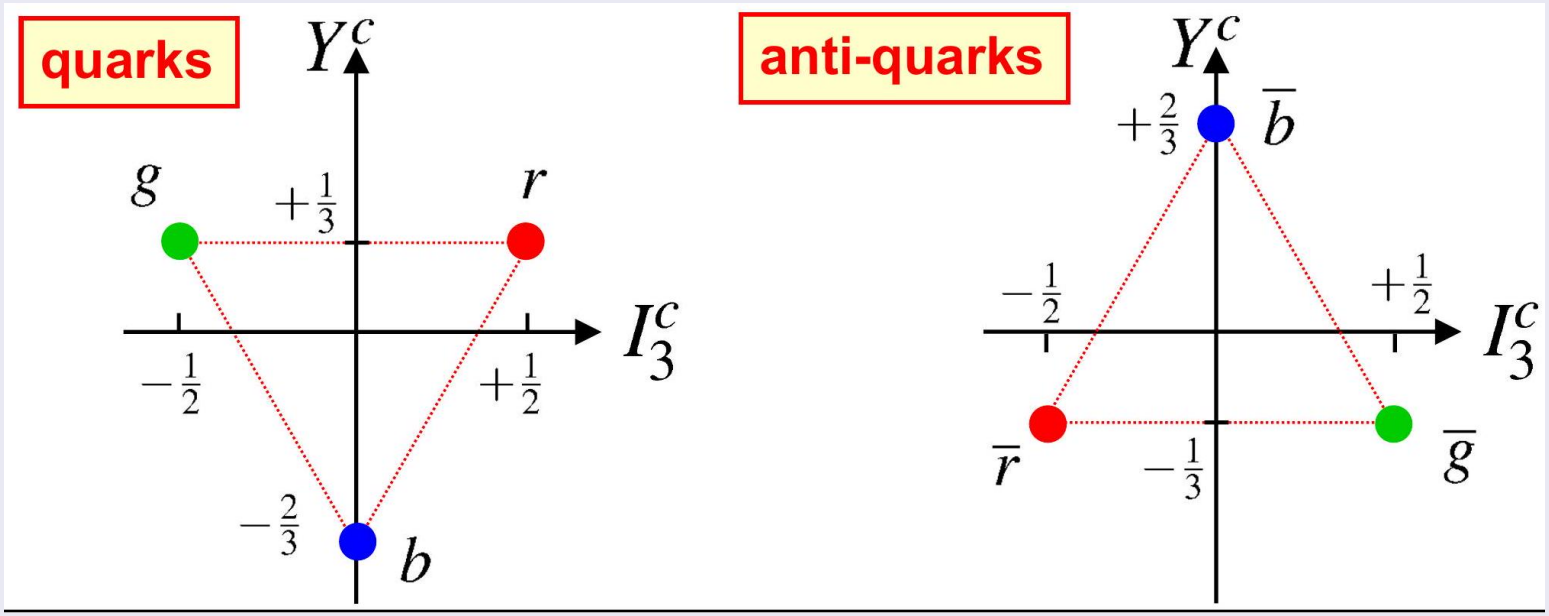
Represent $SU(3)$ colour states r, g, b by:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Analogous to labelling u, d, s flavour states by I_3 and Y, Colour states can be labelled by two quantum numbers:

- I_3^c colour isospin, and
- Y^c colour hypercharge.

Each quark (anti-quark) can have the following colour quantum numbers:



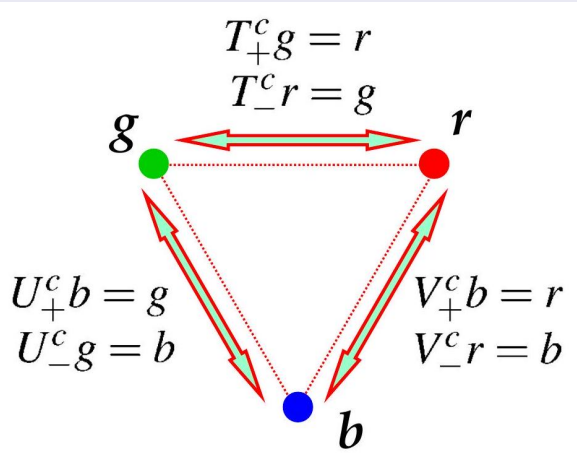
The Colour Confinement Hypothesis

The Colour Confinement Hypothesis is that 'all free particles have colour singlet wavefunctions'.

- Sometimes abbreviated to 'all free particles are colourless'.
- It is suspected that one day this hypothesis will be shown to be derivable from the other principles of the Standard Model.
- If true, then we will never observe free quarks.

We can re-interpret our $SU(3)$ flavour results via an $SU(3)$ colour lens – $(u, d, s) \to (r, g, b)$, but this time we can treat the symmetry as exact rather than approximate!

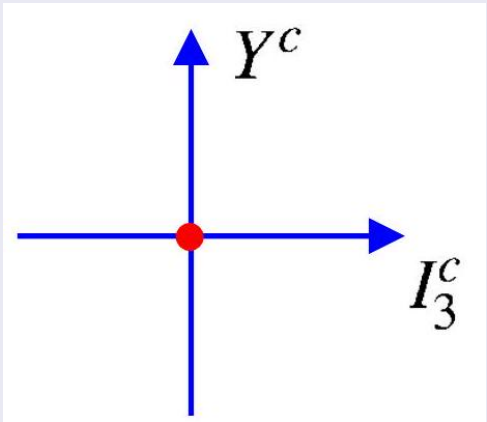
Just as for *uds* flavour symmetry can define colour ladder operators:



Colour Singlets

Reminder of what a colour singlet is:

- colour singlet states have zero colour quantum numbers $I_3^c = 0, Y^c = 0$,
- colour singlet states are invariant under SU(3) colour transformations, and
- all ladder operators $T_\pm, U_\pm, V_\pm$ yield zero when applied to a colour singlet state.

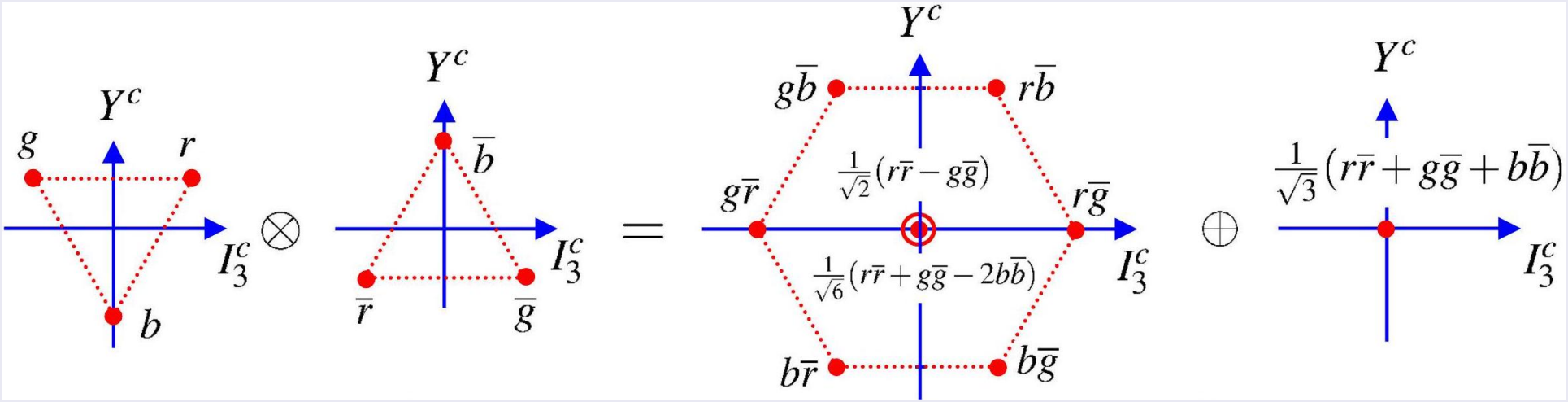


It is not sufficient to have just $I_3^c = 0, Y^c = 0$.

This alone does not signal a colour singlet state.

Meson (i.e. $q\bar{q}$) Colour Wave-functions

The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with *uds* flavour symmetry.



i.e. we get a COLOURED OCTET and a COLOURLESS SINGLET.

- Colour confinement implies that hadrons only exist in colour singlet states so the colour singlet wave-function for mesons is:

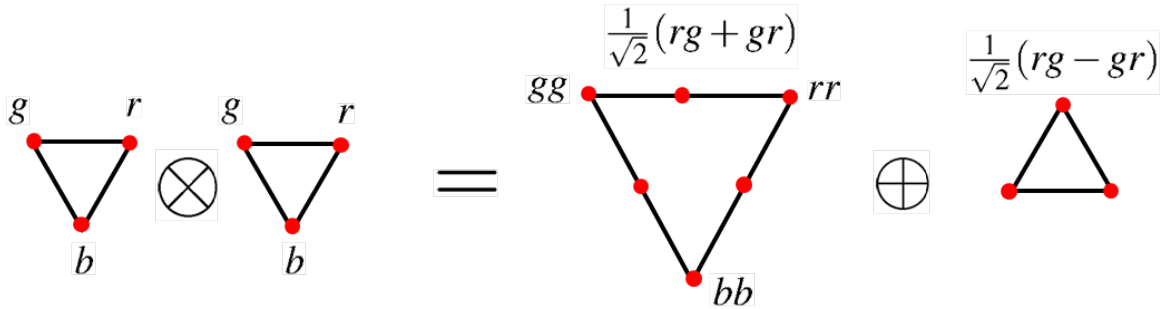
$$\boxed{\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})}.$$

$qq\bar{q}$ bound states do not exist in nature . . .

. . . because there are no $qq\bar{q}$ state with $Y^c = 0; I_3^c = 0$, let alone a colour singlet!

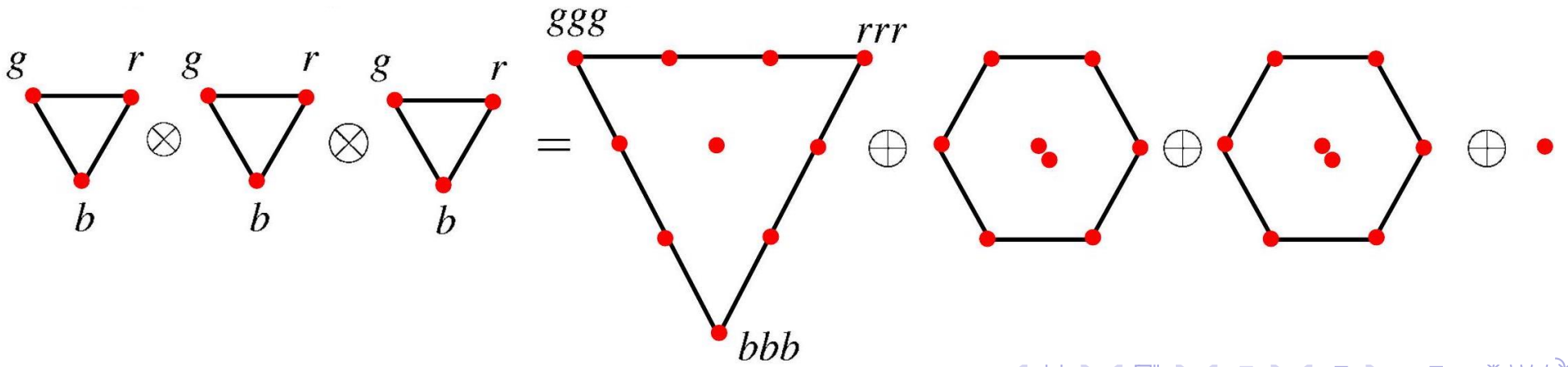
Baryon Colour Wave-Function

- Do qq bound states exist? This is equivalent to asking whether it possible to form a colour SINGLET from two colour TRIPLETs.
- Following the discussion of construction of baryon wave-functions in $SU(3)$-flavour symmetry obtain



- No qq colour SINGLET state!
- Colour confinement $\Longrightarrow$ bound states of qq do not exist!

BUT: combination of three quarks (three colour TRIPLETs) gives a colour SINGLET state (pages 214-216).



The COLOUR SINGLET wave-function is:

$$\boxed{\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)}.$$

Make absolutely sure that this is a COLOUR SINGLET:

- It has $I_3^c = 0, Y^c = 0$: a necessary but not sufficient condition.
- Apply ladder operators, e.g. T_+ (recall $T_+ g = r$)

$$T_+\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0.$$

- Similarly $T_-\psi_c^{qqq} = 0$; $V_\pm\psi_c^{qqq} = 0$; $U_\pm\psi_c^{qqq} = 0$.

ϕ_c^{qqq} definitely is a colourless singlet!

- qqq bound states can exist, and
- The qqq colour singlet wave-function is anti-symmetric.

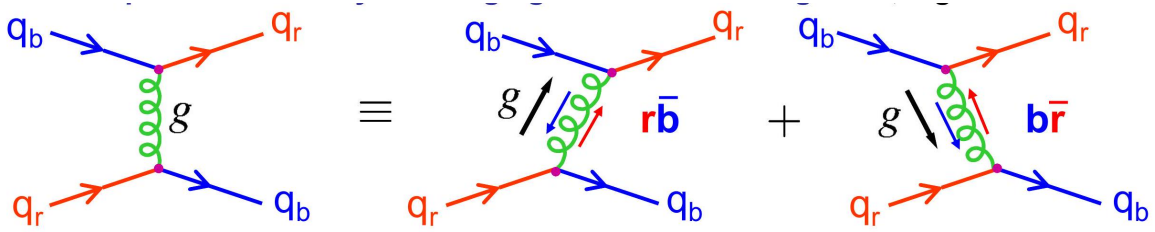
The possible hadrons (i.e. the possible colour singlet states) are therefore:

- $q\bar{q}, qqq$: **Mesons** and **Baryons**
- $q\bar{q}q\bar{q}, qqqq\bar{q}, \ldots$: i.e. **Tetraquarks**, **Pentaquarks**, ...

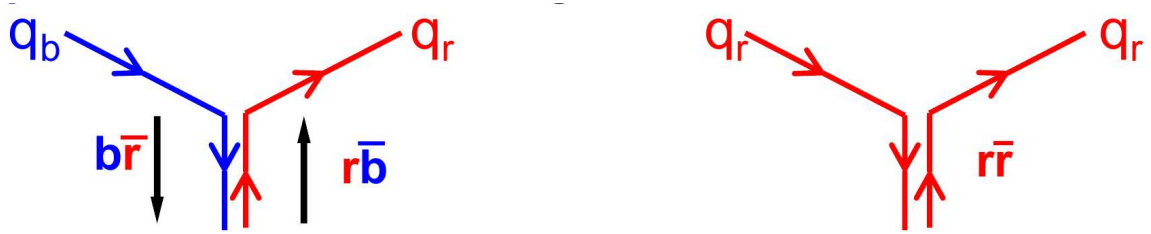
Until 2015, all discovered hadrons had been mesons or baryons. The first pentaquarks were discovered in 2015, 2019 and 2022 by LHCb. Tetraquarks have also since been found.

Gluons

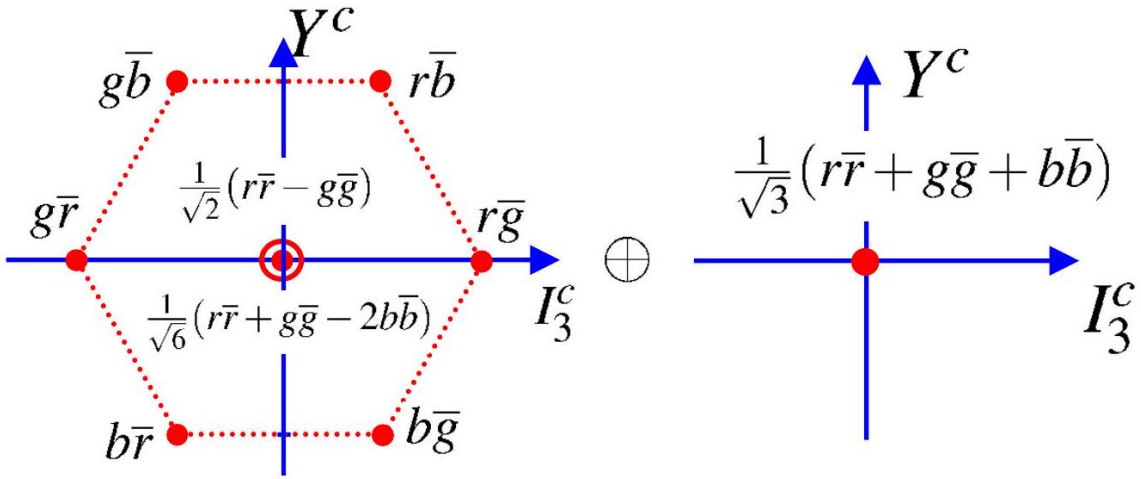
* In QCD quarks interact by exchanging virtual massless gluons, e.g.:



- Gluons carry colour and anti-colour, e.g.:



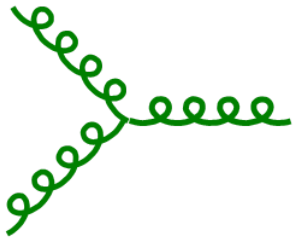
$\star$ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour) where we saw an OCTET and a COLOURLESS SINGLET:

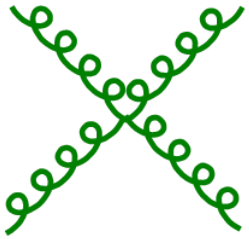


Gluon-Gluon Interactions

- In QED the photon does not carry the charge of the EM interaction (photons are electrically neutral).
- In contrast, in QCD the gluons do carry colour charge $\Longrightarrow$ Gluon Self-Interactions!
- Two new vertices (no QED analogues):

triple-gluon vertex

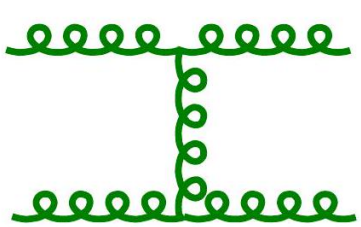


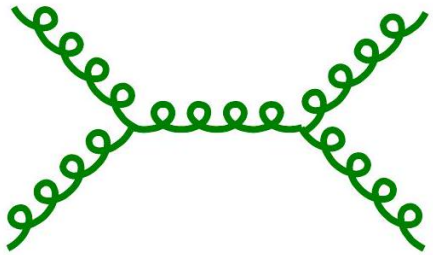


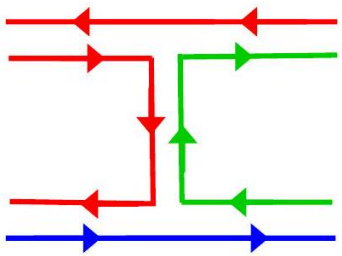
quartic-gluon vertex

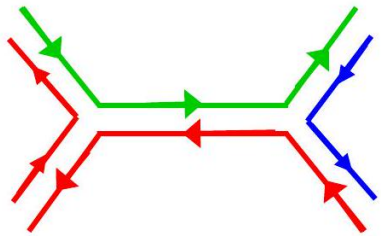
- In addition to quark-quark scattering, therefore can have gluon-gluon scattering:

e.g. possible way of arranging the colour flow:



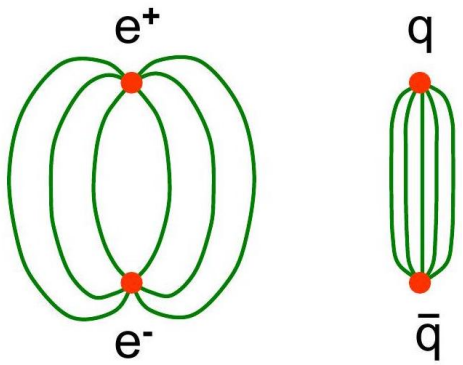




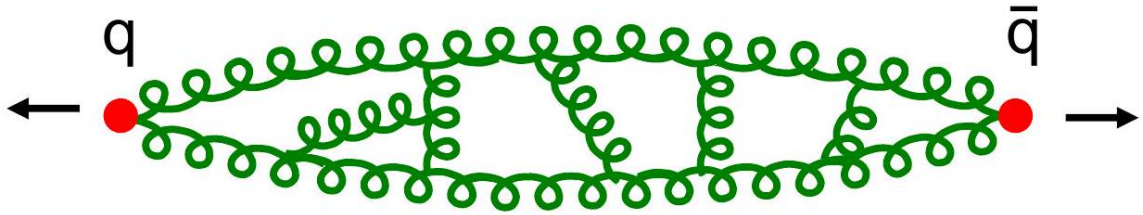


Gluon self-Interactions and Confinement

- Gluon self-interactions are believed to give rise to colour confinement.
- Unlike QED, in QCD gluon self-interactions squeeze lines of force into a 'flux tube'.



- What happens when try to separate two coloured objects e.g. $q\bar{q}$?



- Form a flux tube of interacting gluons of approximately constant energy density $\lambda \sim 1\mathrm{GeV/fm} \sim 160$ kN.

$$\Rightarrow V(r) \sim \lambda r$$

- Require infinite energy to separate coloured objects to infinity.
- Coloured quarks and gluons are always confined within colourless states.
- In this way QCD provides a plausible explanation of confinement - but not yet proven (although there has been recent progress with Lattice QCD).

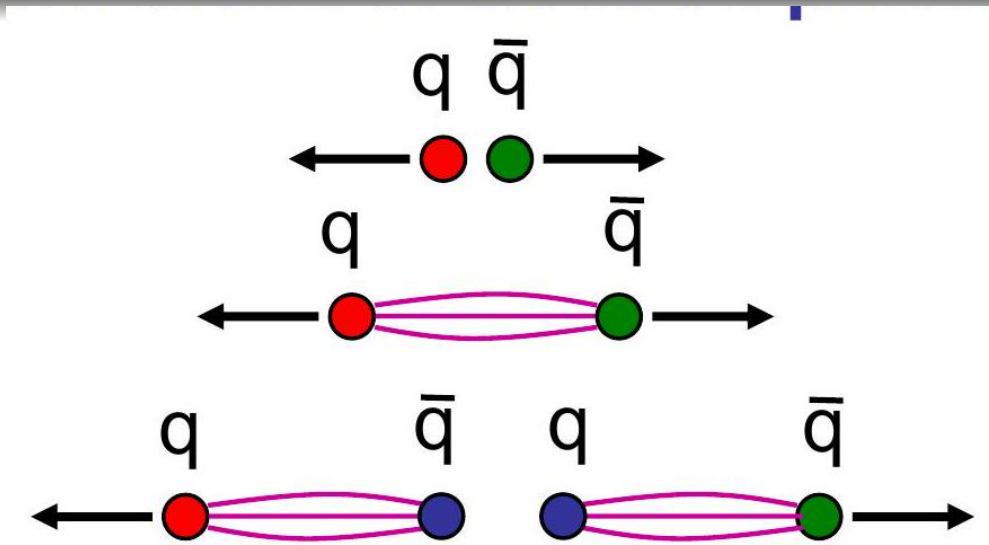
Hadronisation and Jets

Consider a quark and anti-quark produced in electron positron annihilation:

i) Initially Quarks separate at high velocity

ii) Colour flux tube forms between quarks

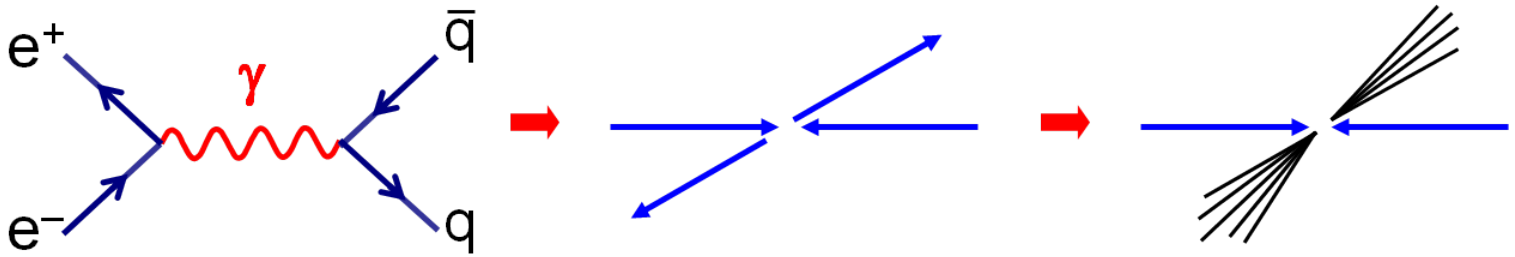
iii) Energy stored in the flux tube sufficient to produce $q\bar{q}$ pairs



iv) Process continues until quarks pair up into jets of colourless hadrons

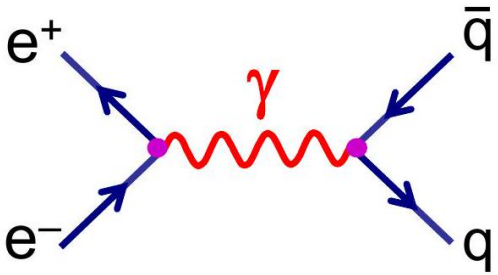


- This process is called hadronisation. It is not (yet) calculable (first principles).
- The main consequence is that at collider experiments quarks and gluons observed as jets of particles.



QCD and Colour in e^+e^- Collisions

e^+e^--colliders are an excellent place to study QCD:



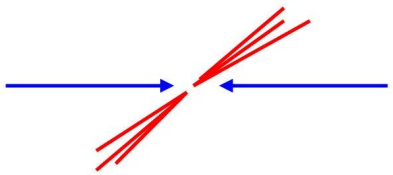
- QED process well-understood.
- No need to know parton structure functions.
- Experimentally very clean - no proton remnants.

⋆ In Handout 5 obtained expressions for the $e^+e^- \to \mu^+\mu^-$ cross-section:

$$\sigma = \frac{4\pi\alpha^2}{3s} \qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

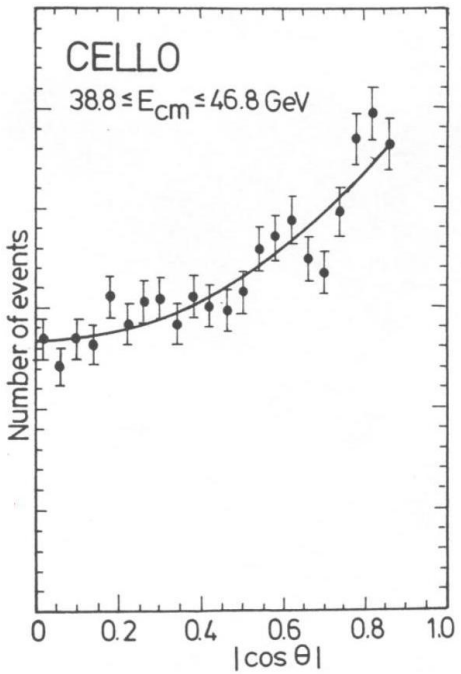
- In e^+e^- collisions produce all quark flavours for which $\sqrt{s} > 2m_q$
- In general, i.e. unless producing a $q\bar{q}$ bound state, produce jets of hadrons.

  Usually can't tell which jet came from the quark and came from anti-quark.



- Angular distribution of jets $\propto \left(1+\cos^2\theta\right) \implies$

  Quarks are spin-$\frac{1}{2}$.



H.J.Behrend et al., Phys Lett 183B (1987) 400

- Colour is conserved and quarks are produced as $r\bar{r}, g\bar{g}, b\bar{b}$.
- For a single quark flavour and single colour:

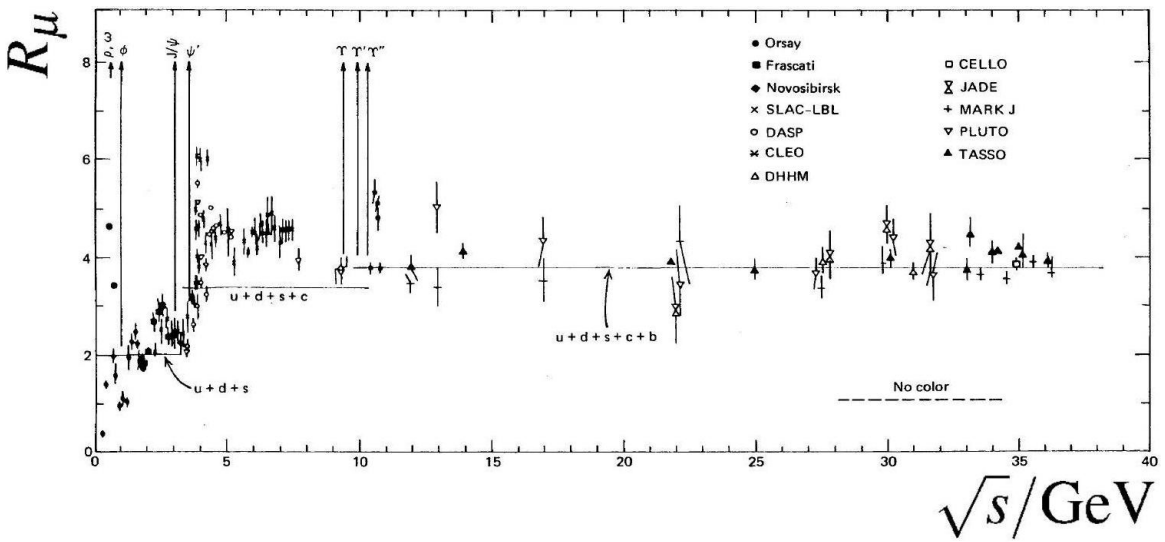
$$\sigma\left(e^+e^- \to q_i\bar{q}_i\right) = \frac{4\pi\alpha^2}{3s}Q_q^2.$$

- Experimentally observe jets of hadrons: (factor 3 is from colour!!)

$$\sigma\left(e^+e^- \to \text{ hadrons }\right) = 3\sum_{u,d,s,\ldots}\frac{4\pi\alpha^2}{3s}Q_q^2.$$

- Usual to express as ratio compared to $\sigma\left(e^+e^- \to \mu^+\mu^-\right)$:

$$R_\mu = \frac{\sigma\left(e^+e^- \to \text{ hadrons }\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\sum_{u,d,s,\ldots}Q_q^2$$



u,d,s: $R_\mu = 3 \times \left(\frac{1}{9} + \frac{4}{9} + \frac{1}{9}\right) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

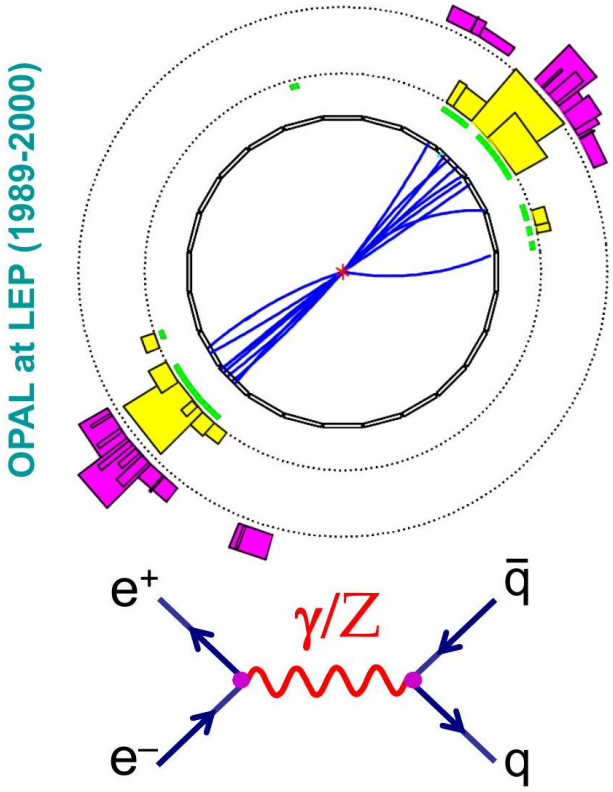
u,d,s,c,b: $R_\mu = \frac{11}{3}$

- Data are consistent with expectation provided there is a factor 3 from colour.

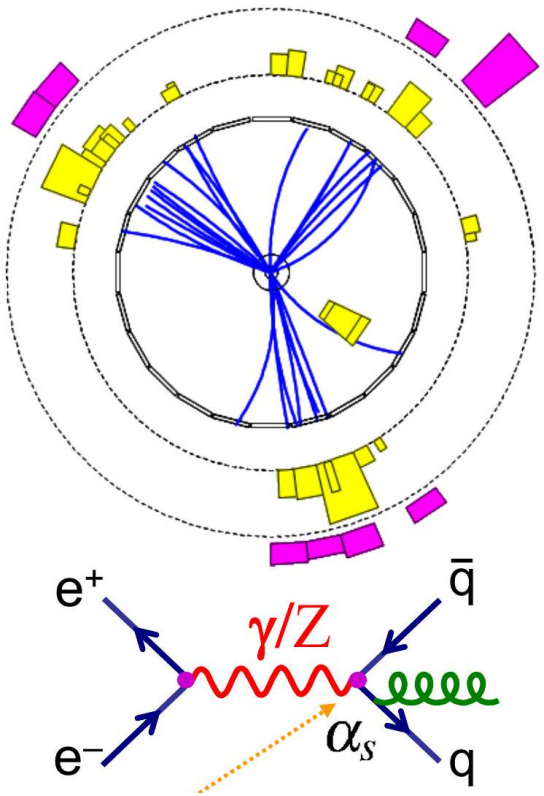
Jet production in e+e-Collisions

e^+e^--colliders are also a good place to study gluons.

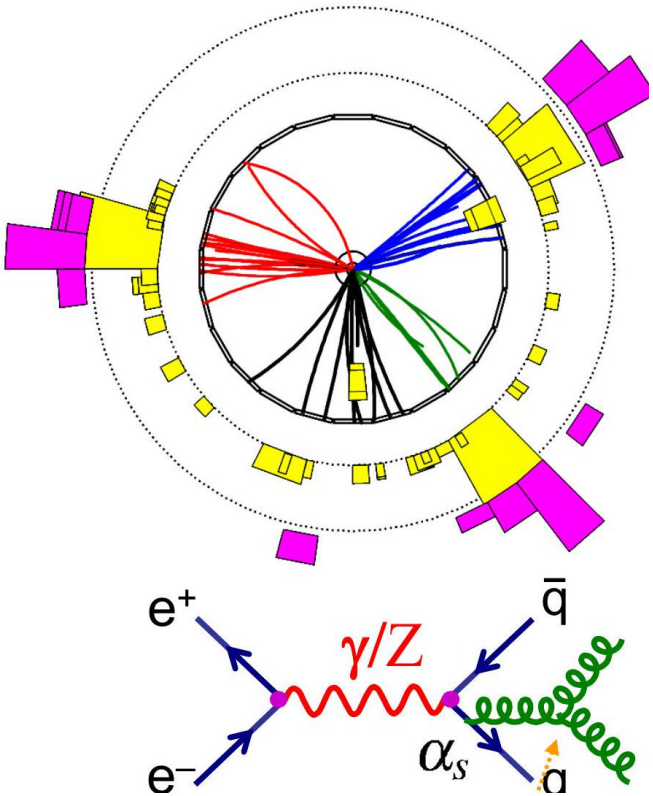
$e^+e^- \to q\bar{q} \to$ 2jets



$e^+e^- \to q\bar{q}g \to$ 3jets



$e^+e^- \to q\bar{q}gg \to$ 4jets



Experimentally:

- Three jet rate $\to$ measurement of α_s.
- Angular distributions $\Rightarrow$ gluons are spin-1.
- Four-jet rate and distributions $\to$ QCD has an underlying $SU(3)$ symmetry.

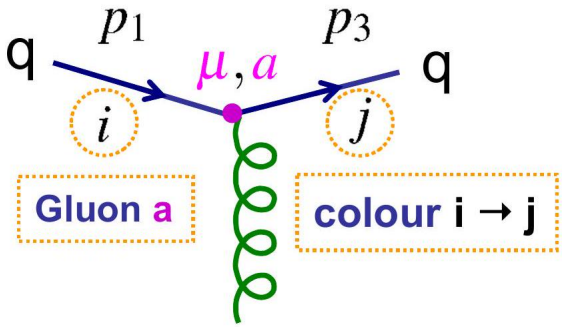
The Quark-Gluon Interaction

- The colour part of the fermion wave-functions are represented by:

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$.
- The QCD qqg vertex is written:

$$\bar{u}(p_3)\, c_j^\dagger \left\{ -\tfrac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_i\, u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3×3 SU(3) Gell-Mann matrices.
- Isolating the colour part:

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix} \lambda^a_{1i} \\ \lambda^a_{2i} \\ \lambda^a_{3i} \end{pmatrix} = \lambda^a_{ji}$$

- Hence the fundamental quark-gluon QCD interaction can be written:

$$\bar{u}(p_3)\, c_j^\dagger \left\{ -\tfrac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_i\, u(p_1) \equiv \boxed{\bar{u}(p_3) \left\{ -\tfrac{1}{2} i g_s \lambda^a_{ji} \gamma^\mu \right\} u(p_1)}.$$

Feynman Rules for QCD

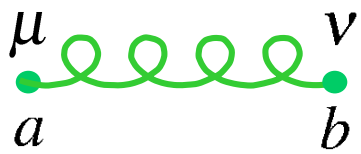
- **External Lines**

spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\bar{u}(p)$	
	incoming anti-quark	$\bar{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\varepsilon^{\mu}(p)$	
	outgoing gluon	$\varepsilon^{\mu}(p)^*$	

- **Internal Lines (propagators)**

spin 1 gluon

$$\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}$$

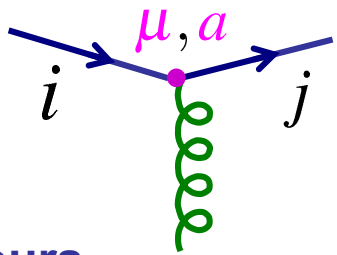


a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$-ig_s\tfrac{1}{2}\lambda^a_{ji}\gamma^{\mu}$$

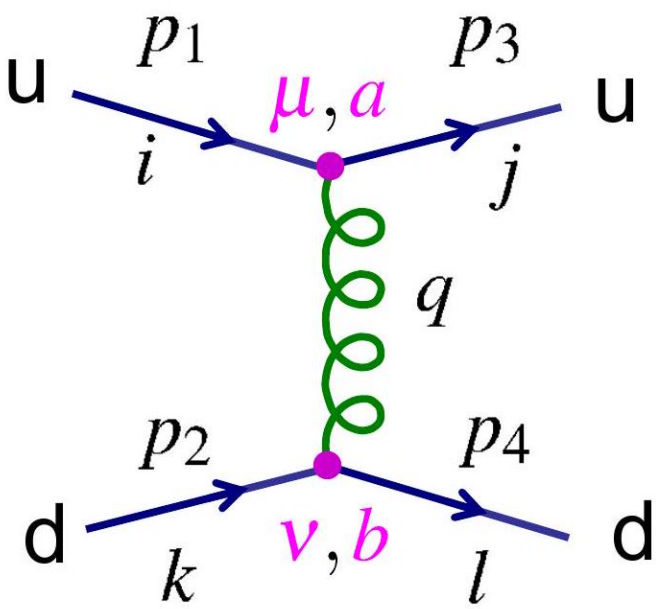


i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ **= product of all factors**

Matrix Element for *ud* quark-quark scattering



- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1, 2, 3\}$ (or $\{r, g, b\}$)
- In terms of colour this scattering is

$$ik \to jl.$$

- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, \ldots, 8$.
- NOTE: the Kronecker δ in the propagator ensures $a = b$, i.e. the gluon "emitted" at a is the same as that "absorbed" at b.

- Applying the Feynman rules:

$$-iM = \left[\bar{u}_u(p_3)\left\{-\frac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\right\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\bar{u}_d(p_4)\left\{-\frac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\right\}u_d(p_2)\right]$$

where summation over a and b (and μ and ν) is implied.

- Summing over a and b using the Kronecker δ gives:

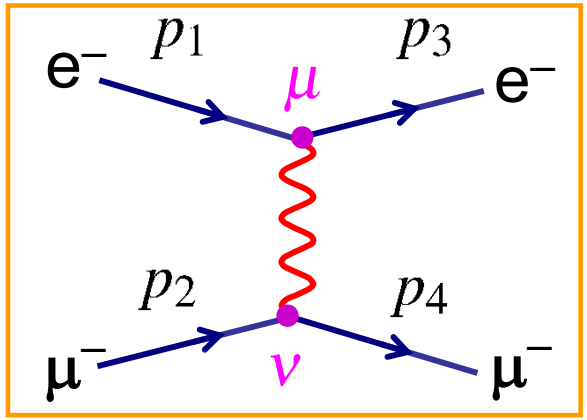
$$\boxed{M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}\left[\bar{u}_u(p_3)\gamma^\mu u_u(p_1)\right]\left[\bar{u}_d(p_4)\gamma^\nu u_d(p_2)\right].}$$

QCD vs QED

QED

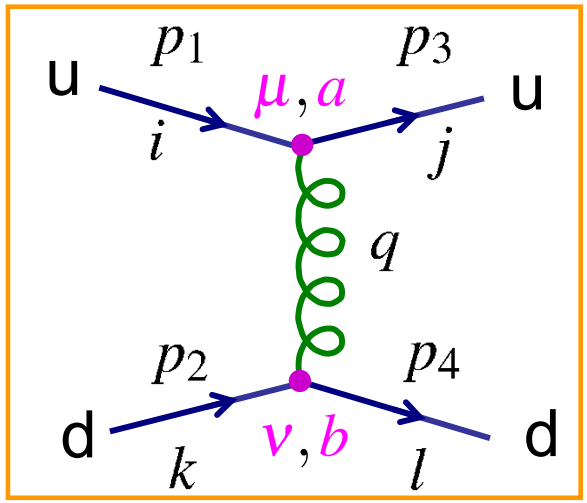
$$-iM = [\bar{u}(p_3) ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_4) ie\gamma^\nu u(p_2)]$$

$$M = -e^2 \frac{1}{q^2} g_{\mu\nu} [\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma^\nu u(p_2)]$$



QCD

$$M = -\frac{g_s^2}{4} \lambda^a_{ji} \lambda^a_{lk} \frac{1}{q^2} g_{\mu\nu} [\bar{u}_u(p_3)\gamma^\mu u_u(p_1)][\bar{u}_d(p_4)\gamma^\nu u_d(p_2)]$$



★ **QCD Matrix Element = QED Matrix Element with:**

- $e^2 \to g_s^2$ or equivalently $\alpha = \frac{e^2}{4\pi} \to \alpha_s = \frac{g_s^2}{4\pi}$

✚ **QCD Matrix Element includes an additional "colour factor"**

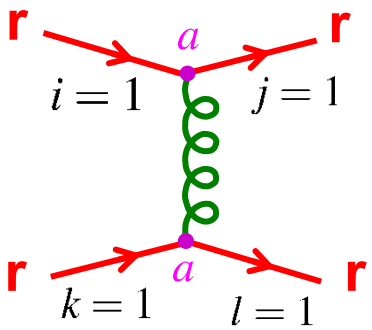
$$C(ik \to jl) \equiv \frac{1}{4} \sum_{a=1}^{8} \lambda^a_{ji} \lambda^a_{lk}$$

Evaluation of QCD Colour Factors

- **QCD colour factors reflect the gluon states that are involved**

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}, g\bar{r}$	$r\bar{b}, b\bar{r}$	$g\bar{b}, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

❶ Configurations involving a single colour



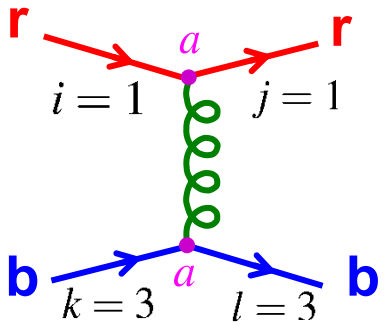
- **Only matrices with non-zero entries in 11 position are involved**

$$C(rr \to rr) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11})$$

$$= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}$$

Similarly find $C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}$

❷ Other configurations where quarks don't change colour e.g. $rb \to rb$



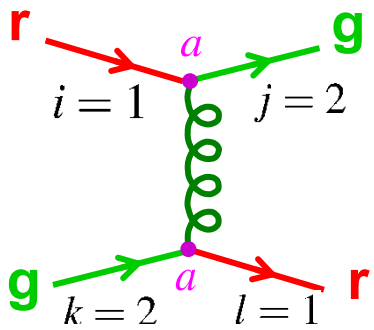
- Only matrices with non-zero entries in **11** and **33** position are involved

$$C(rb \to rb) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{11}^{a}\lambda_{33}^{a} = \frac{1}{4}(\lambda_{11}^{8}\lambda_{33}^{8})$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}.\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb \to rb) = C(rg \to rg) = C(gr \to gr) = C(gb \to gb) = C(br \to br) = C(bg \to bg) = -\frac{1}{6}$

❸ Configurations where quarks swap colours e.g. $rg \to gr$



- Only matrices with non-zero entries in **12** and **21** position are involved

Gluons $r\bar{g}$, $g\bar{r}$

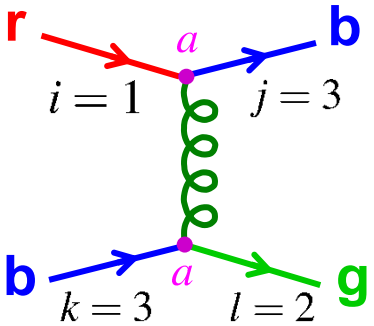
$$C(rg \to gr) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{21}^{a}\lambda_{12}^{a} = \frac{1}{4}(\lambda_{21}^{1}\lambda_{12}^{1} + \lambda_{21}^{2}\lambda_{12}^{2})$$

$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}_{+}^{(ij)}\hat{T}_{-}^{(kl)}$

$$C(rb \to br) = C(rg \to gr) = C(gr \to rg) = C(gb \to bg) = C(br \to rb) = C(bg \to gb) = \frac{1}{2}$$

❹ Configurations involving 3 colours e.g. $rb \to bg$



- Only matrices with non-zero entries in the **13** and **32** position
- But none of the λ matrices have non-zero entries in the **13** and **32** positions. Hence the colour factor is zero

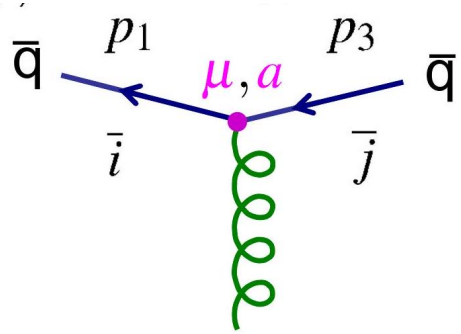
★ **colour is conserved**

Colour Factors : Quarks vs Anti-Quarks

- The QCD qqg vertex was written

$$\bar{u}(p_3)\, c_j^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_i\, u(p_1)$$

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



- Now consider the anti-quark vertex $\bar{q}\bar{q}g$:

$$\bar{v}(p_1)\, c_i^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_j\, v(p_3)$$

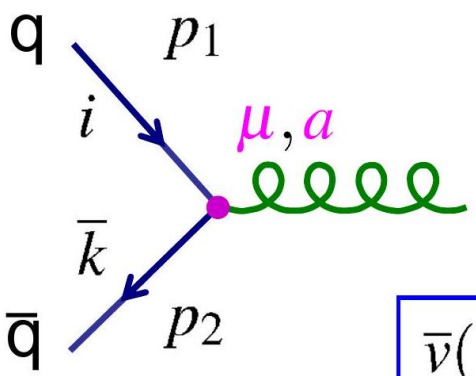
Note that the incoming anti-particle now enters on the LHS of the expression!

The colour part is $c_i^\dagger \lambda^a c_j = c_i^\dagger \begin{pmatrix} \lambda^a_{1j} \\ \lambda^a_{2j} \\ \lambda^a_{3j} \end{pmatrix} = \lambda^a_{ij}$ i.e. indices ij are swapped with respect to the quark case.

- Hence $\boxed{\bar{v}(p_1)\, c_i^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_j\, v(p_3) \equiv \bar{v}(p_1) \left\{ -\frac{1}{2} i g_s \lambda^a_{ij} \gamma^\mu \right\} v(p_3)}$.
- Compare with the quark-gluon QCD interaction:

$$\bar{u}(p_3)\, c_j^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_i\, u(p_1) \equiv \bar{u}(p_3) \left\{ -\frac{1}{2} i g_s \lambda^a_{ji} \gamma^\mu \right\} u(p_1)$$

Finally we can consider the quark - anti-quark annihilation vertex:

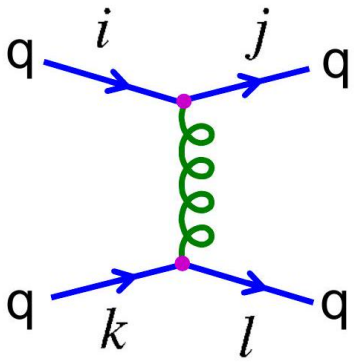


QCD vertex: $\bar{v}(p_2)c_k^\dagger\{-\frac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

with $c_k^\dagger\lambda^a c_i = \lambda^a_{ki}$

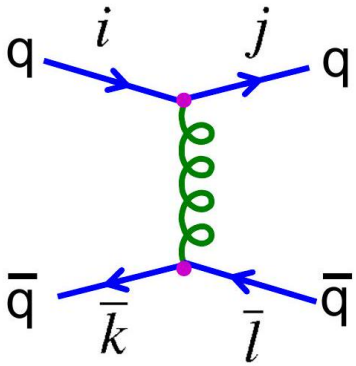
$$\bar{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)$$

- Consequently the colour factors for the different diagrams are:



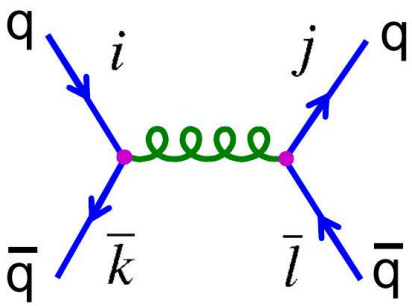
$$C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}$$

$$\begin{aligned} C(rr \to rr) &= \tfrac{1}{3} \\ \text{e.g. } C(rg \to rg) &= -\tfrac{1}{6} \\ C(rg \to gr) &= \tfrac{1}{2} \end{aligned}$$



$$C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}$$

$$\begin{aligned} C(r\bar{r} \to r\bar{r}) &= \tfrac{1}{3} \\ \text{e.g. } C(r\bar{g} \to r\bar{g}) &= -\tfrac{1}{6} \\ C(r\bar{r} \to g\bar{g}) &= \tfrac{1}{2} \end{aligned}$$



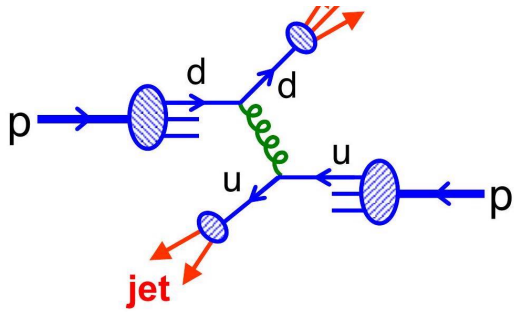
$$C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}$$

$$\begin{aligned} C(r\bar{r} \to r\bar{r}) &= \tfrac{1}{3} \\ \text{e.g. } C(r\bar{g} \to r\bar{g}) &= \tfrac{1}{2} \\ C(r\bar{r} \to g\bar{g}) &= -\tfrac{1}{6} \end{aligned}$$

Colour index of adjoint spinor comes first.

Quark-Quark Scattering

- Consider the process $u + d \to u + d$ which can occur in the high energy proton-proton scattering
- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states



$$\left\langle |M_{fi}|^2 \right\rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^{3} |M_{fi}(ij \to kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\left\langle |C|^2 \right\rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |C(ij \to kl)|^2$$

- For $qq \to qq$

$$\left\langle |C|^2 \right\rangle = \frac{1}{9} \left[3 \times \underbrace{\left(\frac{1}{3}\right)^2}_{rr \to rr, \ldots} + 6 \times \underbrace{\left(-\frac{1}{6}\right)^2}_{rb \to rb, \ldots} + 6 \times \underbrace{\left(\frac{1}{2}\right)^2}_{rb \to br, \ldots} \right] = \frac{2}{9}$$

• Previously derived the Lorentz Invariant cross section for $e^- \mu^- \to e^- \mu^-$ elastic scattering in the ultra-relativistic limit (Handout 6).

QED

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s}\right)^2\right]$$

• For ud $\to$ ud in QCD replace $\alpha \to \alpha_s$ and multiply by $\langle |C|^2 \rangle$

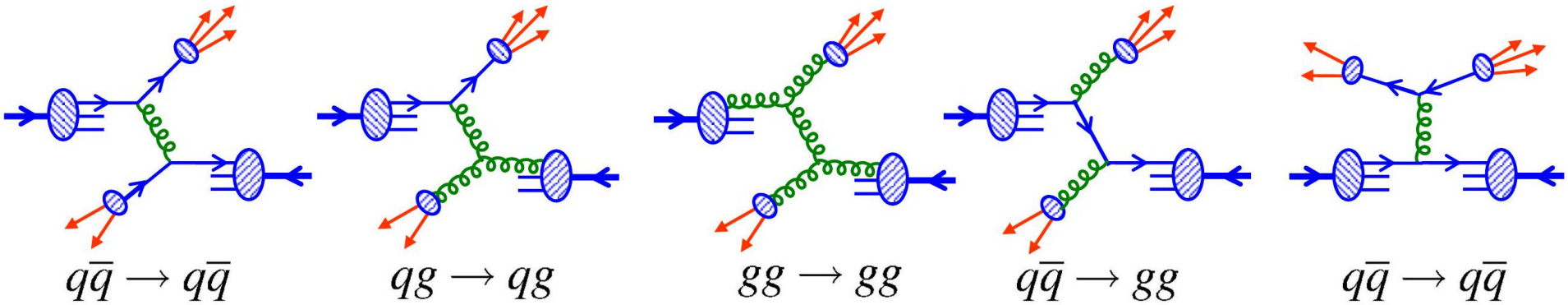
QCD

$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_s^2}{q^4}\left[1 + \left(1 + \frac{q^2}{\hat{s}}\right)^2\right]$$

Never see colour, but enters through colour factors. Can tell QCD is $SU(3)$.

- Here $\hat{S}$ is the centre-of-mass energy of the quark-quark collision

-The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions
e.g. two jet production in proton-antiproton collisions

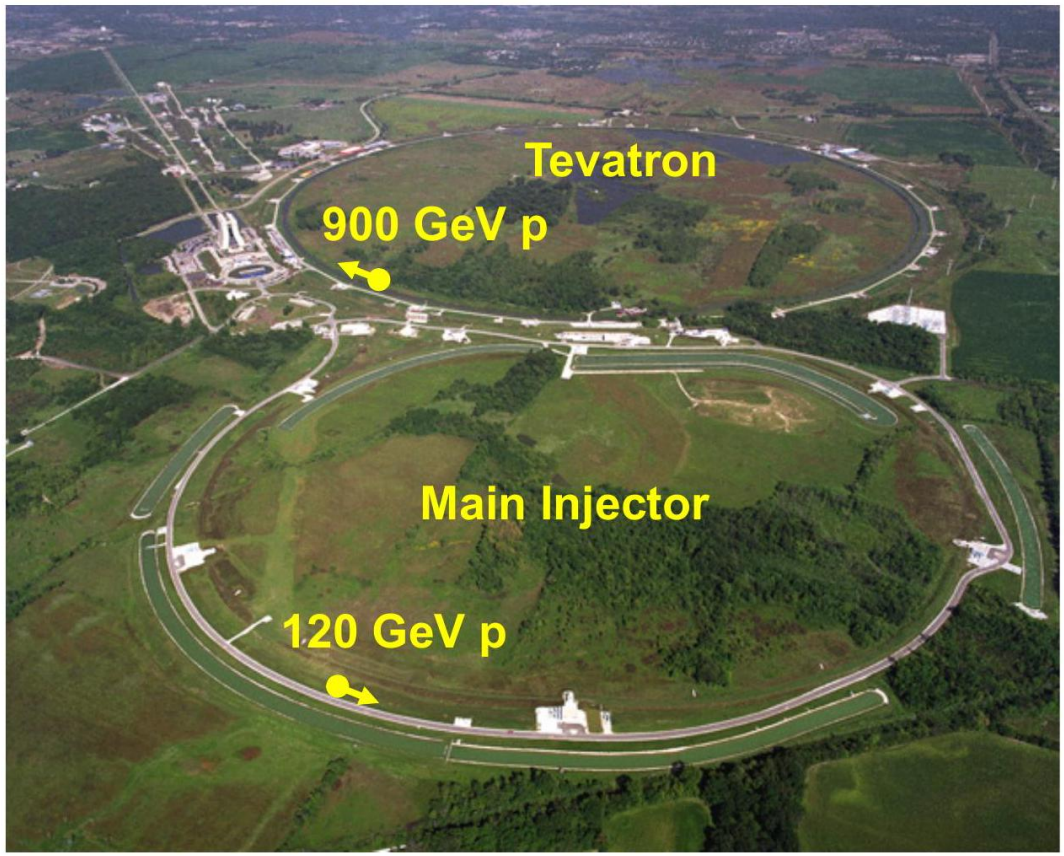


e.g. $p\bar{p}$ collisions at the Tevatron

⋆ Tevatron collider at Fermi National Laboratory (FNAL)

- located $\sim$ 40 miles from Chigaco, US
- started operation in 1987 (ran until 2010)

⋆$\mathrm{p\overline{p}}$ collisions at $\sqrt{\mathrm{s}} = 1.8\ \mathrm{TeV}$ c.f. $14\ \mathrm{TeV}$ at the LHC



Two main accelerators:

⋆ Main Injector

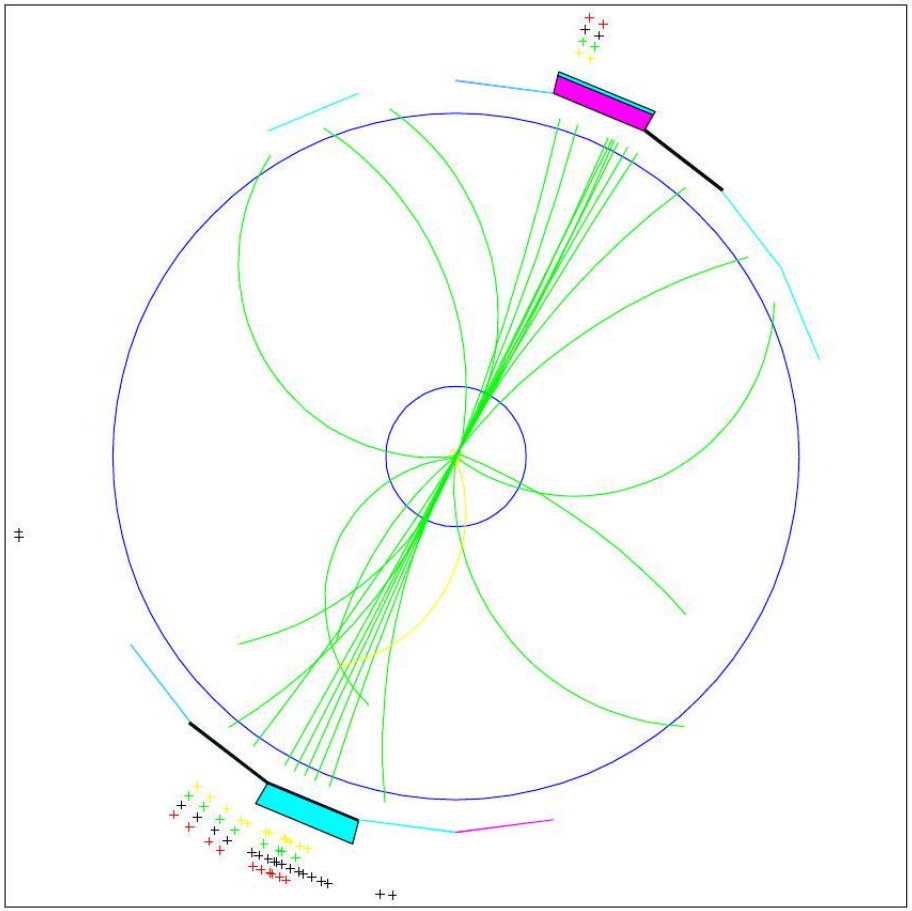
- Accelerated $8\mathrm{GeV}$ p to $120\mathrm{GeV}$
- also $\bar{p}$ to $120\mathrm{GeV}$
- Protons sent to Tevatron & MINOS
- $\bar{p}$ all went to Tevatron

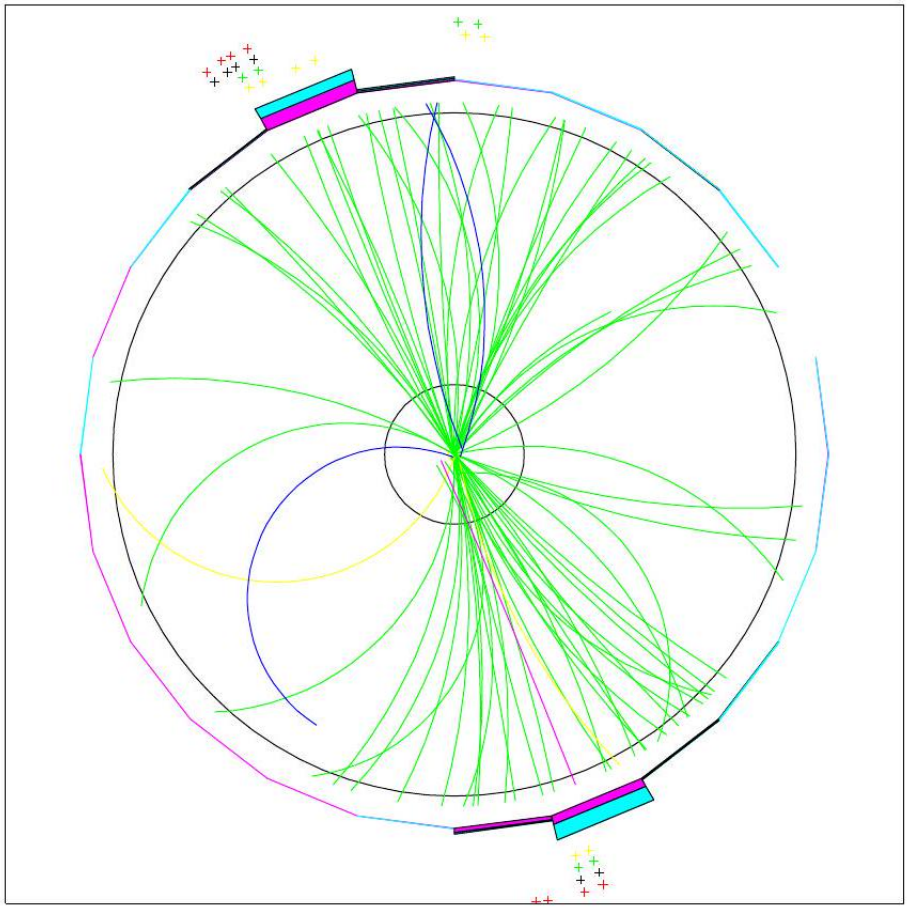
⋆ Tevatron

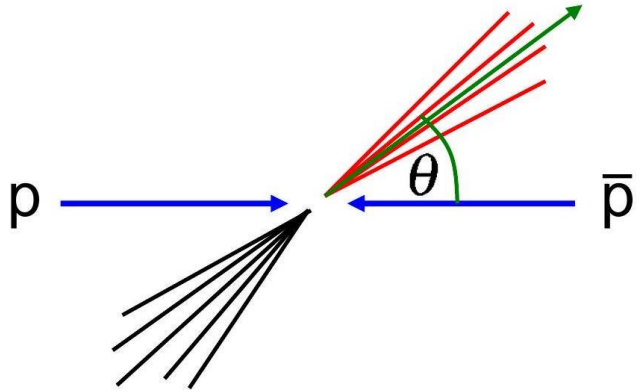
- 4 mile circumference
- accelerated $p/\bar{p}$ from $120\mathrm{GeV}$ to $900\mathrm{GeV}$

Can test QCD predictions by looking at production of pairs of high energy jets

$p\bar{p} \to$ jet jet $+X$:



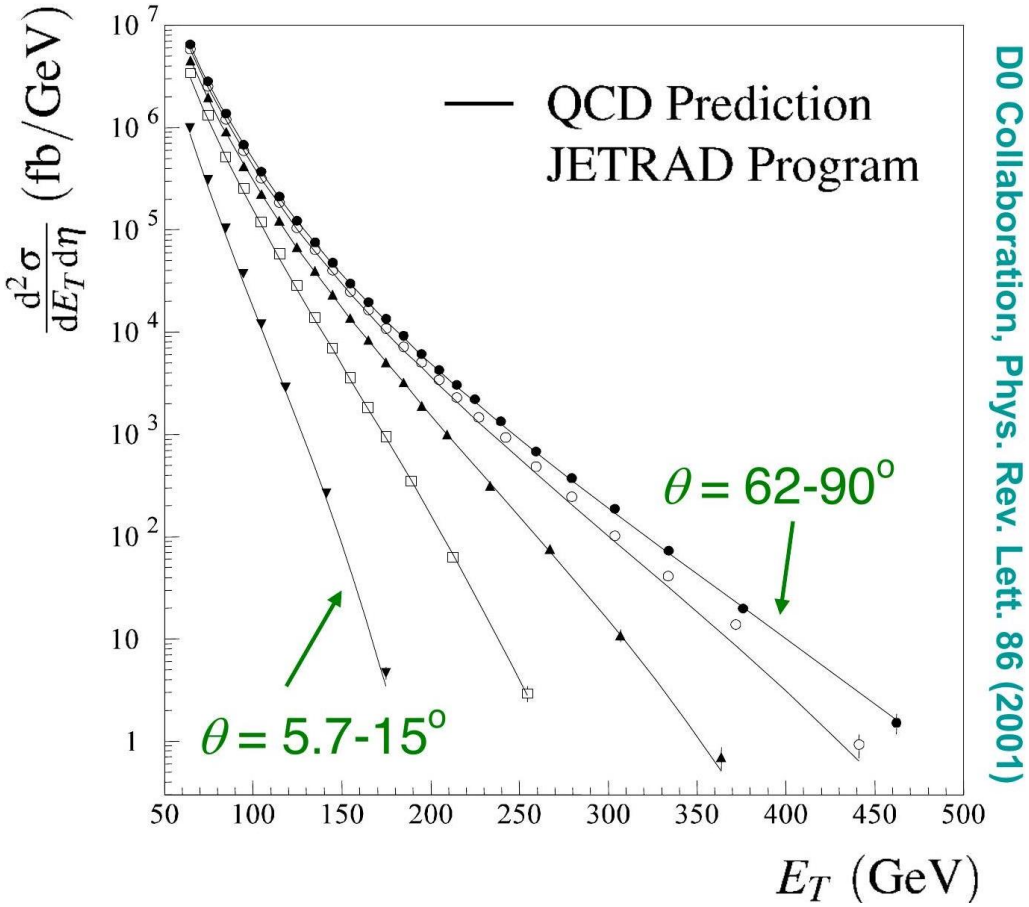




★ **Measure cross-section in terms of**

- **"transverse energy"** $E_T = E_{\text{jet}} \sin\theta$
- **"pseudorapidity"** $\eta = \ln\left[\cot\left(\frac{\theta}{2}\right)\right]$

…don't worry too much about the details here, what matters is that…



D0 Collaboration, Phys. Rev. Lett. 86 (2001)

★ **QCD predictions provide an excellent description of the data**

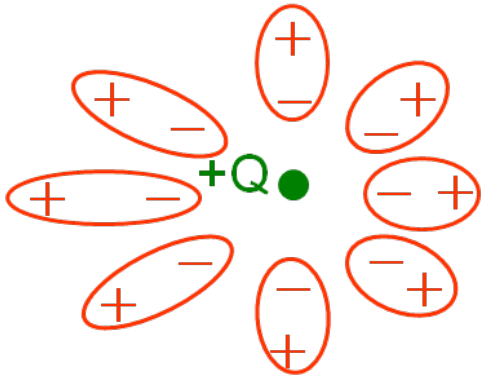
★ **NOTE:**

- **at low E_T cross-section is dominated by low x partons**
  i.e. gluon-gluon scattering
- **at high E_T cross-section is dominated by high x partons**
  i.e. quark-antiquark scattering

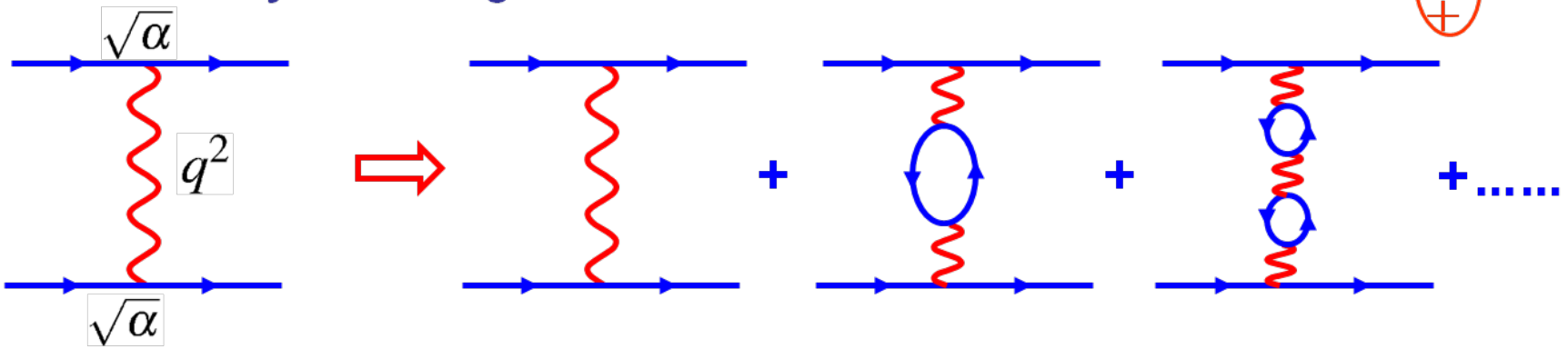
Running Coupling Constants

QED

- "bare" charge of electron screened by virtual e^+e^- pairs
- behaves like a polarizable dielectric



★ In terms of Feynman diagrams:

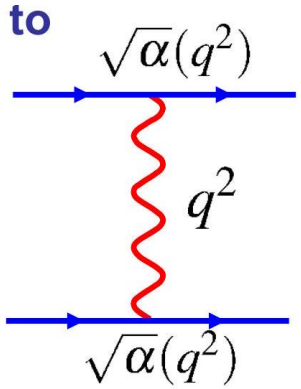


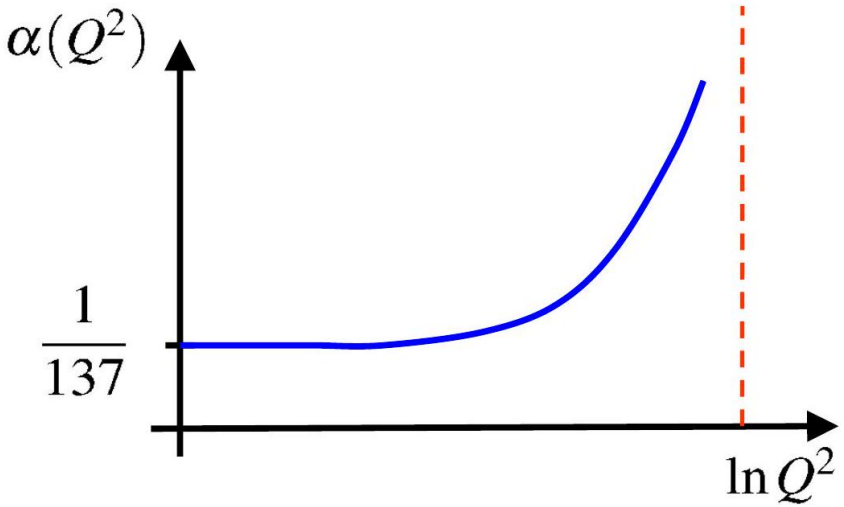
⋆ Same final state so add matrix element amplitudes: $M = M_1 + M_2 + M_3 + \ldots$

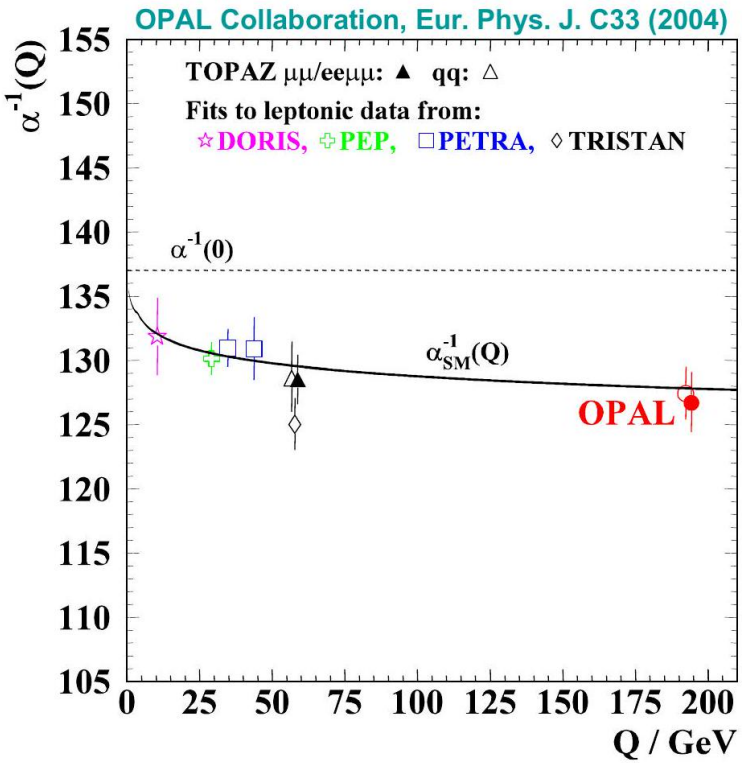
⋆ Giving an infinite series which can be summed and is equivalent to a single diagram with "running" coupling constant

$$\alpha\left(Q^2\right) = \alpha\left(Q_0^2\right) / \left[1 - \frac{\alpha\left(Q_0^2\right)}{3\pi} \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

for $Q^2 \gg Q_0^2$.







Might worry that coupling becomes infinite at

$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at

$$Q \sim 10^{26}\ \mathrm{GeV}.$$

- But quantum gravity effects would come in way below this energy and it is highly unlikely that QED 'as is' would be valid in this regime.

In QED, running coupling increases very slowly:

- Atomic physics: $Q^2 \sim 0$

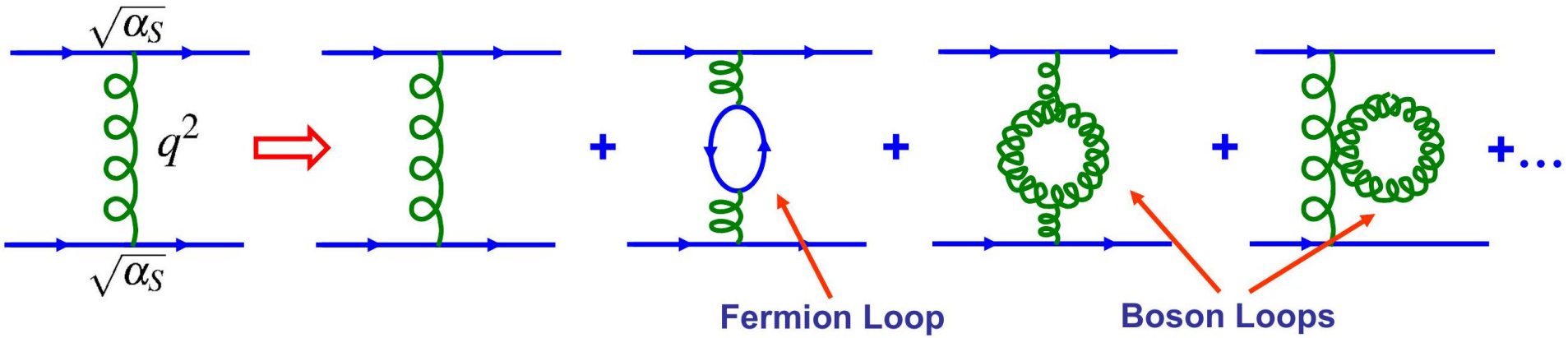
$$1/\alpha = 137.03599976(50)$$

- High energy physics: (e.g. Opal at 193 GeV):

$$1/\alpha = 127.4 \pm 2.1$$

Running of α_s

QCD Similar to QED but also have gluon loops



- Remembering adding amplitudes, so can get negative interference and the sum can be smaller than the original diagram alone.
- Bosonic loops 'interfere negatively':

$$\alpha_S\left(Q^2\right) = \alpha_S\left(Q_0^2\right) / \left[1 + B\alpha_S\left(Q_0^2\right) \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

$$\text{with} \quad B = \frac{11N_c - 2N_f}{12\pi} \quad \begin{cases} N_c = \text{no. of colours} \\ N_f = \text{no. of quark flavours} \end{cases}$$

$$N_c = 3; N_f = 6 \implies B > 0 \quad \text{so}$$

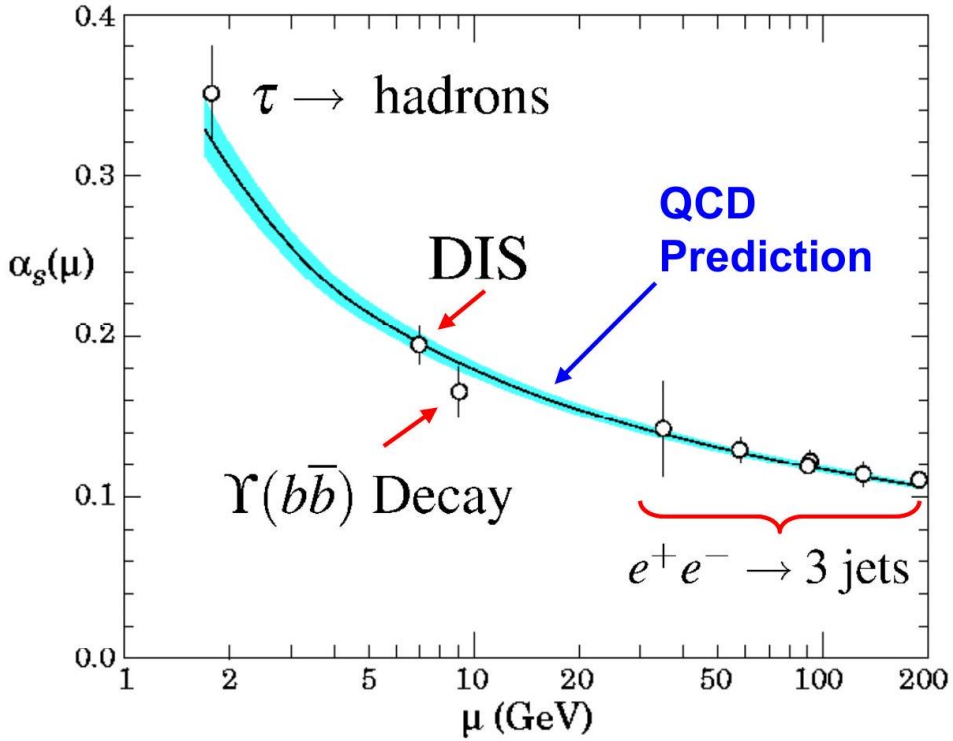
α_s decreases with Q^2.

[Nobel Prize 2004, Gross, Politzer, Wilczek.]

Measure α_s in many ways:

- jet rates
- DIS
- tau decays
- bottomonium decays
- ...

As predicted by QCD, α_s decreases with Q^2.



$\mu \approx Q$

At low Q^2 : α_s is large, e.g. at $Q^2 = 1\ \mathrm{GeV}^2$ find $\alpha_s \sim 1$.

- Can't use perturbation theory! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets.

At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_Z^2$ find $\alpha_s \sim 0.12$.

- Can use perturbation theory and this is the reason that in DIS at high Q^2 quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons).

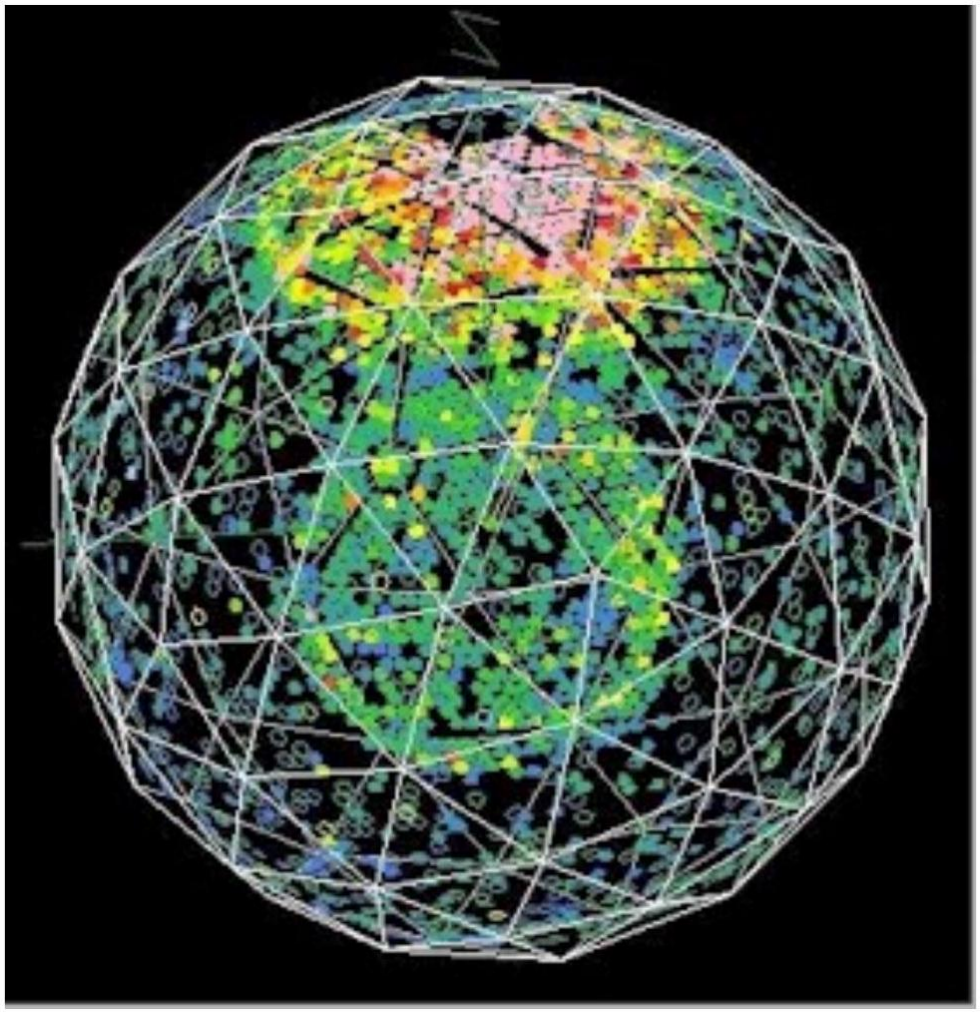
Summary

- Superficially QCD is very similar to QED.
- Gluon self-interactions are believed to result in colour confinement.
- All hadrons are colour singlets which explains why only observe Mesons and Baryons in certain specific flavour combinations.
- At low energies $\alpha_S \sim 1$ so can't use perturbation theory!
- Coupling constant 'runs' becoming smaller at higher energy scales ('Asymptotic Freedom'):

$$\alpha_S(100\ \mathrm{GeV}) \sim 0.1\ .$$

  so at higher energies we can use perturbation theory.
- Where calculations can be performed, QCD provides a good description of relevant experimental data.

Prof. Alex Mitov, 2026



H9: The Weak Interaction and $V - A$

Parity (again/reminder)

- **The parity operator performs spatial inversion through the origin:**

$$\psi'(\vec{x}, t) = \hat{P}\psi(\vec{x}, t) = \psi(-\vec{x}, t).$$

  Doing this **twice** gets us back to the start, so $\hat{P}^{-1} = \hat{P}$.

- To preserve the normalisation of the wave-function

$$\langle \psi \mid \psi \rangle = \langle \psi' | \psi' \rangle = \left\langle \psi \left| \hat{P}^\dagger \hat{P} \right| \psi \right\rangle$$

  so $\hat{P}^\dagger \hat{P} = I$ and thus $\hat{P}$ is **unitary**, i.e. $\hat{P}^{-1} = \hat{P}^\dagger$.

- The last two bullet points taken together mean that $\hat{P}^\dagger = \hat{P}$, i.e. that $\hat{P}$ is Hermitian, which implies that **parity is an observable quantity**. Furthermore, if the interaction Hamiltonian commutes with $\hat{P}$, then **parity is a conserved quantity**.

- If $\psi(\vec{x}, t)$ is an eigenfunction of the parity operator with eigenvalue P. $\hat{P}\hat{P} = I$ forces P to be ± 1. $[(\hat{P}\vec{x} = \lambda\vec{x}) \implies (\hat{P}\hat{P}\vec{x} = \lambda^2\vec{x}) \implies (\vec{x} = \lambda^2\vec{x}) \implies (\lambda = \pm 1).]$

- We will see later (page 263) that QED and QCD are invariant under parity, but that
- **the weak interactions does not conserve parity**.

Intrinsic Parities of fundamental particles

Spin-1 Bosons

From Gauge Field Theory can show that the gauge bosons have $P = -1$

$$P_\gamma = P_g = P_{W^+} = P_{W^-} = P_Z = -1.$$

Spin-$\frac{1}{2}$ Fermions

From the Dirac equation showed (Handout 2):
Spin-$\frac{1}{2}$ particles have opposite parity to spin-$\frac{1}{2}$ anti-particles. Conventional choice is:

Particles: $P_{e^-} = P_{\mu^-} = P_{\tau^-} = P_\nu = P_q = +1,$

Anti-particles: $P_{e^+} = P_{\mu^+} = P_{\tau^+} = P_{\bar{\nu}} = P_{\bar{q}} = -1.$

For Dirac spinors it was shown (in Handout 2) that the parity operator (in our rep) is:

$$\hat{P} = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Vectors vs Axial-Vectors

Under a parity transformation vectors and axial-vectors transform differently:

$$\text{Vectors change sign} \begin{cases} \vec{r} \xrightarrow{\hat{P}} -\vec{r} & \\ \vec{p} \xrightarrow{\hat{P}} -\vec{p} & \left(p_x = \frac{\partial}{\partial x},\ \text{etc.}\right) \end{cases}$$

$$\text{Axial-Vectors are unchanged} \begin{cases} \vec{L} \xrightarrow{\hat{P}} +\vec{L} & (\vec{L} = \vec{r} \wedge \vec{p}) \\ \vec{\mu} \xrightarrow{\hat{P}} +\vec{\mu} & (\vec{\mu} \propto \vec{L}) \end{cases}$$

Parity transformation for adjoint spinors

The transformation we already know for ordinary spinors is:

$$u \xrightarrow{\hat{P}} u' \equiv \hat{P}u = \gamma^0 u. \tag{109}$$

The transformation for adjoint spinors must therefore be:

$$\bar{u} \xrightarrow{\hat{P}} \bar{u}' \equiv \overline{\hat{P}u} = \bar{u}\hat{P} = \bar{u}\gamma^0. \tag{110}$$

Above result follows from invariance of $\bar{u}u$ given $(\hat{P})^2 = 1$, or (if you prefer) because:

$$\begin{aligned}
\bar{u}' &\equiv \overline{\hat{P}u} && \text{(by definition of } u'\text{)} \\
&= (\hat{P}u)^\dagger \gamma^0 && \text{(by definition pf adjoint operation)} \\
&= (\gamma^0 u)^\dagger \hat{P} && \text{(since } \hat{P} = \gamma^0\text{)} \\
&= u^\dagger \gamma^{0\dagger} \hat{P} && \text{(by } (AB)^\dagger = B^\dagger A^\dagger\text{)} \\
&= u^\dagger \gamma^0 \hat{P} && \text{(since } \gamma^0 \text{ is Hermitiarn)} \\
&= \bar{u}\hat{P} && \text{(by definition of } \bar{u}\text{).}
\end{aligned}$$

Parity transformations for currents $j^\mu = \bar{\phi}\gamma^\mu\psi$:

Appendix VII already proved that all currents transform as Lorentz four-vectors. However, that proof relied on Lorentz transformations being continuously connected to the identity.

Parity transformations are not continuously connected to the identity, so we need to **consider how currents transform under parity separately**.

From (109) and (110) we deduce the following transformations under Parity:

$$j^\mu = \bar{\phi}\gamma^\mu\psi \xrightarrow{\hat{P}} (j')^\mu = (\bar{\phi}\gamma^0)\gamma^\mu(\gamma^0\psi) = \bar{\phi}(\gamma^0\gamma^\mu\gamma^0)\psi$$

and so
$$\textbf{time:}\quad j^0 \longrightarrow \bar{\phi}\gamma^0\gamma^0\gamma^0\psi = \bar{\phi}\gamma^0(\gamma^0\gamma^0)\psi = \bar{\phi}\gamma^0\psi = j^0, \quad \text{and}$$
$$\textbf{space:}\quad j^i \longrightarrow \bar{\phi}\gamma^0\gamma^i\gamma^0\psi = -\bar{\phi}\gamma^i(\gamma^0\gamma^0)\psi = -\bar{\phi}\gamma^i\psi = -j^i.$$

Thus, **under parity, currents still behave like bona fide four-vectors**:
the time part is unchanged and the space part changes sign:

$$\boxed{\left(j^0, +\vec{j}\right) \xrightarrow{\hat{P}} \left(j^0, -\vec{j}\right)} \tag{111}$$

thus **scalar products between currents** (or any four-vectors) **are invariant under parity**:

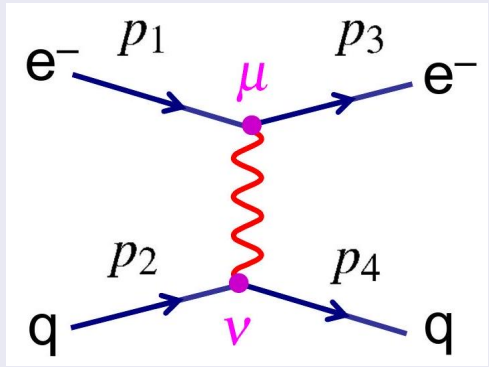
$$j_e \cdot j_q = j_e^0 j_q^0 - \vec{j_e}\cdot\vec{j_q} \xrightarrow{\hat{P}} j_e^0 j_q^0 - \left(-\vec{j_e}\right)\cdot\left(-\vec{j_q}\right) = j_e \cdot j_q. \tag{112}$$

Parity Conservation in QED and QCD

Consider the QED process $e^- q \to e^- q$:

The Feynman rules give:

$$-iM \propto \left[\bar{u}_e(p_3)\, ie\gamma^\mu u_e(p_1)\right] \frac{-ig_{\mu\nu}}{q^2} \left[\bar{u}_q(p_4)\, ie\gamma^\nu u_q(p_2)\right]$$



which can be re-cast in terms of the electron and quark currents as

$$M \propto -\frac{e^2}{q^2} g_{\mu\nu} j_e^\mu j_q^\nu = -\frac{e^2}{q^2} j_e \cdot j_q.$$

with: $\boxed{j_e^\mu = \bar{u}_e(p_3)\,\gamma^\mu u_e(p_1)}$ and $\boxed{j_q^\mu = \bar{u}_q(p_4)\,\gamma^\mu u_q(p_2)}$.

We saw in (111) and (112) that such currents are *bona fide* four-vectors and that their dot products (and thus M above) are therefore invariant under parity.
The QCD vertex has the same spinor structure as that of QED, and the result is also valid for higher order matrix elements in QED and QCD, therefore:

QED and QCD conserve parity, and the predictions of these theories do not change when and experiment is parity-inverted (i.e. when it is mirrored).

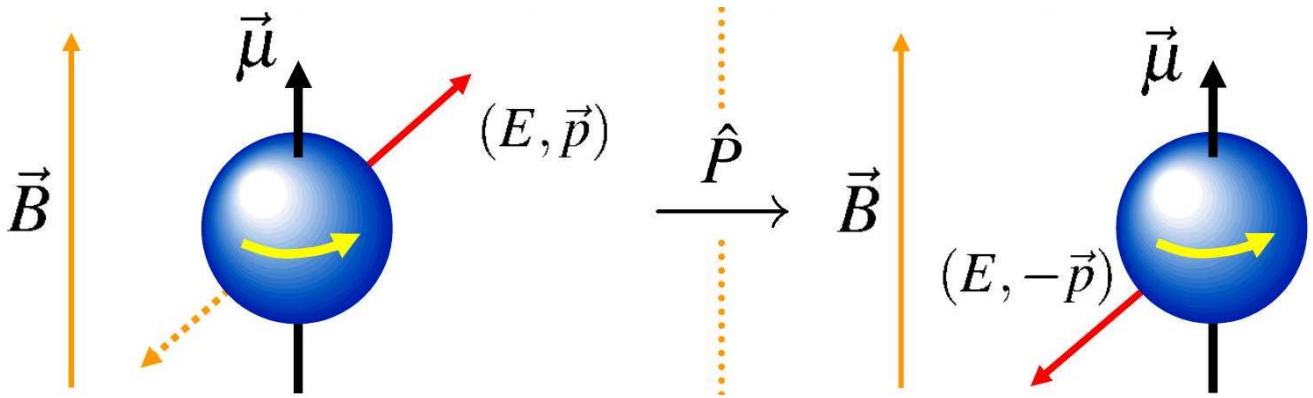
Parity Violation in β-Decay

- In 1957 C.S.Wu et al. studied beta decay of polarized cobalt-60 nuclei:

$$^{60}\mathrm{Co} \rightarrow {}^{60}Ni^{*} + e^{-} + \bar{\nu}_e.$$

  and observed that **electrons were emitted preferentially in direction opposite to applied B-field** !

- Having such a preference (*irrespective of whether it favours 'opposite to' or 'in the same direction as'* $\vec{B}$) is incompatible with parity being a symmetry of nature because parity transforms vector $\vec{p}$ and axial-vector $\vec{B}$ differently:



- If parity were a symmetry of nature we would expect equal rate for producing e^{-} in directions along and opposite to the nuclear spin.

Conclusion: parity is somehow violated by the weak interaction.

$\Longrightarrow$ The weak interaction vertex is not of the form $\bar{u}_e \gamma^{\mu} u_{\nu}$!

Bilinear Covariants

- The requirement of Lorentz invariance of the matrix element severely restricts the form of the interaction vertex.
- QED and QCD are called **vector interactions** because their currents:

$$j^\mu = \bar{\psi}\gamma^\mu\phi$$

transform as **non-axial four-vectors** (Appendix VII and (111)).

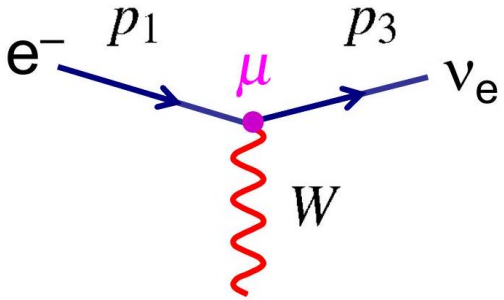
- In general, there are only 5 possible combinations of two spinors and the gamma matrices that form Lorentz covariant currents. They are the so-called "bilinear covariants":

Type	Form	Components	Boson Spin
Scalar	$\bar{\psi}\phi$	1	0
Pseudoscalar	$\bar{\psi}\gamma^5\phi$	1	0
Vector	$\bar{\psi}\gamma^\mu\phi$	4	1
Axial Vector	$\bar{\psi}\gamma^\mu\gamma^5\phi$	4	1
Tensor	$\bar{\psi}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\phi$	6	2

- Note that in total the sixteen components correspond to the 16 elements of a general 4x4 matrix: "decomposition into Lorentz covariant combinations".

'$V - A$'-structure of the Weak Interaction

- The most general form for the interaction between a fermion and a boson is a linear combination of bilinear covariants.
- For an interaction corresponding to the exchange of a spin-1 boson the most general form is a linear combination of VECTOR AND AXIAL-VECTOR currents.
- Experimentally, the weak interaction's current is determined VECTOR minus AXIAL VECTOR — called '$V - A$' for short:



$$j^\mu \propto \bar{u}_{\nu_e}(\gamma^\mu - \gamma^\mu\gamma^5)u_e$$

V – A

- Can this account for parity violation? First consider parity transformation of a pure AXIAL-VECTOR current

$$j_A = \bar{\psi}\gamma^\mu\gamma^5\phi \quad \text{with} \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3; \quad \gamma^5\gamma^0 = -\gamma^0\gamma^5$$

$$j_A = \bar{\psi}\gamma^\mu\gamma^5\phi \xrightarrow{\hat{P}} \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\phi = -\bar{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^5\phi$$

$$j_A^0 \xrightarrow{\hat{P}} -\bar{\psi}\gamma^0\gamma^0\gamma^0\gamma^5\phi = -\bar{\psi}\gamma^0\gamma^5\phi = -j_A^0$$

$$j_A^k \xrightarrow{\hat{P}} -\bar{\psi}\gamma^0\gamma^k\gamma^0\gamma^5\phi = +\bar{\psi}\gamma^k\gamma^5\phi = +j_A^k \quad k = 1, 2, 3.$$

- The space-like components remain unchanged and the time-like components change sign (the opposite to the parity properties of a vector-current):

$$\boxed{\text{Vector}: \quad \left(j^0, \vec{j}\right) \xrightarrow{\hat{P}} \left(j^0, -\vec{j}\right)} \quad \text{versus} \quad \boxed{\text{Axial}: \quad \left(j^0, \vec{j}\right) \xrightarrow{\hat{P}} \left(-j^0, \vec{j}\right)}$$

- Now consider the matrix element:

$$M \propto g_{\mu\nu} j_1^\mu j_2^\nu = j_1^0 j_2^0 - \vec{j}_1 \cdot \vec{j}_2.$$

- An M containing a **combination of a two axial-vector currents is invariant under parity** because:

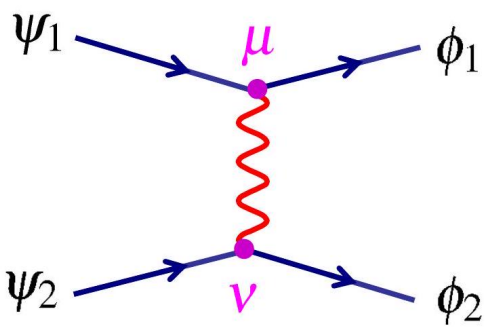
$$j_{A1} \cdot j_{A2} \xrightarrow{\hat{P}} \left(\left(-j_{A1}^0\right)\left(-j_{A2}^0\right) - \left(\vec{j}_{A1}\right) \cdot \left(\vec{j}_{A2}\right)\right) = j_{A1} \cdot j_{A2}$$

- Consequently parity is conserved for both a **pure vector** and **pure axial-vector** interactions.
- However, a scalar product between a **vector** current **and** an **axial vector** current:

$$j_{V1} \cdot j_{A2} \xrightarrow{\hat{P}} \left(\left(j_{V1}^0\right)\left(-j_{A2}^0\right) - \left(-\vec{j}_{V1}\right) \cdot \left(\vec{j}_{A2}\right)\right) = -j_{V1} \cdot j_{A2}$$

changes sign under parity. This can **interfere** with parity invariant terms in an M having both, to give parity violating cross sections!

Consider a general linear combination of VECTOR and AXIAL-VECTOR (note this is relevant for the Z-boson vertex)



$$j_1 = \overline{\phi}_1(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)\psi_1 = g_V j_1^V + g_A j_1^A$$

$$\frac{g_{\mu\nu}}{q^2 - m^2}$$

$$j_2 = \overline{\phi}_2(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)\psi_2 = g_V j_2^V + g_A j_2^A$$

$$M_{fi} \propto j_1 \cdot j_2 = g_V^2\, j_1^V \cdot j_2^V + g_A^2\, j_1^A \cdot j_2^A + g_V g_A \left(j_1^V \cdot j_2^A + j_1^A \cdot j_2^V\right).$$

- Consider the parity transformation of this scalar product

$$j_1 \cdot j_2 \xrightarrow{\hat{P}} g_V^2\, j_1^V \cdot j_2^V + g_A^2\, j_1^A \cdot j_2^A - g_V g_A \left(j_1^V \cdot j_2^A + j_1^A \cdot j_2^V\right).$$

- If **either** g_A **or** g_V **is zero, parity is conserved**. (Theories with pure vector or pure axial-vector interactions have parity symmetric cross sections.)
- Relative strength of parity violating part is proportional to:

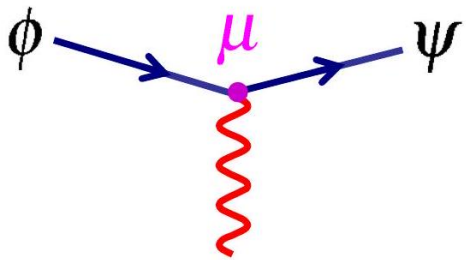
$$\frac{g_V g_A}{g_V^2 + g_A^2}.$$

Chiral Structure of QED (reminder of material from Handout 4)

- Chiral projections operators are: $P_R = \frac{1}{2}\left(1+\gamma^5\right)$; $\quad P_L = \frac{1}{2}\left(1-\gamma^5\right)$.
- In the ultra-relativistic limit, chiral states correspond to helicity states.

- Any spinor can be expressed as:

$$\psi \equiv \tfrac{1}{2}\left(1+\gamma^5\right)\psi + \tfrac{1}{2}\left(1-\gamma^5\right)\psi \equiv P_R\psi + P_L\psi \equiv \psi_R + \psi_L.$$

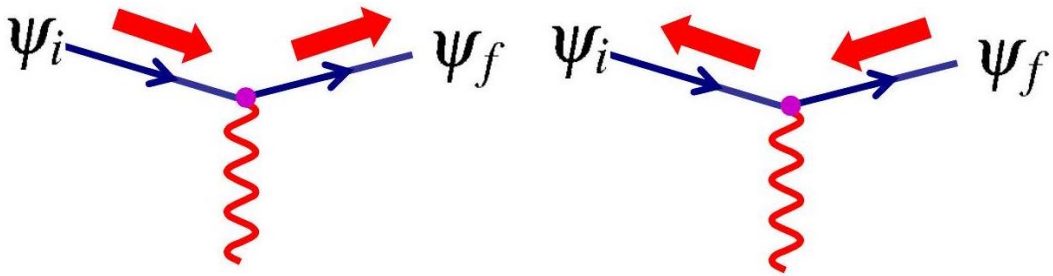


- The QED current $\bar{\psi}\gamma^\mu\phi$ could be written (see page 122) in terms of chiral states as:

$$\bar{\psi}\gamma^\mu\phi \equiv \bar{\psi}_R\gamma^\mu\phi_R + \bar{\psi}_L\gamma^\mu\phi_L.$$

since $\bar{\psi}_R\gamma^\mu\phi_L \equiv 0$ and $\bar{\psi}_L\gamma^\mu\phi_R \equiv 0$.

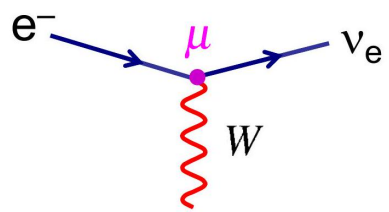
- A consequence of the above is that in the ultra-relativistic limit only two helicity combinations are non-zero:



Chiral and Helicity Structure of the Weak Interaction

⋆ The charged current $\left(W^{\pm}\right)$ weak vertex is:

$$\frac{-ig_w}{\sqrt{2}}\frac{1}{2}\gamma^{\mu}\left(1-\gamma^5\right).$$



⋆ Since $\frac{1}{2}\left(1-\gamma^5\right)$ projects out left-handed chiral particle states:

$$\bar{\psi}\frac{1}{2}\gamma^{\mu}\left(1-\gamma^5\right)\phi=\bar{\psi}\gamma^{\mu}\phi_L.$$

⋆ Writing $\bar{\psi}=\bar{\psi}_R+\bar{\psi}_L$ and from discussion of QED, $\bar{\psi}_R\gamma^{\mu}\phi_L=0$ gives

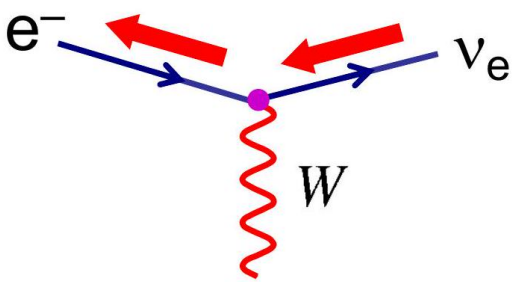
$$\bar{\psi}\frac{1}{2}\gamma^{\mu}\left(1-\gamma^5\right)\phi=\bar{\psi}_L\gamma^{\mu}\phi_L.$$

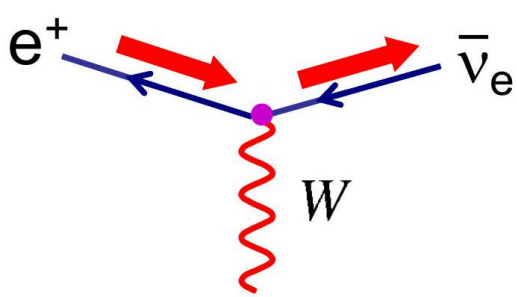
Only the left-handed chiral components of spinors participate in charged current weak interactions.

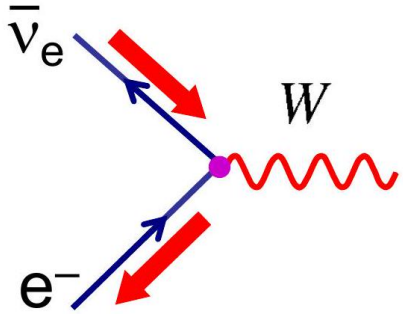
At very high energy ($E \gg m$), the left-handed chiral components are helicity eigenstates, so:

In the ultra-relativistic limit only left-handed particles and right-handed antiparticles participate in charged current weak interactions.

E.g.: in the relativistic limit, the only possible electron-neutrino interactions are:

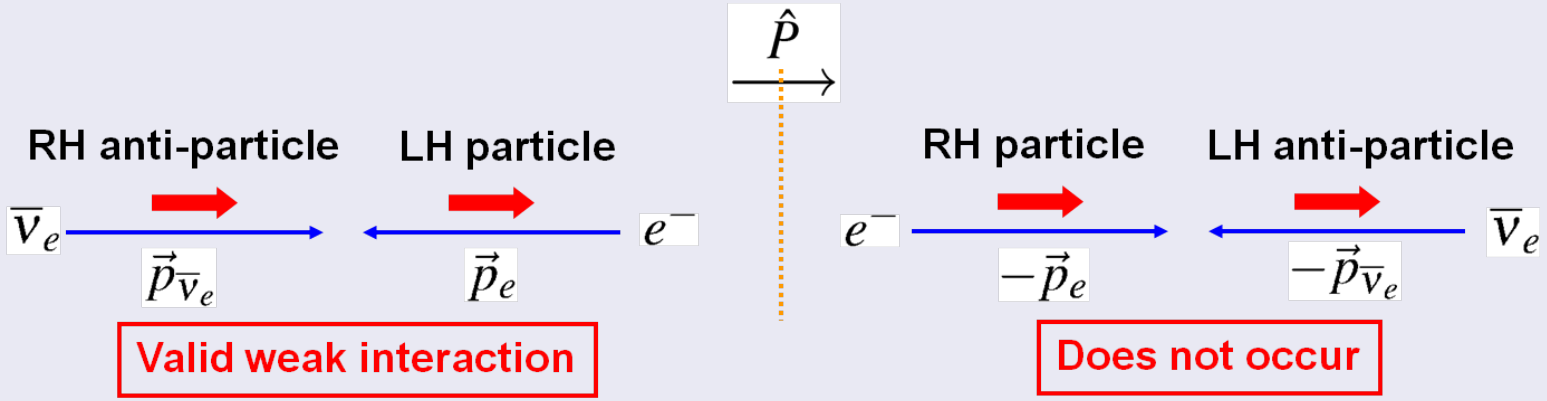






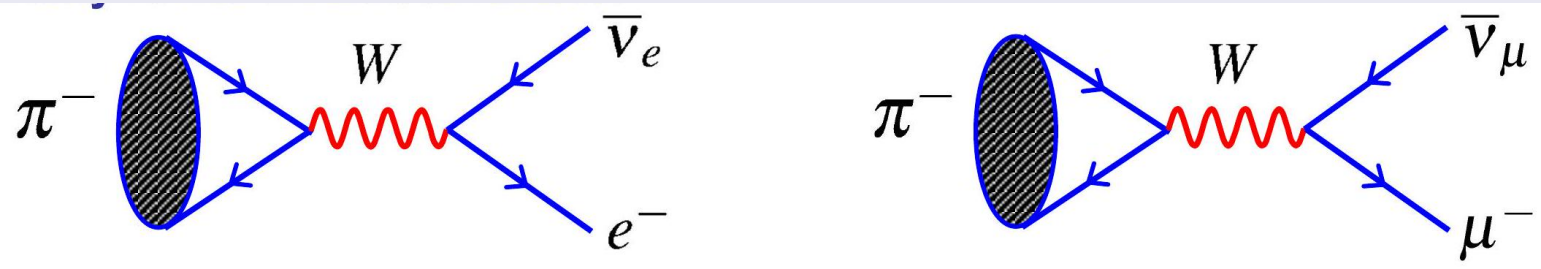
Example of parity violating consequence:

Annihilation of the form: e.g. $\bar{\nu}_e + e^- \rightarrow W^-$ is spin-dependent:



Helicity in Pion Decay

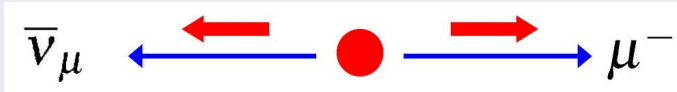
Pion decay demonstrates importance of helicity in the weak interaction.



- Might expect the decay to electrons to dominate due to increased phase space, but the opposite happens. The electron decay is helicity suppressed by a large factor:
- $\frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} = 1.23 \times 10^{-4}$. (See example sheet question 6 and next slides.)

Consider decay in pion rest frame.

- Pion is spin zero: so the spins of the $\bar{\nu}$ and μ are opposite.
- Weak interaction only couples to RH chiral anti-particle states. Since neutrinos are (almost) massless, neutrino must be in RH Helicity state.
- Therefore, to conserve angular mom. muon is emitted in a RH HELICITY state:

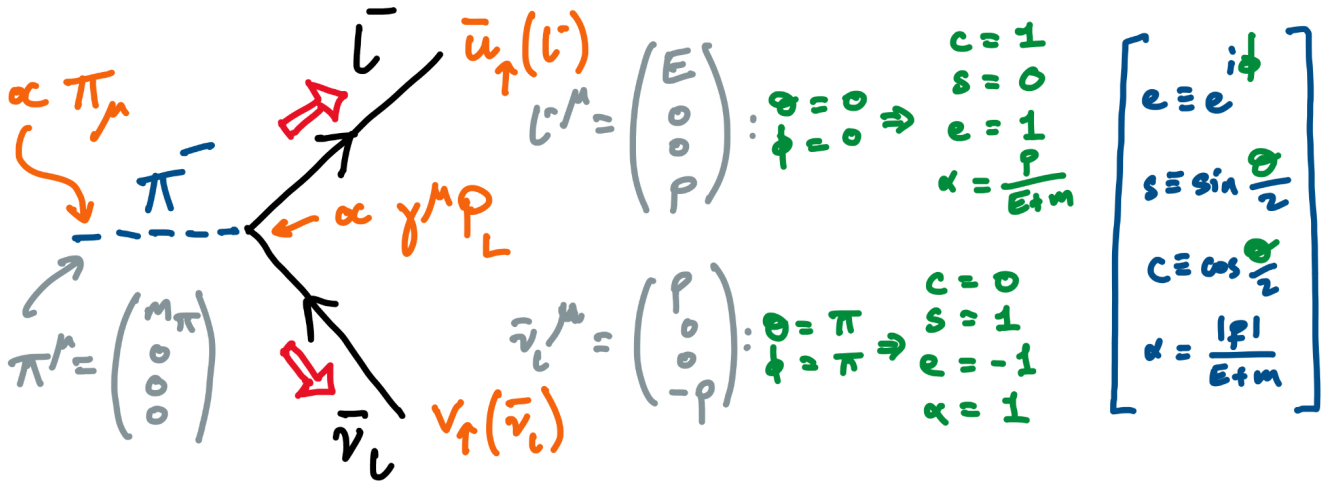


... but only left-handed CHIRAL particle states participate in weak interaction!

Helicity in pion decay (cont.)

Thus need to calculate matrix element M for the process shown right.

(The $\propto$ symbols drop factors not depending on m_l.)



$$
\begin{aligned}
M &\propto \pi_\mu \bar{u}_\uparrow(l^-)\gamma^\mu P_L v_\uparrow(\bar{\nu}_l) && \text{(from Feynam rules and diagram above)} \\
&= \pi_\mu \bar{u}_\uparrow(l^-)\gamma^\mu v_\uparrow(\bar{\nu}_l) && \text{(massless } \uparrow \text{ anti-neutrino already left chiral)} \\
&= m_\pi \bar{u}_\uparrow(l^-)\gamma^0 v_\uparrow(\bar{\nu}_l) && \text{(choosing to evaluate in } \pi^- \text{ rest frame)} \\
&\propto u_\uparrow^\dagger(l^-) v_\uparrow(\bar{\nu}_l) && \text{(since } (\gamma^0)^2 = 1 \text{ and } m_\pi \text{ is independent of } m_l\text{)} \\
&= \sqrt{E+m_l} \begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix}^\dagger \sqrt{p} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} && \text{(using page 79's } u_\uparrow = \begin{pmatrix} c \\ es \\ \alpha c \\ \alpha es \end{pmatrix} \text{ and } v_\uparrow = \begin{pmatrix} \alpha s \\ -\alpha ec \\ -s \\ ec \end{pmatrix}\text{)} \\
&= \sqrt{(E+m_l)p} \cdot \left(1 - \frac{p}{E+m_l}\right). && (113)
\end{aligned}
$$

Helicity in pion decay (cont.)

We can simplify the pion decay matrix element M on the previous page by using some of the two-body decay kinematics results of example sheet Q4. That question tells us that:

$$p = \frac{m_\pi^2 - m_l^2}{2m_\pi} \quad \text{and} \quad E = \frac{m_\pi^2 + m_l^2}{2m_\pi} \tag{114}$$

so

$$\begin{aligned}
|M|^2 &\propto (E + m_l) \cdot p \cdot \left(1 - \frac{p}{E + m_l}\right)^2 \qquad \text{(from (113))} \\
&= \frac{p}{E + m_l} \cdot (E + m_l - p)^2 \\
&= \frac{p}{E + m_l} \cdot \left(E^2 + m_l^2 + p^2 - 2p(E + m_l) + 2Em_l\right) \\
&= \frac{p}{E + m_l} \cdot \left(2E^2 - 2p(E + m_l) + 2Em_l\right) \\
&= \frac{2p}{E + m_l} \cdot (E(E + m_l) - p(E + m_l)) \\
&= 2p\,(E - p) \\
&\propto 2\left(m_\pi^2 - m_l^2\right)\left((m_\pi^2 + m_l^2) - (m_\pi^2 - m_l^2)\right) \qquad \text{(using (114))} \\
&\propto m_l^2\left(m_\pi^2 - m_l^2\right). \qquad \text{(This result without omitted factors begins Ex.Sheet Q6.)}
\end{aligned}$$

Helicity in pion decay (cont.)

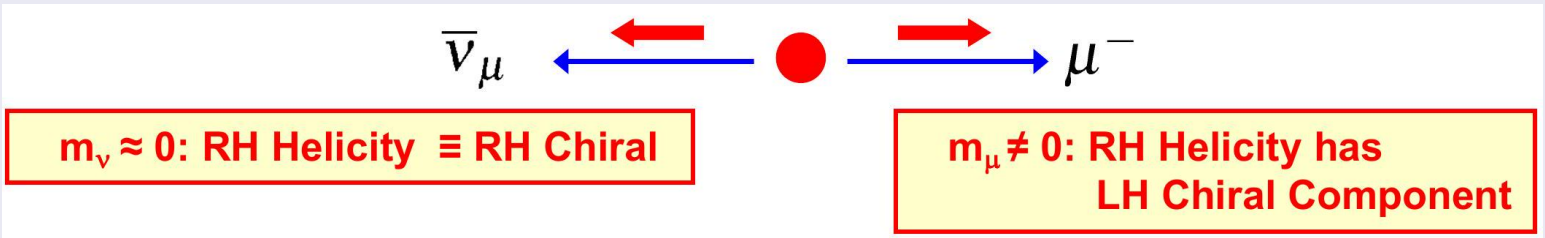
Example sheet Q6 adds in the phase space factors which need to be combined with our matrix element:

$$|M(\pi^- \to l^- \bar{\nu}_l)|^2 \propto m_l^2 \left(m_\pi^2 - m_l^2\right)$$

to predict the actual suppression of $\pi^- \to e^- \bar{\nu}_e$ relative to $\pi^- \to \mu^- \bar{\nu}_\mu$ given at the top of page 272.

Take home messages:

1. Although only LH chiral particles participate in the weak interaction, the contribution from RH Helicity particle states is not always zero, and
2. sometimes (such as in pion decay) the RH helicity particle states are the only thing that allows the decay to happen!



Examples of evidence for V-A

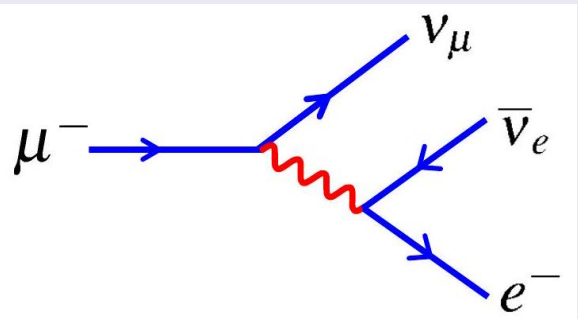
Charged pion decay (Ex.Sheet Q17)

Experimentally measure: $\frac{\Gamma\left(\pi^- \to e^- \bar{\nu}_e\right)}{\Gamma\left(\pi^- \to \mu^- \bar{\nu}_\mu\right)} = (1.230 \pm 0.004) \times 10^{-4}$.

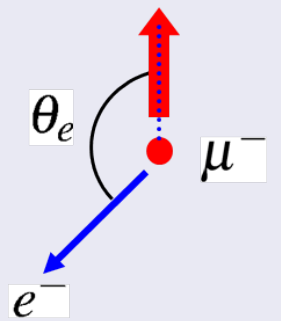
Theoretical predictions depend on Lorentz Structure of the interaction:

- **V-A and V+A** $(\bar{\psi}\gamma^\mu(1 \mp \gamma^5)\phi)$ predict ..... $\frac{\Gamma\left(\pi^- \to e^- \bar{\nu}_e\right)}{\Gamma\left(\pi^- \to \mu^- \bar{\nu}_\mu\right)} \approx 1.3 \times 10^{-4}$.
- **Scalar** $(\bar{\psi}\phi)$ **and Pseudo-Scalar** $\left(\bar{\psi}\gamma^5\phi\right)$ predict ..... $\frac{\Gamma\left(\pi^- \to e^- \bar{\nu}_e\right)}{\Gamma\left(\pi^- \to \mu^- \bar{\nu}_\mu\right)} \approx 5.5$.

Muon decay



Measure electron energy and angular distributions relative to muon spin direction. Results expressed in terms of general S+P+V+A+T form in 'Michel Parameters' (see PDG 57: Muon Decay Parameters [6] and see TWIST [7]).



Measurement of TWIST expt ($6 \times 10^9 \mu$ decays, Phys. Rev. Lett. 95 (2005) 101805) was ..... $\rho = 0.75080 \pm 0.00105$.

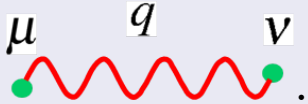
V-A prediction: ..... $\rho = 0.75$.

Weak Charged Current Propagator

The charged-current weak interaction is different from QED and QCD in that it is mediated by massive W-bosons ($M_W \approx 80.3\ \mathrm{GeV}$). The W propagator is thus different:

- As seen in Handout 4 denominator changes: $\frac{1}{q^2} \longrightarrow \frac{1}{q^2 - m^2}$.
- In addition the sum over **W** boson polarization states modifies the numerator.

Resulting W-**boson propagator** is: $\dfrac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m_W^2\right]}{q^2 - m_W^2}$



However in the limit where q^2 is small compared with $m_W = 80.3\ \mathrm{GeV}$ the interaction takes a simpler form:

W-boson propagator in the limit $\left(q^2 \ll m_W^2\right)$:

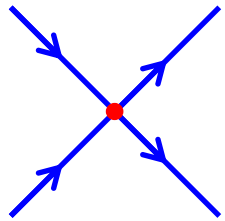
$$\frac{i g_{\mu\nu}}{m_W^2}.$$



The interaction appears point-like (i.e. no q^2 dependence).

Connection to Fermi Theory

In 1934, before the discovery of parity violation, Fermi proposed, in analogy with QED, that the invariant matrix element for β-decay was of the form:



$$M_{fi} = G_F \cdot g_{\mu\nu} \left[\bar{\psi}\gamma^\mu\psi\right] \left[\bar{\psi}\gamma^\nu\psi\right]$$

where $G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$.

Note the absence of a propagator: this represents an interaction at a point!

After the discovery of parity violation in 1957 this was modified to

$$\boxed{M_{fi} = \frac{G_F}{\sqrt{2}} g_{\mu\nu} \left[\bar{\psi}\gamma^\mu \left(1-\gamma^5\right)\psi\right] \left[\bar{\psi}\gamma^\nu \left(1-\gamma^5\right)\psi\right]}$$

(the factor of $\sqrt{2}$ was included so the numerical value of G_F did not need to be changed).

Compare to the prediction for W-boson exchange:

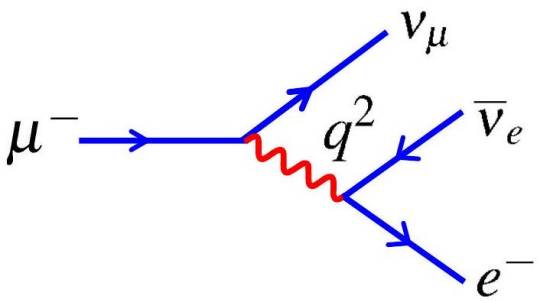
$$M_{fi} = \left[\frac{g_W}{\sqrt{2}}\bar{\psi}\frac{1}{2}\gamma^\mu \left(1-\gamma^5\right)\psi\right] \frac{g_{\mu\nu} - q_\mu q_\nu / m_W^2}{q^2 - m_W^2} \left[\frac{g_W}{\sqrt{2}}\bar{\psi}\frac{1}{2}\gamma^\nu \left(1-\gamma^5\right)\psi\right]$$

which for $q^2 \ll m_W^2$ becomes: $\boxed{M_{fi} = \frac{g_W^2}{8m_W^2} g_{\mu\nu} \left[\bar{\psi}\gamma^\mu \left(1-\gamma^5\right)\psi\right] \left[\bar{\psi}\gamma^\nu \left(1-\gamma^5\right)\psi\right]}$

and so consistency requires $\boxed{\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}}$.

Still often use G_F to express strength of weak interaction as the is the quantity that is precisely determined in muon decay.

Strength of Weak Interaction as measured in muon decay



- Here $q^2 < m_\mu \sim 0.106\ \mathrm{GeV}$.
- To a very good approximation the W-boson propagator can be written: $\dfrac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m_W^2\right]}{q^2 - m_W^2} \approx \dfrac{i g_{\mu\nu}}{m_W^2}$.
- In muon decay measure g_W^2/m_W^2.
- Convert to G_F measurement using $\boxed{\dfrac{G_\mathrm{F}}{\sqrt{2}} = \dfrac{g_W^2}{8 m_W^2}}$.

- Muon decay measurements find $G_F = 1.16639(1) \times 10^{-5}\ \mathrm{GeV}^{-2}$
- To obtain the intrinsic strength of weak interaction need to know mass of W-boson: $m_W = 80.403 \pm 0.029$ GeV

$$\Rightarrow \quad \alpha_W = \frac{g_W^2}{4\pi} = \frac{8 m_W^2 G_\mathrm{F}}{4\sqrt{2}\pi} \approx \frac{1}{30}.$$

The intrinsic strength of the weak interaction is similar to, but greater than, the EM interaction! It is the massive W-boson in the propagator which makes it appear weak. For $q^2 \gg m_W^2$ weak interactions are more likely than EM.

Summary

- Weak interaction vertex has 'vector minus axial-vector' parts and so is termed a '$V - A$'-interaction:

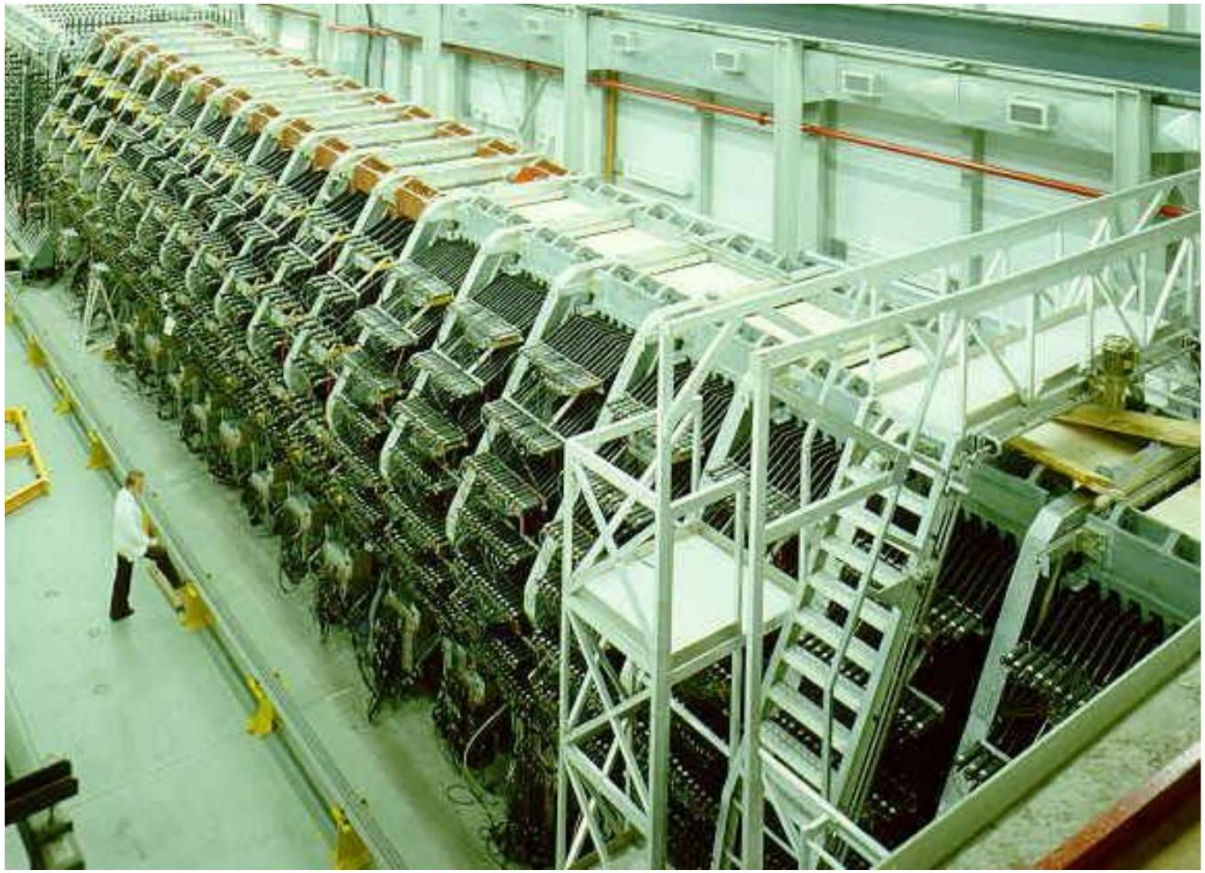
$$(\text{weak vertex factor}) = \frac{-ig_w}{\sqrt{2}} \frac{1}{2} \gamma^\mu \left(1 - \gamma^5\right).$$

- Consequently only left-handed chiral particle states and right-handed chiral anti-particle states participate in the weak interaction.
- $V - A$ is therefore also a form of 'Maximal Parity Violation'.
- The Weak interaction also violates charge conjugation symmetry.
- At low q^2 the weak interaction is only weak because of the large W-boson mass:

$$\frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}.$$

- Intrinsic strength of weak interaction is similar to that of QED.

Prof. Alex Mitov, 2026



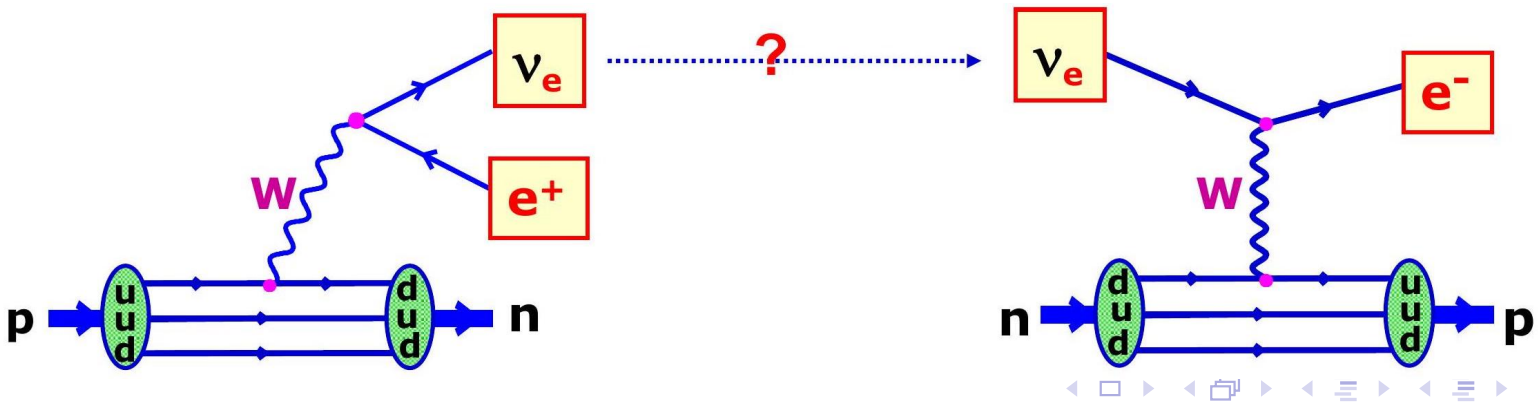
H10: Leptonic Weak Interactions and Neutrino Deep Inelastic Scattering

Aside : Neutrino Flavours

- Recent experiments (see Handout 11) imply neutrinos have mass (albeit very small).
- The textbook neutrino states, ν_e, ν_μ, ν_τ, are not the fundamental particles; these are ν_1, ν_2, ν_3.
- Concepts like 'electron number' conservation are now known not to hold.
- So what are ν_e, ν_μ, ν_τ ?
- Never directly observe neutrinos — can only detect them by their weak interactions. Hence by definition ν_e is the neutrino state produced along with an electron. Similarly, charged current weak interactions of the state ν_e produce an electron.

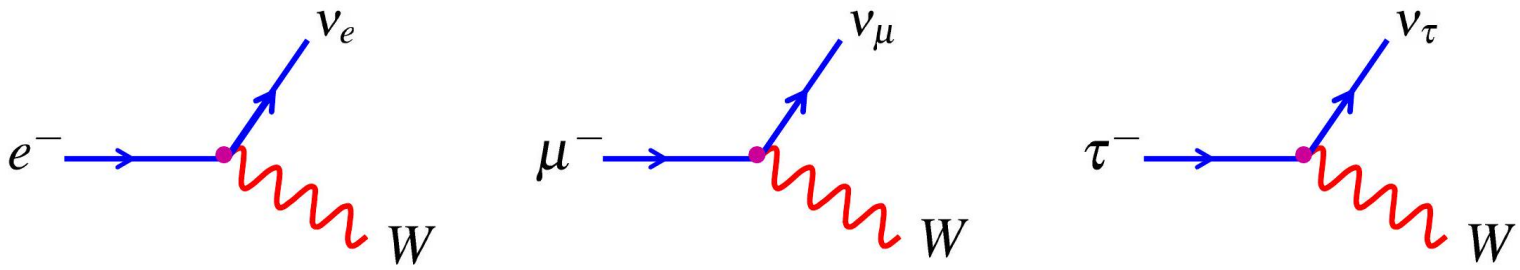
$$\boxed{\{\nu_e, \nu_\mu, \nu_\tau\} = \text{weak eignestates}}$$

- Unless dealing with very large distances: the neutrino produced with a positron will interact to produce an electron. For the discussion of the weak interaction continue to use ν_e, ν_μ, ν_τ as if they were the fundamental particle states.

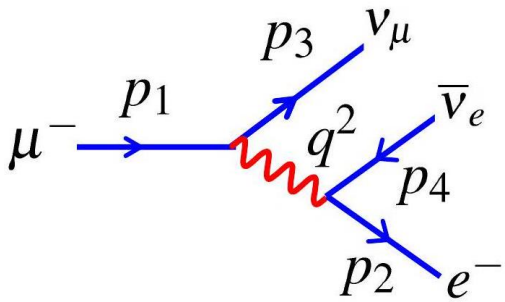


Muon Decay and Lepton Universality

- The leptonic charged current ($W^{\pm}$) interaction vertices are:



- Consider muon decay:



- It is straight-forward to write down the matrix element

$$M_{fi} = \frac{g_W^{(e)} g_W^{(\mu)}}{8 m_W^2} [\bar{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)] g_{\mu\nu} [\bar{u}(p_2)\gamma^\nu(1-\gamma^5)v(p_4)].$$

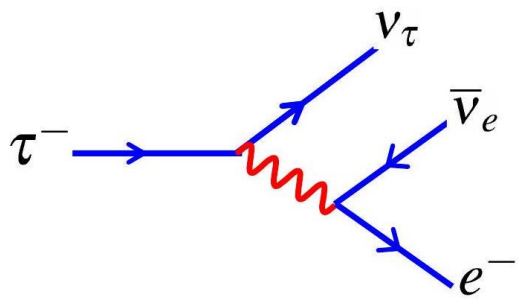
- Note: for lepton decay $q^2 \ll m_W^2$ so propagator is a constant (Fermi theory limit!).
- The evaluation of this matrix element together with the appropriate three-body phase space to get the muon decay rate leads to the result (so-called 'Sargent's Rule') quoted over page:

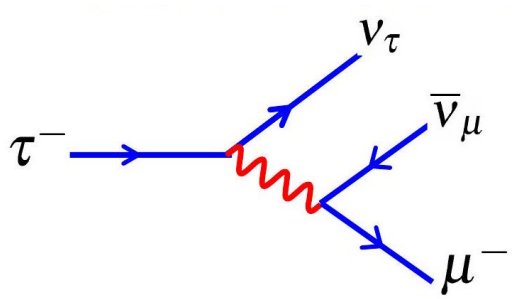
Measuring the ratio of G_F^τ to G_F^μ.

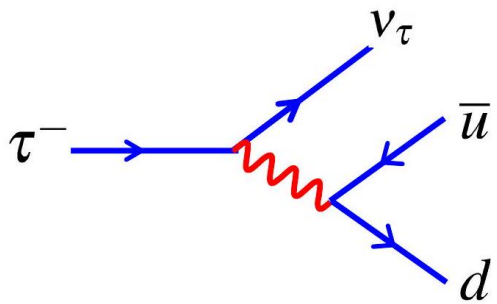
- The muon to electron decay rate is:

$$\Gamma(\mu \to e\nu\bar{\nu}) = \frac{G_F^e G_F^\mu m_\mu^5}{192\pi^3} = \frac{1}{\tau_\mu} \qquad \text{with} \qquad G_F = \frac{g_W^2}{4\sqrt{2}m_W^2}.$$

- Similarly for tau to electron $\Gamma(\tau \to e\nu\bar{\nu}) = \frac{G_F^e G_F^\tau m_\tau^5}{192\pi^3}$ however, the tau can decay to a number of final states:







- Taking the ratio of the above decay rates gives:

$$\frac{\Gamma(\tau \to e\nu\bar{\nu})}{\Gamma(\mu \to e\nu\bar{\nu})} = \frac{G_F^e G_F^\tau m_\tau^5}{G_F^e G_F^\mu m_\mu^5}$$

so

$$\frac{G_F^\tau}{G_F^\mu} = \frac{m_\mu^5 \cdot \Gamma(\tau \to e\nu\bar{\nu})}{m_\tau^5 \cdot \Gamma(\mu \to e\nu\bar{\nu})} = \frac{m_\mu^5}{m_\tau^5} \cdot \frac{\Gamma_\tau BR(\tau \to e\nu\bar{\nu})}{\Gamma_\mu \cdot 1} = \frac{m_\mu^5 \tau_\mu}{m_\tau^5 \tau_\tau} \cdot BR(\tau \to e\nu\bar{\nu}).$$

Charged Current Lepton Universality

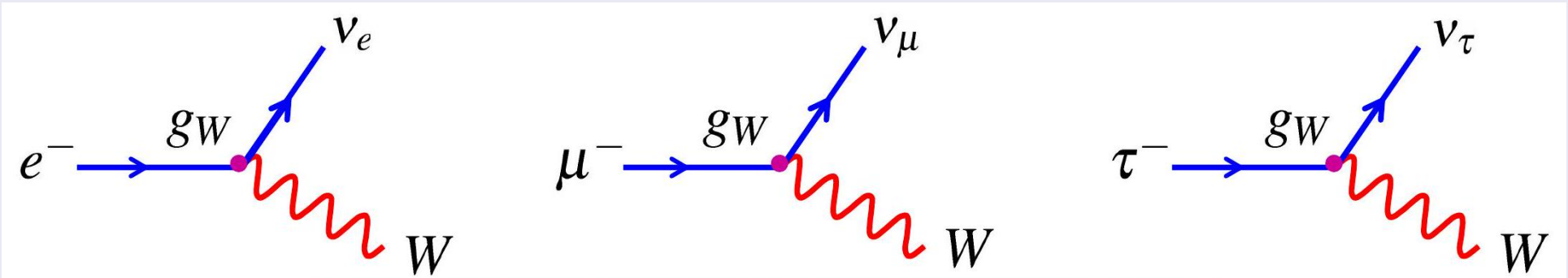
- All these quantities are precisely measured: $\left\{ \begin{array}{l} m_\mu = 0.1056583692(94)\,\mathrm{GeV} \\ m_\tau = 1.77699(28)\,\mathrm{GeV} \\ \tau_\tau = 0.2906(10) \times 10^{-12}\,\mathrm{s} \\ \tau_\mu = 2.19703(4) \times 10^{-6}\,\mathrm{s} \\ Br(\tau \to e\nu\nu) = 0.1784(5) \end{array} \right\}$ so

$$\frac{G_F^\tau}{G_F^\mu} = \frac{m_\mu^5 \tau_\mu}{m_\tau^5 \tau_\tau} Br(\tau \to e\nu\nu) = 1.0024 \pm 0.0033.$$

- Similarly by comparing $Br(\tau \to \mu\nu\nu)$ and $Br(\tau \to e\nu\nu)$

$$\frac{G_F^e}{G_F^\mu} = 1.000 \pm 0.004.$$

The above demonstrates the weak charged current is the same for all leptonic vertices. This is referred to as 'Charged Current Lepton Universality'.



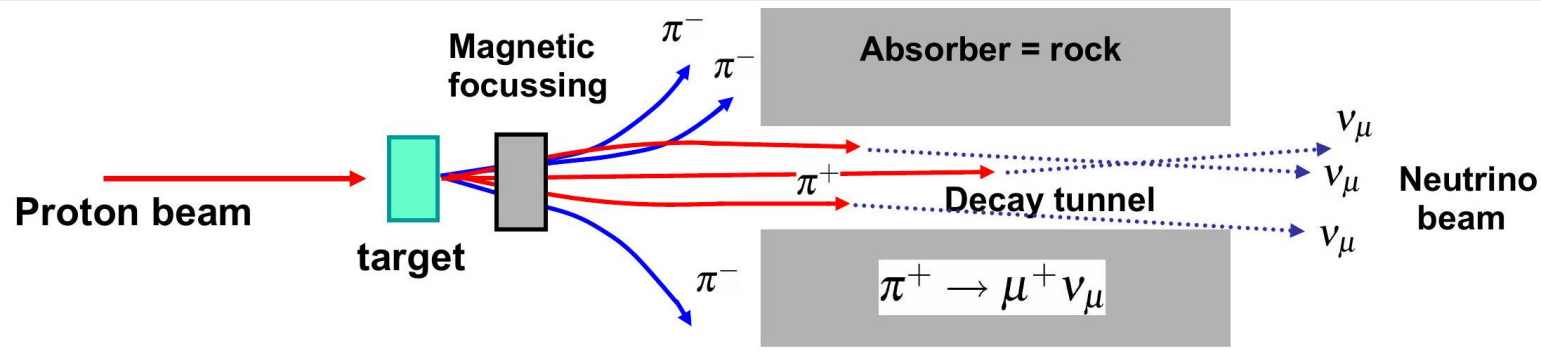
Neutrino Scattering

In Handout 6 considered electron-proton Deep Inelastic Scattering where a virtual photon is used to probe nucleon structure.

- Can also consider the weak interaction equivalent: Neutrino Deep Inelastic Scattering where a virtual W-boson probes the structure of nucleons. **This provides additional information about parton structure functions.**
- provides a good example of calculations of weak interaction cross sections.

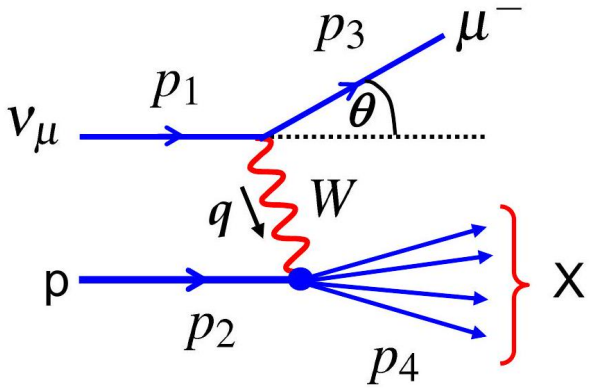
Neutrino beams are needed. To make them:

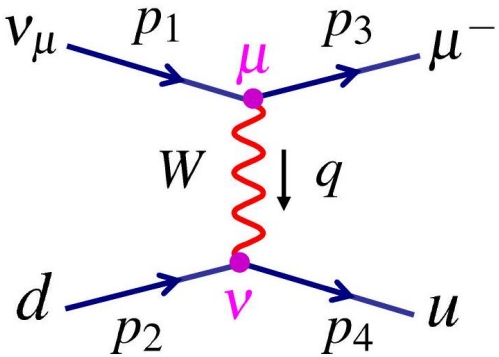
- Smash high energy protons into a fixed target to make hadrons.
- Focus positive pions/kaons.
- Allow them to decay $\pi^+ \to \mu^+\nu_\mu + K^+ \to \mu^+\nu_\mu (BR \approx 64\%)$;
- this gives a beam of "collimated" ν_μ.
- Focus negative pions/kaons to give beam of $\bar{\nu}_\mu$.



Neutrino-Quark Scattering

- For ν_μ-proton Deep Inelastic Scattering the underlying process is $\nu_\mu d \to \mu^- u$





- In the limit $q^2 \ll m_W^2$ the W-boson propagator is $\approx i g_{\mu\nu}/m_W^2$. so the Feynman rules give:

$$-iM_{fi} = \left[-i\frac{g_W}{\sqrt{2}}\bar{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1)\right]\frac{ig_{\mu\nu}}{m_W^2}\left[-i\frac{g_W}{\sqrt{2}}\bar{u}(p_4)\tfrac{1}{2}\gamma^\nu(1-\gamma^5)u(p_2)\right]$$

so

$$\boxed{M_{fi} = \frac{g_W^2}{2m_W^2}g_{\mu\nu}\left[\bar{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1)\right]\left[\bar{u}(p_4)\tfrac{1}{2}\gamma^\nu(1-\gamma^5)u(p_2)\right].}$$

- Evaluate the matrix element in the extreme relativistic limit where the muon and quark masses can be neglected.

- In this limit the helicity states are equivalent to the chiral states. Furthermore

$$\frac{1}{2}(1-\gamma^5)u_\uparrow(p_1)=0 \quad \text{and} \quad \frac{1}{2}(1-\gamma^5)u_\downarrow(p_1)=u_\downarrow(p_1)$$

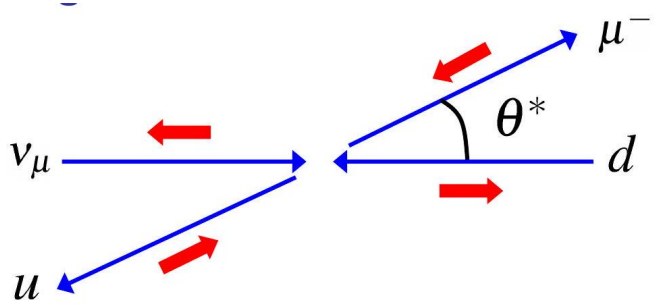
  so $M_{fi}=0$ for $u_\uparrow(p_1)$ and $u_\uparrow(p_2)$.

- Since the weak interaction 'conserves the helicity', the only helicity combination where the matrix element is non-zero is

$$M_{fi}=\frac{g_W^2}{2m_W^2}g_{\mu\nu}\left[\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right]\left[\bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)\right].$$

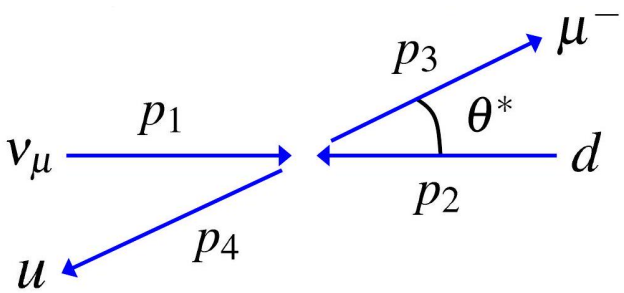
  (We could have written this down straight away as in the ultra-relativistic limit only LH helicity particle states participate in the weak interaction.)

- We will next consider this scattering in the C.o.M frame:



Evaluation of Neutrino-Quark Scattering Matrix Element

- Go through the calculation in gory detail (fortunately only one helicity combination)
- In the $\nu_\mu d$ CMS frame, neglecting particle masses:



$$p_1 = (E, 0, 0, E),$$
$$p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$$
$$p_4 = (E, -E\sin\theta^*, 0, -E\cos\theta^*)$$

- Dealing with LH helicity particle spinors. From Handout 2 (page 79), for a massless particle travelling in direction θ, ϕ:

$$u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix} \qquad c = \cos\frac{\theta}{2}; \qquad s = \sin\frac{\theta}{2}$$

- Here $(\theta_1, \phi_1) = (0, 0)$; $(\theta_2, \phi_2) = (\pi, 0)$; $(\theta_3, \phi_3) = (\theta^*, 0)$; $(\theta_4, \phi_4) = (\pi - \theta^*, \pi)$ giving:

$$u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}; \quad u_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}; \quad u_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} -c \\ -s \\ c \\ s \end{pmatrix}.$$

Neutrino-Quark scattering (cont.)

- To calculate

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu} \left[\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right] \left[\bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)\right]$$

need twice to evaluate terms of the form

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4, \\
\overline{\psi}\gamma^1\phi &= \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1, \\
\overline{\psi}\gamma^2\phi &= \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1), \\
\overline{\psi}\gamma^3\phi &= \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2.
\end{aligned}$$

- Using

$$u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix};\quad u_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix};\quad u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix};\quad u_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}-c\\-s\\c\\s\end{pmatrix}$$

we get

$$\boxed{\begin{aligned}\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2E(c, s, -is, c)\\ \bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2) &= 2E(c, -s, -is, -c)\end{aligned}}$$

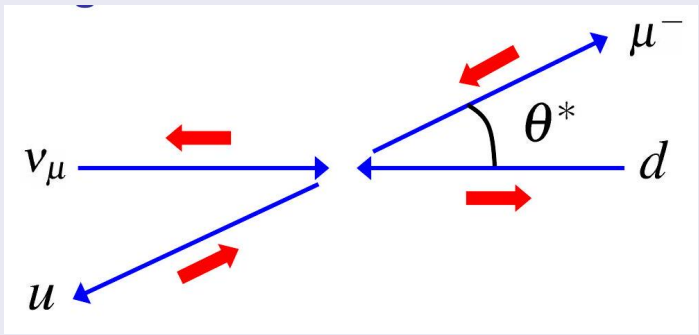
$$\Longrightarrow \boxed{M_{fi} = \frac{g_W^2}{2m_W^2} 4E^2(c^2 + s^2 + s^2 + c^2) = \frac{g_W^2 \hat{s}}{m_W^2}} \qquad \text{where} \quad \hat{s} = (2E)^2.$$

Neutrino-Quark scattering (cont.)

Note that the Matrix Element is isotropic:

$$M_{fi} = \frac{g_W^2}{m_W^2}\hat{s}.$$

We could have anticipated this since the helicity combination (spins anti-parallel) has $S_z = 0$ so expect no preferred polar angle.



- As before need to sum over all possible spin states and average over all possible initial state spin states. Here only one possible spin combination ($LL \to LL$) and only two possible initial state combinations (the neutrino is always produced in a LH helicity state).

$$\left\langle |M_{fi}|^2 \right\rangle = \frac{1}{2}\left|\frac{g_W^2}{m_W^2}\hat{s}\right|^2$$

  The factor of a half arises because half of the time the quark will be in a RH states and won't participate in the charged current Weak interaction.
- From Handout 1, in the extreme relativistic limit, the cross section for any $2 \to 2$ body scattering process is

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2\hat{s}}\langle |M_{fi}|^2 \rangle.$$

Neutrino-Quark scattering (conclusion).

- Therefore:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2\hat{s}}\langle|M_{fi}|^2\rangle = \frac{1}{64\pi\hat{s}}\frac{1}{2}\left(\frac{g_W^2\hat{s}}{m_W^2}\right)^2 = \left(\frac{g_W^2}{8\sqrt{2}\pi m_W^2}\right)^2\hat{s}.$$

- Using $\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}$ the above simplifies to

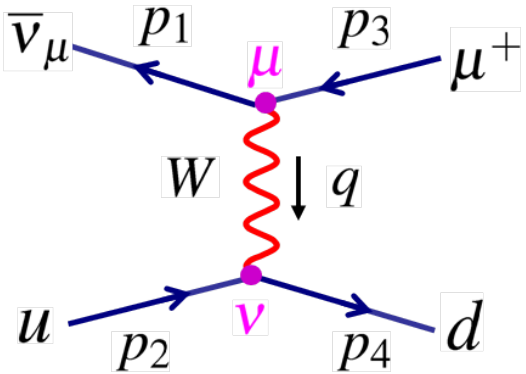
$$\boxed{\frac{d\sigma}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}} \tag{115}$$

and integrating this isotropic distribution over $d\Omega^*$ gives

$$\boxed{\sigma_{\nu q} = \frac{G_F^2\hat{s}}{\pi}} \tag{116}$$

- Since the cross section is a (longitudinally) Lorentz invariant, (116) is also the cross section for scattering in the lab frame.

Antineutrino-Quark Scattering



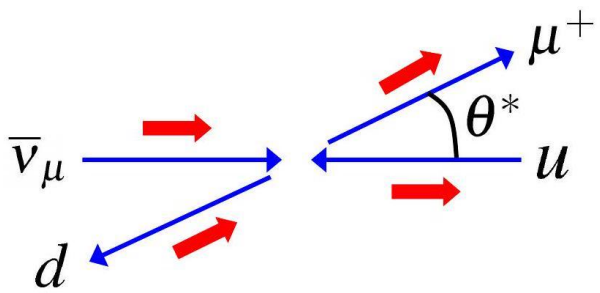
- In the ultra-relativistic limit, the charged-current interaction matrix element is:

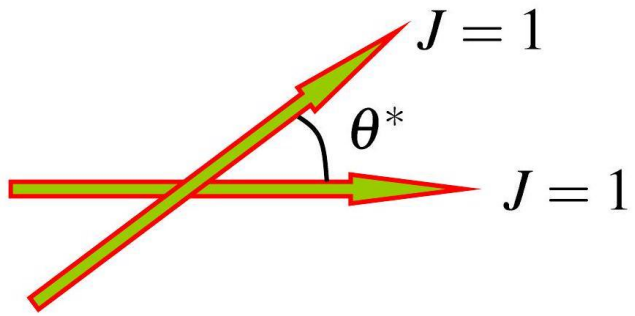
$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu} \left[\bar{v}(p_1)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_3)\right] \left[\bar{u}(p_4)\gamma^\nu \tfrac{1}{2}(1-\gamma^5)u(p_2\right.$$

- In the extreme relativistic limit only LH Helicity particles and RH Helicity anti-particles participate in the charged current weak interaction:

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu} \left[\bar{v}_\uparrow(p_1)\gamma^\mu v_\uparrow(p_3)\right] \left[\bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)\right]$$

In terms of the particle spins it can be seen that the interaction occurs in a total angular momentum $J = 1$ state:

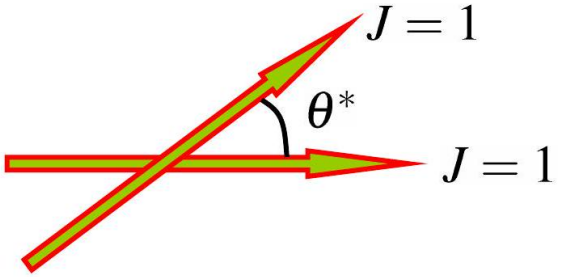




Antineutrino-Quark Scattering (cont.)

- Similarly to the neutrino-quark scattering calculation obtain:

$$\frac{d\sigma_{\bar{\nu}q}}{d\Omega^*} = \frac{d\sigma_{\nu q}}{d\Omega^*}\frac{1}{4}(1+\cos\theta^*)^2$$



- The factor $\frac{1}{4}(1+\cos\theta^*)^2$ can be understood in terms of the overlap of the initial and final angular momentum wave-functions.
- Integrating over solid angle:

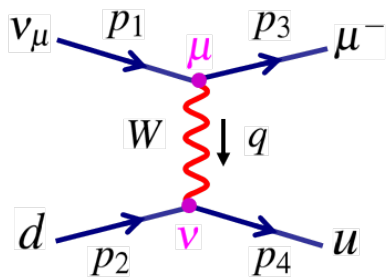
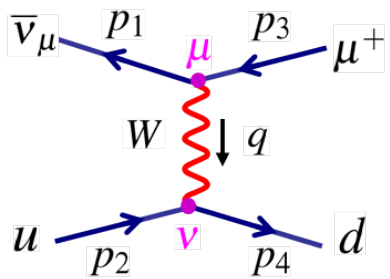
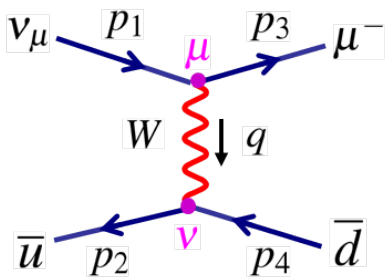
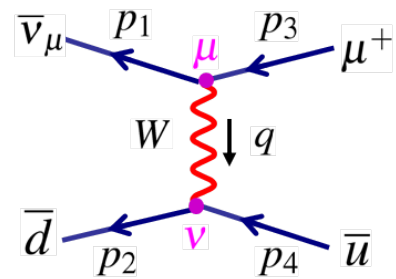
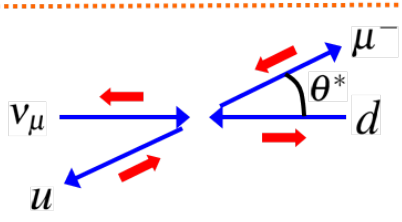
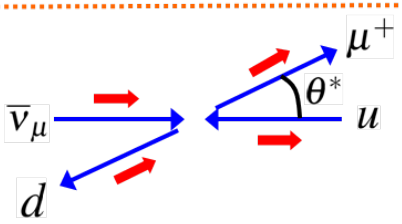
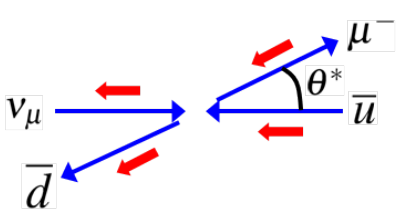
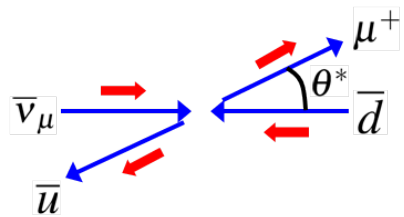
$$\int (1+\cos\theta^*)^2\, d\Omega^* = 2\pi \int_{-1}^{+1} (1+\cos\theta^*)^2\, d(\cos\theta^*) = \frac{16\pi}{3}$$

$$\Longrightarrow \qquad \boxed{\sigma_{\bar{\nu}q} = \frac{G_F^2 \hat{s}}{3\pi}}.$$

- This is a factor three smaller than the neutrino quark cross-section: $\boxed{\frac{\sigma_{\bar{\nu}q}}{\sigma_{\nu q}} = \frac{1}{3}}$.

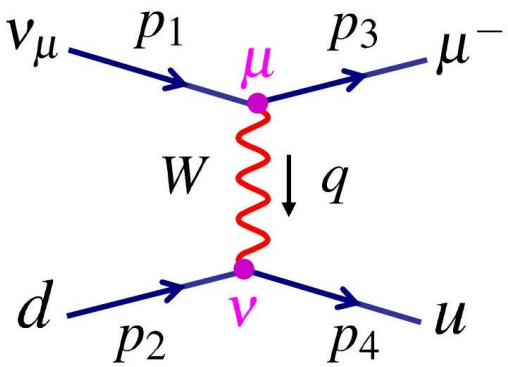
(Anti)neutrino-(Anti)quark Scattering

- Non-zero anti-quark component to the nucleon also consider scattering from q
- Cross-sections can be obtained immediately by comparing with quark scattering and remembering to only include LH particles and RH anti-particles

			
			
$S_z = 0$	$S_z = +1$	$S_z = -1$	$S_z = 0$
$\frac{d\sigma_{\nu q}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$	$\frac{d\sigma_{\bar{\nu} q}}{d\Omega^*} = \frac{G_F^2}{16\pi^2}(1+\cos\theta^*)^2\hat{s}$	$\frac{d\sigma_{\nu \bar{q}}}{d\Omega^*} = \frac{G_F^2}{16\pi^2}(1+\cos\theta^*)^2\hat{s}$	$\frac{d\sigma_{\bar{\nu}\bar{q}}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$
$\sigma_{\nu q} = \frac{G_F^2\hat{s}}{\pi}$	$\sigma_{\bar{\nu} q} = \frac{G_F^2\hat{s}}{3\pi}$	$\sigma_{\nu \bar{q}} = \frac{G_F^2\hat{s}}{3\pi}$	$\sigma_{\bar{\nu}\bar{q}} = \frac{G_F^2\hat{s}}{\pi}$

Differential Cross Section in form $d\sigma/dy$

- We derived the differential neutrino scattering cross sections in C.o.M frame. Let us convert them to Lorentz invariant form.



- As for DIS use Lorentz invariant $y \equiv \dfrac{p_2.q}{p_2.p_1}$.
- In relativistic limit y can be expressed in terms of the C.o.M. scattering angle: $y = \frac{1}{2}(1 - \cos\theta^*)$.
- In the lab. frame $y = 1 - \dfrac{E_3}{E_1}$.

- Convert from $\dfrac{d\sigma}{d\Omega^*} \to \dfrac{d\sigma}{dy}$ using

$$\frac{d\sigma}{dy} = \left|\frac{d\cos\theta^*}{dy}\right| \frac{d\sigma}{d\cos\theta^*} = \left|\frac{d\cos\theta^*}{dy}\right| 2\pi \frac{d\sigma}{d\Omega^*} = 4\pi \frac{d\sigma}{d\Omega^*}.$$

- But we already found (in (115)) that: $\dfrac{d\sigma}{d\Omega^*} = \dfrac{G_F^2}{4\pi^2}\hat{s}$ hence:

$$\boxed{\frac{d\sigma_{\nu q}}{dy} = \frac{d\sigma_{\bar{\nu}\bar{q}}}{dy} = \frac{G_F^2}{\pi}\hat{s}}.$$

Similarly

$$\frac{d\sigma_{\bar{\nu}q}}{d\Omega^*} = \frac{d\sigma_{\nu\bar{q}}}{d\Omega^*} = \frac{G_F^2}{16\pi^2}(1+\cos\theta^*)^2\hat{s}$$

becomes

$$\boxed{\frac{d\sigma_{\bar{\nu}q}}{dy} = \frac{d\sigma_{\nu\bar{q}}}{dy} = \frac{G_F^2}{\pi}(1-y)^2\hat{s}}$$

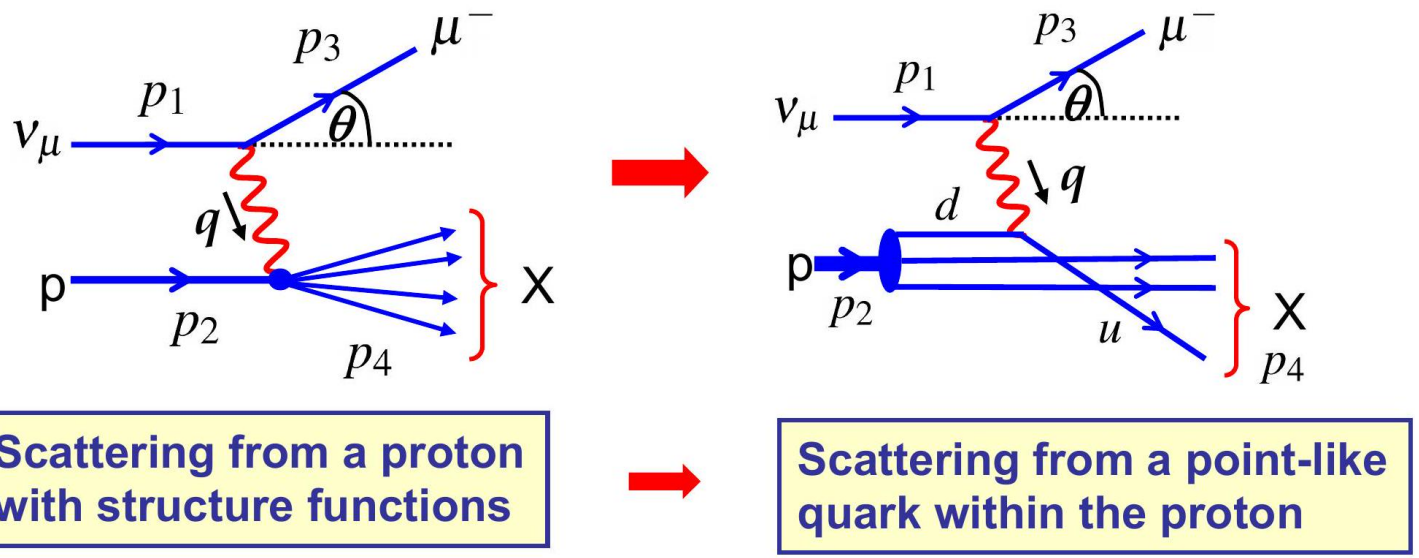
through use of $y = \frac{1}{2}(1-\cos\theta^*) \;\rightarrow 1-y = \frac{1}{2}(1+\cos\theta^*)$.
For comparison, the electro-magnetic $e^{\pm}q \rightarrow e^{\pm}q$ cross section is:

QED: $$\frac{d\sigma_{e^{\pm}q}}{dy} = \frac{2\pi\alpha^2}{q^4}e_q^2\left[1+(1-y)^2\right]\hat{s}$$

Weak: $$\frac{d\sigma_{\bar{\nu}q}}{dy} = \frac{d\sigma_{\nu\bar{q}}}{dy} = \frac{G_F^2}{\pi}(1-y)^2\hat{s}.$$

(Differences are in (i) Helicity Structure, and (ii) in interaction+propagator.)

Parton Model For Neutrino Deep Inelastic Scattering



- **Neutrino**-proton scattering can occur via scattering from a **down-quark** or from an **up-antiquark**.
- In the parton model, number of down quarks within the proton in the $x \to x + dx$ momentum fraction range is $d^p(x)\,dx$. Their contribution to the neutrino scattering cross-section is obtained by multiplying by the $\nu_\mu d \to \mu^- u$ cross-section derived previously

$$\frac{d\sigma^{\nu p}}{dy} = \frac{G_F^2}{\pi}\hat{s}d^p(x)\,dx$$

where $\hat{s}$ is the centre-of-mass energy of the $\nu_\mu d$.

- Similarly for the $\bar{u}$ contribution

$$\frac{d\sigma^{\nu p}}{dy} = \frac{G_F^2}{\pi}\hat{s}(1-y)^2\bar{u}^p(x)\,dx.$$

- The neutrino-**proton** scattering cross section is obtained by summing the two contributions and using $\hat{s} = xs$ to give:

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2}{\pi}sx\left[d^p(x) + (1-y)^2\bar{u}^p(x)\right].$$

- The anti-neutrino **proton** differential cross section can be obtained in the same manner:

$$\frac{d^2\sigma^{\bar{\nu} p}}{dx\,dy} = \frac{GF^2}{\pi}sx\left[(1-y)^2u^p(x) + \bar{d}^p(x)\right].$$

- For (anti)neutrino-**neutron** scattering:

$$\frac{d^2\sigma^{\nu n}}{dx\,dy} = \frac{G_F^2}{\pi}sx\left[d^n(x) + (1-y)^2\bar{u}^n(x)\right]$$

$$\frac{d^2\sigma^{\bar{\nu} n}}{dx\,dy} = \frac{G_F^2}{\pi}sx\left[(1-y)^2u^n(x) + \bar{d}^n(x)\right].$$

- As before, define neutron distributions functions in terms of those of the proton
  $u(x) \equiv u^{\mathrm{p}}(x) = d^{\mathrm{n}}(x); \qquad d(x) \equiv d^{\mathrm{p}}(x) = u^{\mathrm{n}}(x)$
  $\overline{u}(x) \equiv \overline{u}^{\mathrm{p}}(x) = \overline{d}^{\mathrm{n}}(x); \qquad \overline{d}(x) \equiv \overline{d}^{\mathrm{p}}(x) = \overline{u}^{\mathrm{n}}(x)$

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2}{\pi} sx \left[d(x) + (1-y)^2 \bar{u}(x) \right] \tag{117}$$

$$\frac{d^2\sigma^{\bar{\nu} p}}{dx\,dy} = \frac{G_F^2}{\pi} sx \left[(1-y)^2 u(x) + \bar{d}(x) \right] \tag{118}$$

$$\frac{d^2\sigma^{\nu n}}{dx\,dy} = \frac{G_F^2}{\pi} sx \left[u(x) + (1-y)^2 \bar{d}(x) \right] \tag{119}$$

$$\frac{d^2\sigma^{\bar{\nu} n}}{dx\,dy} = \frac{G_F^2}{\pi} sx \left[(1-y)^2 d(x) + \bar{u}(x) \right] \tag{120}$$

- Because neutrino cross sections are very small, need massive detectors. These are usually made of iron, hence, one experimentally measurea a combination of proton/neutron scattering cross sections.

- For an isoscalar target (i.e. equal numbers of protons and neutrons), the mean cross section per nucleon, N:

$$\frac{d^2\sigma^{\nu N}}{dx\,dy} = \frac{1}{2}\left(\frac{d^2\sigma^{\nu p}}{dx\,dy} + \frac{d^2\sigma^{\nu n}}{dx\,dy}\right)$$

$$\Longrightarrow \boxed{\frac{d^2\sigma^{\nu N}}{dx\,dy} = \frac{G_F^2}{2\pi} sx\left[u(x) + d(x) + (1-y)^2(\bar{u}(x) + \bar{d}(x))\right]}.$$

- Integrate over momentum fraction x:

$$\boxed{\frac{d\sigma^{\nu N}}{dy} = \frac{G_F^2}{2\pi} s\left[f_q + (1-y)^2 f_{\bar{q}})\right]} \tag{121}$$

where f_q and $f_{\bar{q}}$ are the total momentum fractions carried by the quarks and by the anti-quarks within a nucleon:

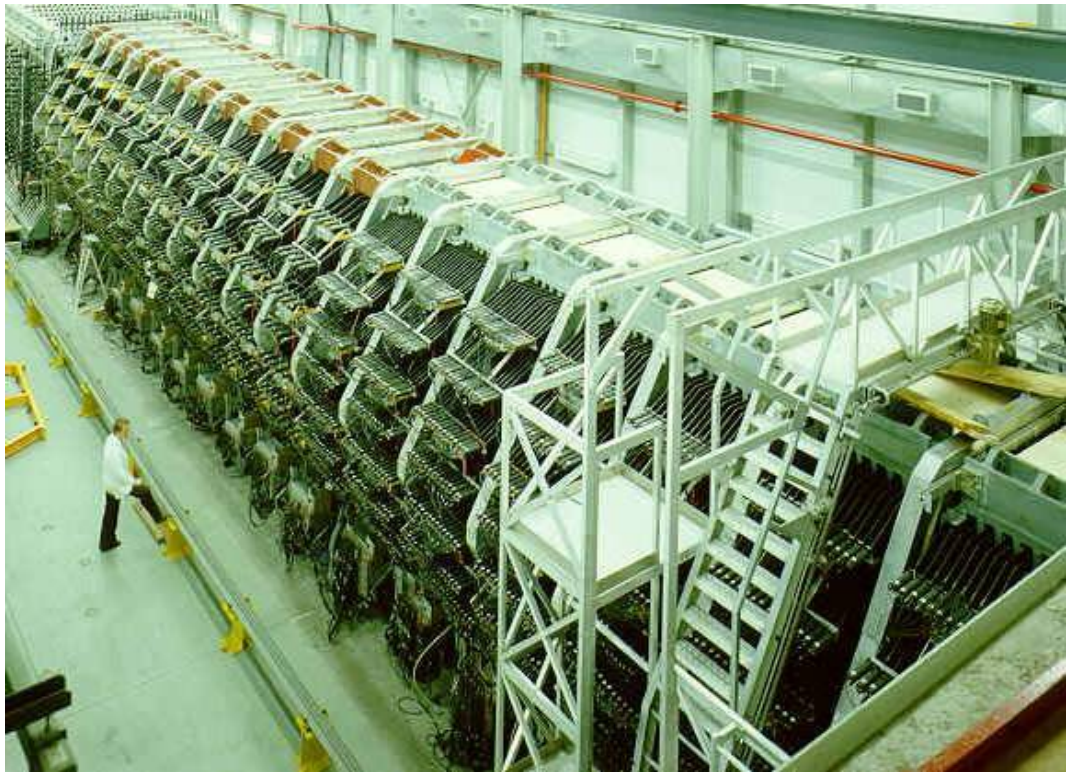
$$f_q \equiv f_d + f_u = \int_0^1 x\left[u(x) + d(x)\right]\,dx; \qquad f_{\bar{q}} \equiv f_{\bar{d}} + f_{\bar{u}} = \int_0^1 x\left[\bar{u}(x) + \bar{d}(x)\right]\,dx.$$

- Similarly

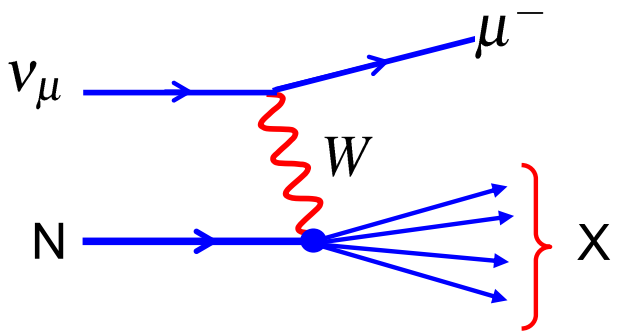
$$\boxed{\frac{d\sigma^{\bar{\nu} N}}{dy} = \frac{G_F^2}{2\pi} s\left[(1-y)^2 f_q + f_{\bar{q}})\right]}. \tag{122}$$

CDHS Experiment (CERN 1976-1984)

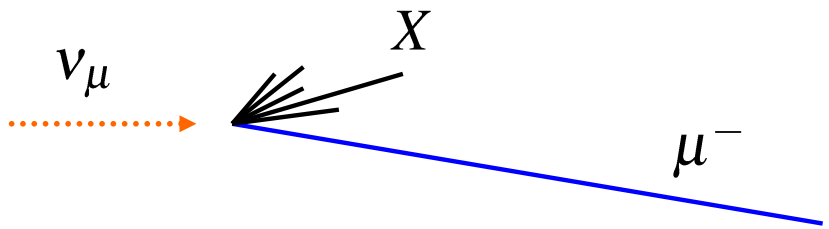
- 1250 tons
- Magnetized iron modules
- Separated by drift chambers



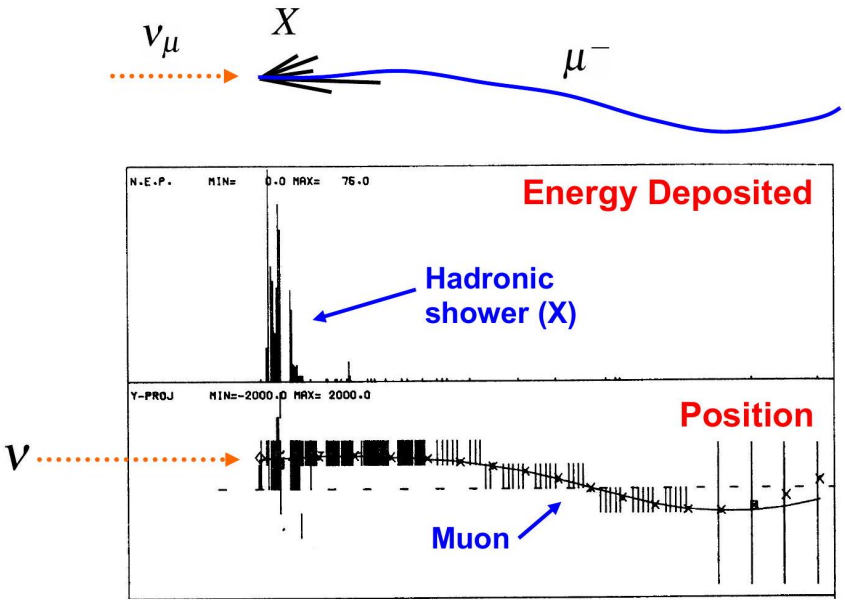
Study Neutrino Deep Inelastic Scattering



Experimental Signature:



CDHS (cont.)



- Measure energy of X, and call it E_X,
- Measure muon momentum from curvature in B-field. and call it E_μ.
- For each event can then determine neutrino energy and y. We get the former by summing the hadronic energy, the muon energy, and subtracting the nucleon rest mass:

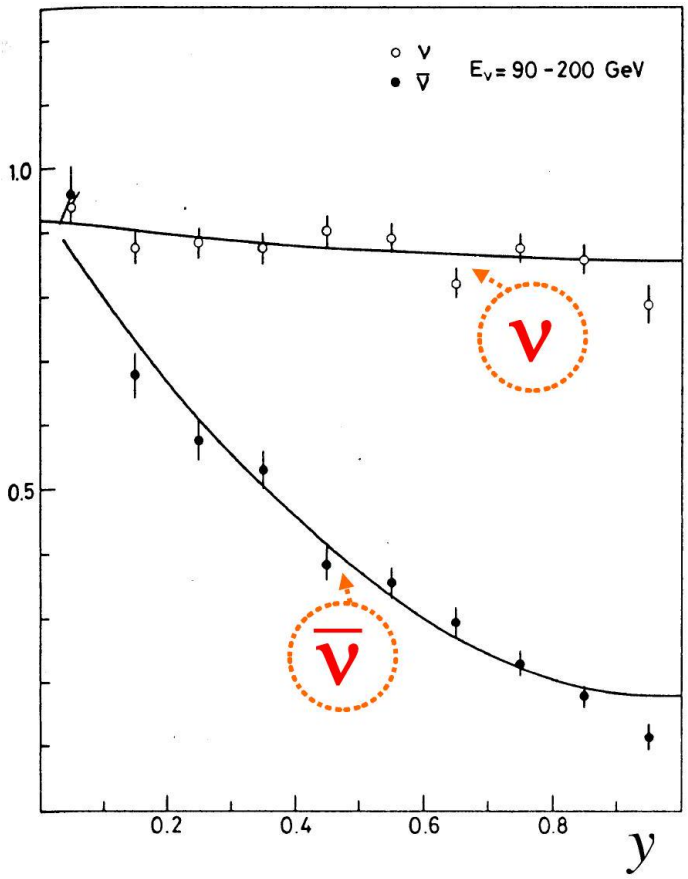
$$E_\nu = E_X + E_\mu - M_N.$$

The latter may be obtained from (98) i.e. from

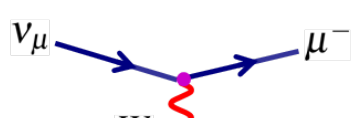
$$y = \left(1 - \frac{E_\mu}{E_\nu}\right).$$

Measured y Distribution

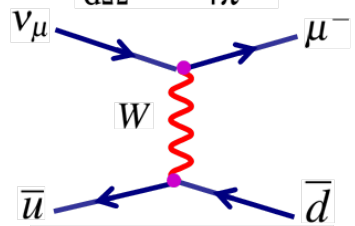
- CDHS measured y distribution
- Shapes can be understood in terms of (anti)neutrino – (anti)quark scattering



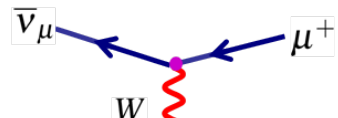
J. de Groot et al., Z.Phys. C1 (1979) 143



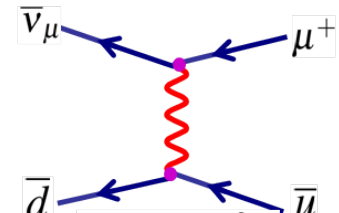
$$\frac{d\sigma_{\nu d}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$$



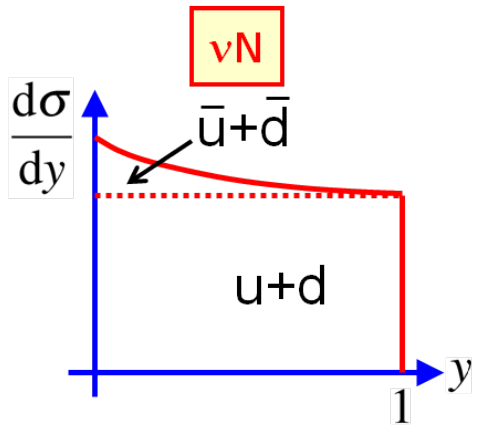
$$\frac{d\sigma_{\nu\bar{u}}}{d\Omega^*} = \frac{G_F^2}{\pi}(1-y)^2\hat{s}$$

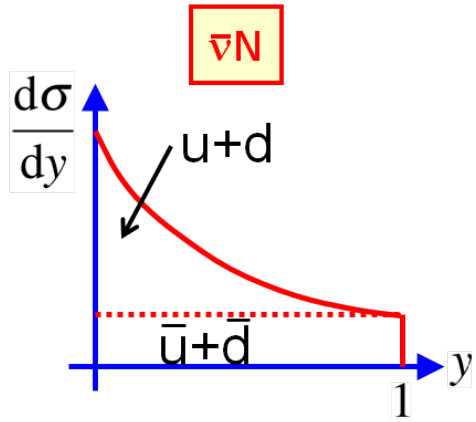


$$\frac{d\sigma_{\bar{\nu}u}}{d\Omega^*} = \frac{G_F^2}{\pi}(1-y)^2\hat{s}$$



$$\frac{d\sigma_{\bar{\nu}\bar{d}}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}$$





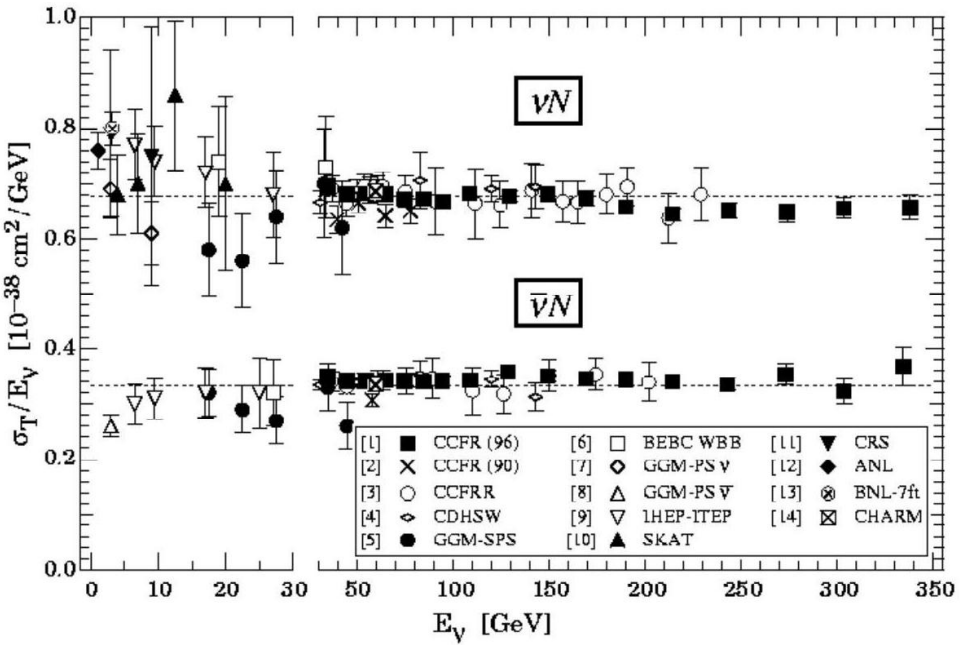
Measured Total Cross Sections

- Integrating the expressions for $\frac{d\sigma}{dy}$ (equations (121) and (122)):

$$\boxed{\sigma^{\nu N} = \frac{G_F^2 s}{2\pi}\left[f_q + \frac{1}{3}f_{\bar{q}}\right]} \quad \text{and} \quad \boxed{\sigma^{\bar{\nu} N} = \frac{G_F^2 s}{2\pi}\left[\frac{1}{3}f_q + f_{\bar{q}}\right]}.$$

- In lab frame $p^\mu(\nu) = (E_\nu, 0, 0, +E_\nu)$ and $p^\mu(p) = (m_p, 0, 0, 0)$ so $s = (E_\nu + m_p)^2 - E_\nu^2 = 2E_\nu m_p + m_p^2 \approx 2E_\nu m_p$ therefore **the Neutrino DIS cross section is approximately proportional to lab. frame neutrino energy!**

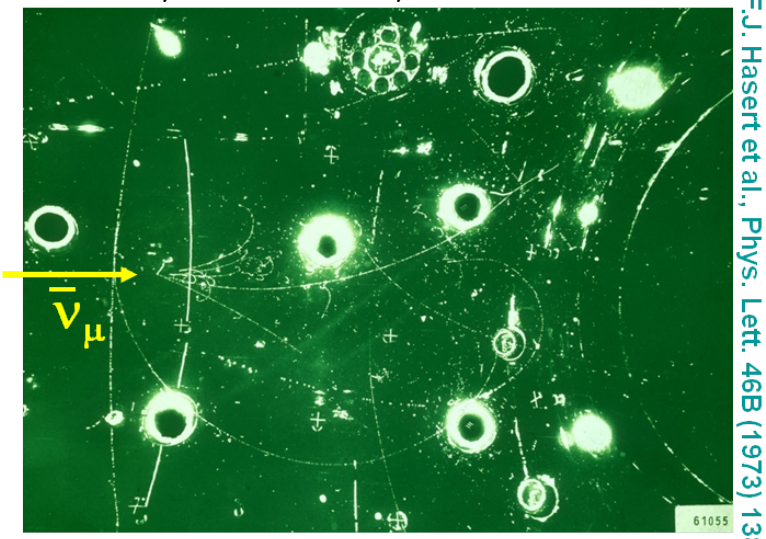
- Measure cross sections can be used to determine fraction of protons momentum carried by quarks, f_q, and fraction carried by anti-quarks, $f_{\bar{q}}$.
- Find: $f_q \approx 0.41$; $f_{\bar{q}} \approx 0.08$.
- $\sim 50\%$ of momentum carried by gluons (which don't interact with virtual W-boson).
- If no anti-quarks in nucleons expect $\frac{\sigma^{\nu N}}{\sigma^{\bar{\nu} N}} = 3$.
- Including anti-quarks $\frac{\sigma^{\nu N}}{\sigma^{\bar{\nu} N}} \approx 2$.



Weak Neutral Current

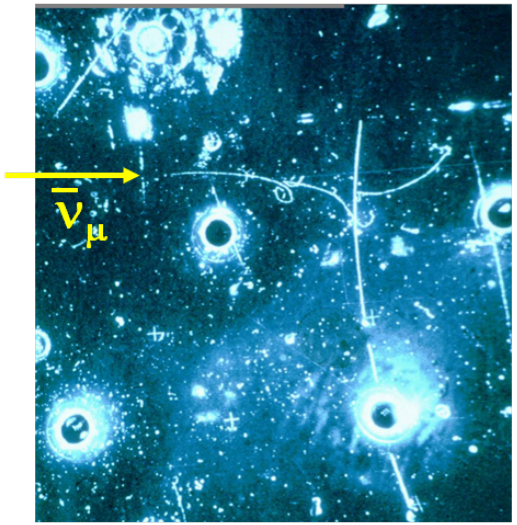
- Neutrinos also interact via the Neutral Current. First observed in the Gargamelle bubble chamber in 1973. Interaction of muon neutrinos produce a final state muon.

$\bar{\nu}_\mu + N \to \bar{\nu}_\mu + \text{hadrons}$



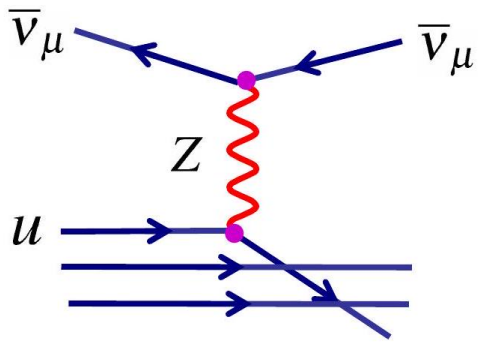
F.J. Hasert et al., Phys. Lett. 46B (1973) 138

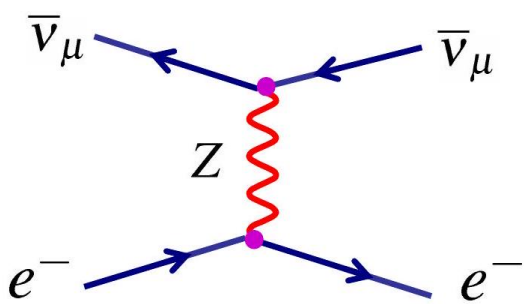
$\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$



F.J. Hasert et al., Phys. Lett. 46B (1973) 121

- Cannot be due to W exchange – this is first evidence for Z-boson!

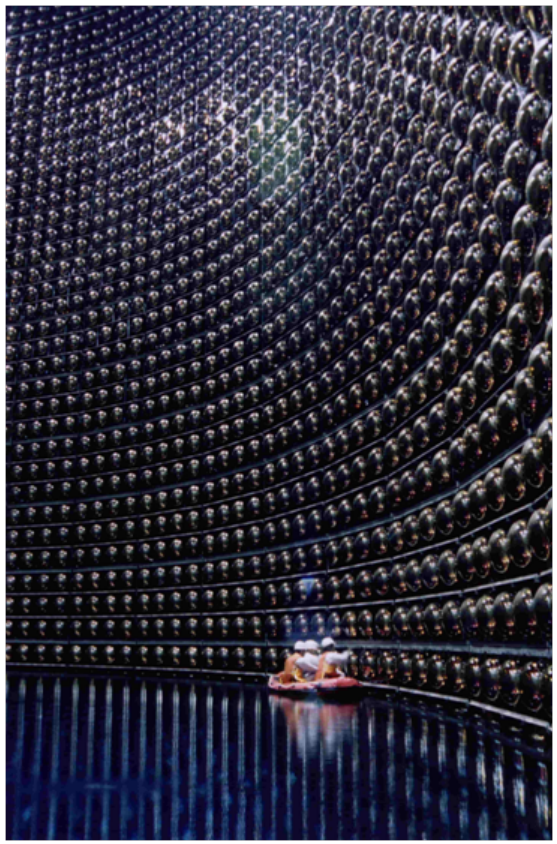




Summary

- Derived neutrino/anti-neutrino – quark/anti-quark weak charged current (CC) interaction cross sections.
- Neutrino-nucleon scattering allows us to measure anti-quark content of protons nad neutrons because:
  - ν couples to d and $\bar{u}$,
  - $\bar{\nu}$ couples to u and $\bar{d}$,
  - $\nu\bar{q}$ scattering is suppressed by factor $(1-y)^2$ compared with νq, and
  - $\bar{\nu}q$ scattering is suppressed by factor $(1-y)^2$ compared with $\bar{\nu}\bar{q}$.
- Further aspects of neutrino deep-inelastic scattering (expressed in general structure functions) are covered in Appendix XIX.
- Finally observe that neutrinos interact via weak neutral currents (NC).

Prof. Alex Mitov, 2026



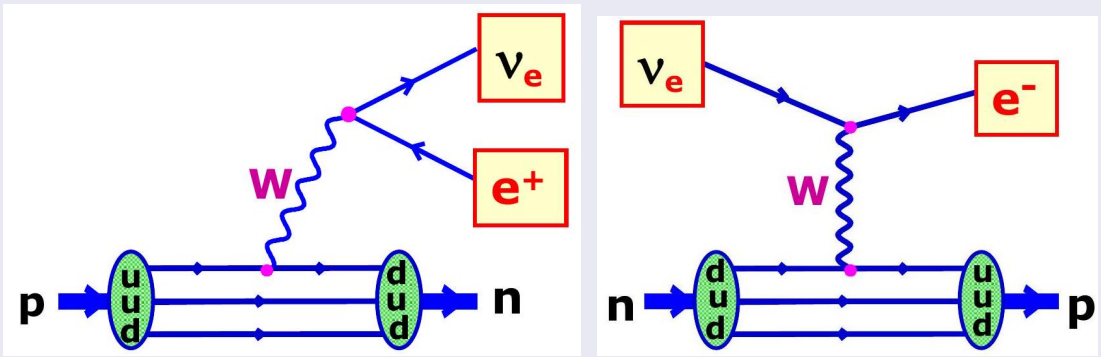


H11: Neutrino Oscillations

Neutrino Flavours Revisited

'Weak' eigenstates: ν_e, ν_μ, ν_τ

Earlier we defined the neutrino weak eigenstates to be the states produced or absorbed in association with an electron, muon or tau lepton respectively.



Mass eigenstates: ν_1, ν_2, ν_3

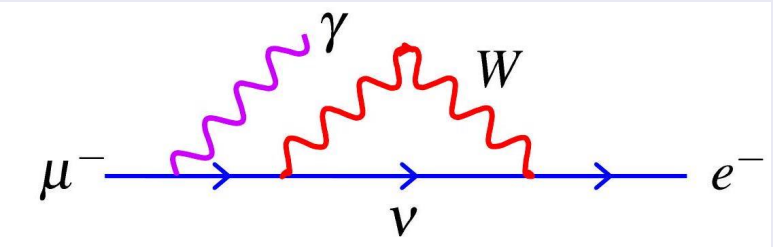
We can define the neutrino mass eigenstates to be the states which have a well defined mass. These propagate without changing as eigenstates of the free Hamiltonian.

Are the mass eigenstates the same as the weak eigenstates?

Evidence at short length-scales appeared to suggest that the mass and flavour eigenstates were identical.

Evidence in support of that historical view included:

- The non-observation of the decay $\mu \to e\gamma$.
  (observations suggest that $\mathrm{BR}\left(\mu^- \to e^-\gamma\right) < 10^{-12}$)



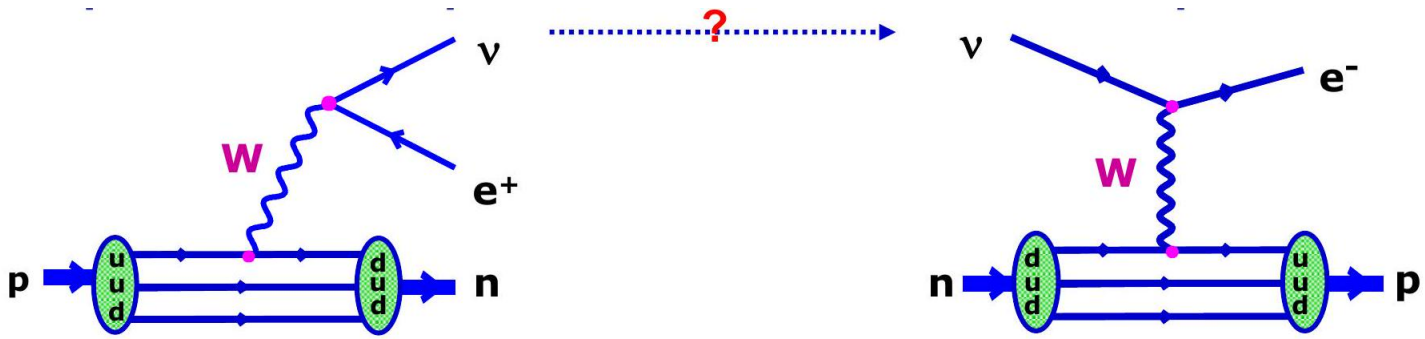
- Neutrinos produced with a given flavour (e.g. ν_e produce with e) that were later observed to undergo a nearby weak interaction with matter were seen to generate leptons which were always the same flavour as the progenitor.

Alas, the mass and weak eigenstates are not the same!

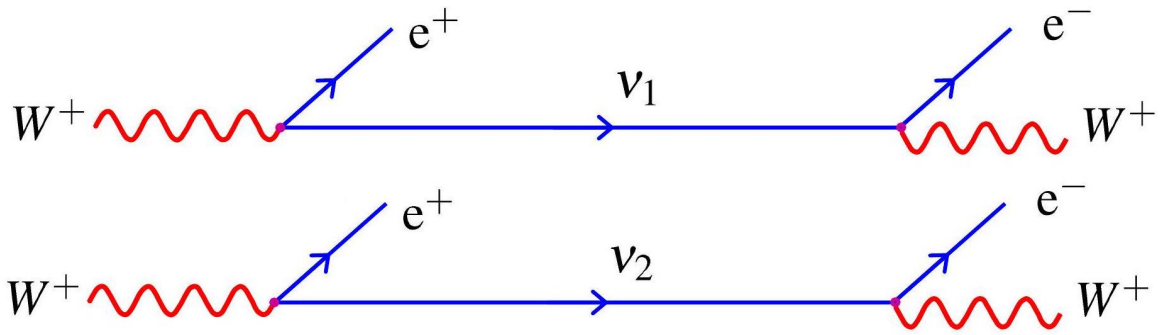
We will later see that **neutrinos change flavour on longer length-scales**, and so we will need a framework in which to account for that difference.

Mass Eigenstates and Weak Eigenstates

- The neutrino 'crossing the gap' between production and destruction remains unobserved while in transit.
- Just as the total amplitude the **double-slit** experiment is an **interference** between ampltides for transists of each slit, the amplitude for a neutrino production-and-destruction process like the one shown below is an interference between amplitudes for the mass eigenstates that could have 'crossed the gap'.
- In other words, a process like this:



can be thought of as an intereference between competing processes like these:



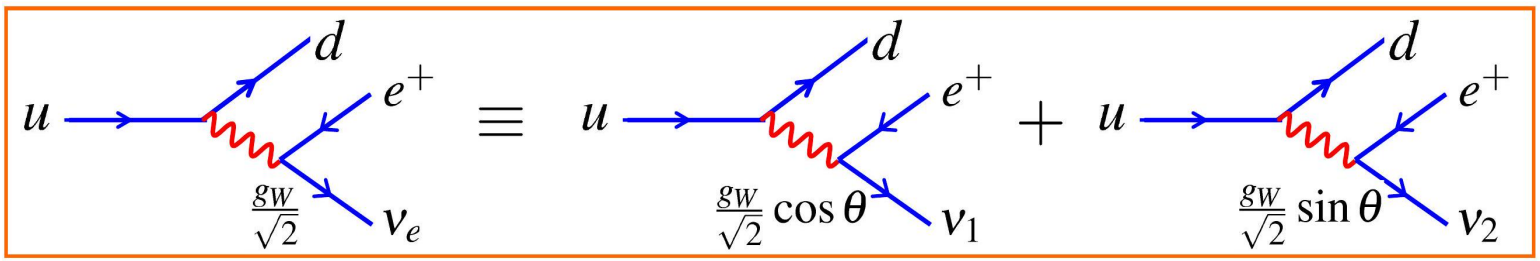
Neutrino Oscillations for Two Flavours

- Neutrinos are produced and interact as weak eigenstates, ν_e, ν_μ.
- The weak eigenstates are coherent linear combinations of the fundamental mass eigenstates ν_1, ν_2.
- There must be a 2×2 unitary change-of-basis matrix that relates the weak and mass eigenstates:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{123}$$

and so

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \tag{124}$$



- The mass eigenstates are the free particle solutions to the wave-equation and evolve as plane waves in the usual way since they have well defined masses:

$$|\nu_1(t)\rangle = |\nu_1\rangle\, e^{i\vec{p}_1\cdot\vec{x} - iE_1 t} \qquad |\nu_2(t)\rangle = |\nu_2\rangle\, e^{i\vec{p}_2\cdot\vec{x} - iE_2 t}.$$

-Suppose at time $t = 0$ a neutrino is produced in a pure ν_e state in a decay $u \to de^+\nu_e$. Then:

$$|\psi(0)\rangle = |\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle\,.$$

-Take the z-axis to be along the neutrino direction.
-The wave-function evolves according to the time-evolution of the mass eigenstates (free particle solutions to the wave equation):

$$|\psi(t)\rangle = \cos\theta\,|\nu_1\rangle\,e^{-ip_1\cdot x} + \sin\theta\,|\nu_2\rangle\,e^{-ip_2\cdot x}$$

where $p_i \cdot x = E_i t - \vec{p}_i \cdot \vec{x} = E_i t - |\vec{p}_i|\,z$.
-Suppose the neutrino interacts in a detector at a distance L and at a time T. Then

$$\boxed{|\psi(L,T)\rangle = \cos\theta\,|\nu_1\rangle\,e^{-i\phi_1} + \sin\theta\,|\nu_2\rangle\,e^{-i\phi_2}}$$

for $\phi_i = p_i \cdot x = E_i T - |\vec{p}_i|\,L$.
-Expressing the mass eigenstates, $|\nu_1\rangle\,, |\nu_2\rangle$, in terms of weak eigenstates using (124) gives:

$$|\psi(L,T)\rangle = \cos\theta\,(\cos\theta\,|\nu_e\rangle - \sin\theta\,|\nu_\mu\rangle)\,e^{-i\phi_1} + \sin\theta\,(\sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle)\,e^{-i\phi_2}$$

and so

$$\boxed{|\psi(L,T)\rangle = |\nu_e\rangle\left(\cos^2\theta e^{-i\phi_1} + \sin^2\theta e^{-i\phi_2}\right) + |\nu_\mu\rangle\sin\theta\cos\theta\left(-e^{-i\phi_1} + e^{-i\phi_2}\right)}\,.$$

- If the masses of $|\nu_1\rangle, |\nu_2\rangle$ are the same, the mass eigenstates remain in phase, $\phi_1 = \phi_2$, and the state remains the linear combination corresponding to $|\nu_e\rangle$ and in a weak interaction will produce an electron
- If the masses are different, the wave-function no longer remains a pure $|\nu_e\rangle$:

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |\langle \nu_\mu \mid \psi(L,T)\rangle|^2 \\ &= \cos^2\theta \sin^2\theta \left(-e^{-i\phi_1} + e^{-i\phi_2}\right)\left(-e^{+i\phi_1} + e^{+i\phi_2}\right) \\ &= \frac{1}{4}\sin^2 2\theta \left(2 - 2\cos\left(\phi_1 - \phi_2\right)\right) \\ &= \sin^2 2\theta \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right). \end{aligned}$$

- Evaluation of the phase difference:

$$\Delta\phi_{12} = \phi_1 - \phi_2 = (E_1 - E_2)\,T - (|p_1| - |p_2|)\,L$$

needs care ...

- (i) One could assume $|p_1| = |p_2| = p$ in which case

$$\Delta\phi_{12} = (E_1 - E_2)\,T = \left[\left(p^2 + m_1^2\right)^{1/2} - \left(p^2 + m_2^2\right)^{1/2}\right] L \text{ using } L \approx (c)\,T.$$

Thus:

$$\Delta\phi_{12} = p\left[\left(1+\frac{m_1^2}{p^2}\right)^{1/2} - \left(1+\frac{m_2^2}{p^2}\right)^{1/2}\right] L \approx \frac{m_1^2 - m_2^2}{2p} L.$$

- However we have neglected that fact that for the same momentum, different mass eigenstates will propagate at different velocities and be observed at different times. A full derivation requires a wave-packet treatment which we will not provide – but the same result obtained above would still be found.
- The last statement may be partly by justified by noting that the phase difference can **also** be written:

$$\Delta\phi_{12} = (E_1 - E_2)\, T - \left(\frac{|p_1|^2 - |p_2|^2}{|p_1| + |p_2|}\right) L$$

i.e. as:

$$\Delta\phi_{12} = (E_1 - E_2)\left[T - \left(\frac{E_1 + E_2}{|p_1| + |p_2|}\right) L\right] + \left(\frac{m_1^2 - m_2^2}{|p_1| + |p_2|}\right) L$$

and the first term on the RHS **vanishes if we assume** (ii) $E_1 = E_2$ **or** (iii) $\beta_1 = \beta_2$.

- **In all three eases:**

$$\boxed{\Delta\phi_{12} \approx \frac{m_1^2 - m_2^2}{2\mathrm{p}} L \approx \frac{\Delta m^2}{2E} L}.$$

⋆ Hence the two-flavour oscillation probability is:

$$P\left(\nu_e \to \nu_\mu\right) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right)$$

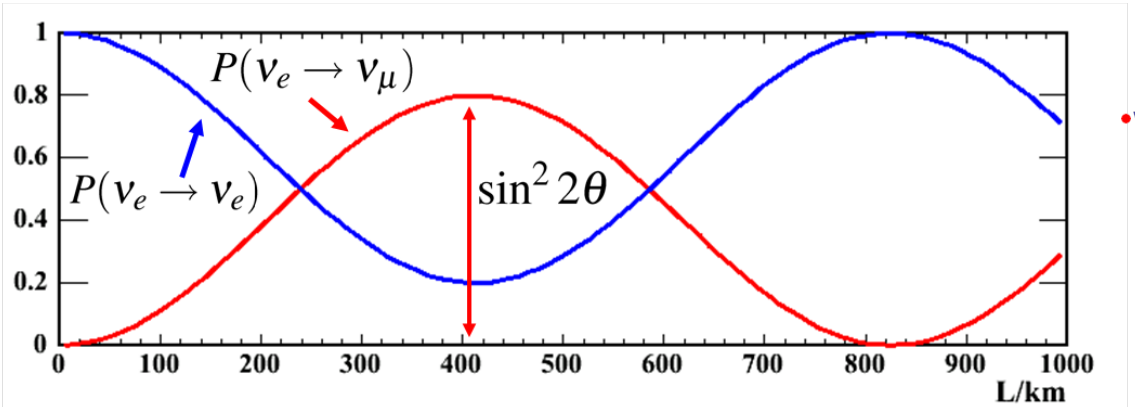
with

$$\Delta m^2_{21} = m^2_2 - m^2_1.$$

⋆ The corresponding two-flavour survival probability is:

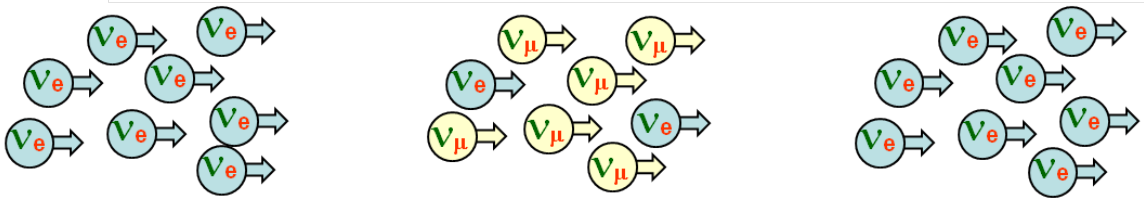
$$P\left(\nu_e \to \nu_e\right) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right).$$

For example, with $\Delta m^2 = 0.003\ \mathrm{eV}^2$, $\sin^2 2\theta = 0.8$ and $E_\nu = 1\ \mathrm{GeV}$ get:



- **wavelength**

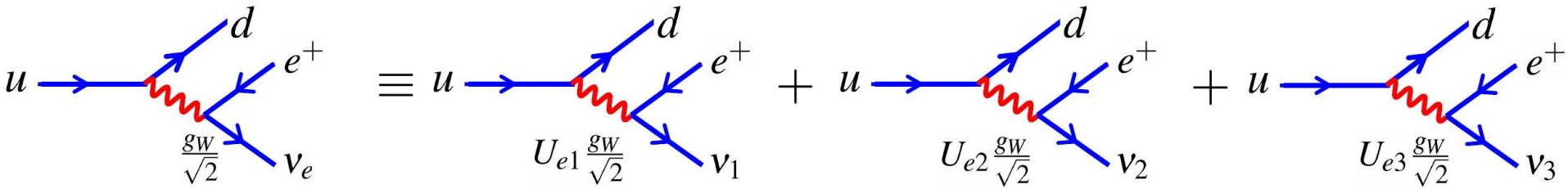
$$\lambda_{\mathrm{osc}} = \frac{4\pi E}{\Delta m^2}$$



Neutrino Oscillations for Three Flavours

⋆ It is simple to extend this treatment to three generations of neutrinos.
⋆ In this case we have:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$



⋆ This 3×3 Unitary matrix U is known as the Pontecorvo-Maki-Nakagawa-Sakata matrix, usually abbreviated 'PMNS Matrix'.
⋆ It has to be unitary to conserve state normalizations.
-Using $U^\dagger U = I \quad \Rightarrow \quad U^{-1} = U^\dagger = (U^*)^T$

gives $$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Unitarity Relations

⋆ The unitarity of the PMNS matrix gives several useful relations:

$$UU^\dagger = I$$

implies that

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so

$$U_{e1}U^*_{e1} + U_{e2}U^*_{e2} + U_{e3}U^*_{e3} = 1 \quad (125)$$
$$U_{\mu 1}U^*_{\mu 1} + U_{\mu 2}U^*_{\mu 2} + U_{\mu 3}U^*_{\mu 3} = 1 \quad (126)$$
$$U_{\tau 1}U^*_{\tau 1} + U_{\tau 2}U^*_{\tau 2} + U_{\tau 3}U^*_{\tau 3} = 1 \quad (127)$$
$$U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} = 0 \quad (128)$$
$$U_{e1}U^*_{\tau 1} + U_{e2}U^*_{\tau 2} + U_{e3}U^*_{\tau 3} = 0 \quad (129)$$
$$U_{\mu 1}U^*_{\tau 1} + U_{\mu 2}U^*_{\tau 2} + U_{\mu 3}U^*_{\tau 3} = 0. \quad (130)$$

Three flavour oscillations (cont.) I

$\star$ To calculate the oscillation probability proceed as before by first considering a state which is produced at $t = 0$ as a $|\nu_e\rangle$:

$$|\psi(t=0)\rangle = |\nu_e\rangle = U_{e1}\,|\nu_1\rangle + U_{e2}\,|\nu_2\rangle + U_{e3}\,|\nu_3\rangle$$

-The wave-function evolves as:

$$|\psi(t)\rangle = U_{e1}\,|\nu_1\rangle\, e^{-ip_1\cdot x} + U_{e2}\,|\nu_2\rangle\, e^{-ip_2\cdot x} + U_{e3}\,|\nu_3\rangle\, e^{-ip_3\cdot x}$$

where

$$p_i \cdot x = E_i t - \vec{p}_i \cdot \vec{x} = E_i t - |\vec{p}| z.$$

-After travelling a distance L in the z-direction:

$$|\psi(L)\rangle = U_{e1}\,|\nu_1\rangle\, e^{-i\phi_1} + U_{e2}\,|\nu_2\rangle\, e^{-i\phi_2} + U_{e3}\,|\nu_3\rangle\, e^{-i\phi_3}$$

where $\phi_i = p_i.x = E_i t - |\vec{p}| L = (E_i - |\vec{p}_i|)\, L$.
-As before we can approximate

$$\phi_i \approx \frac{m_i^2}{2E_i} L.$$

Three flavour oscillations (cont.) II

-Expressing the mass eigenstates in terms of the weak eigenstates we find

$$\begin{aligned}|\psi(L)\rangle = U_{e1}\left(U_{e1}^{*}|\nu_e\rangle + U_{\mu 1}^{*}|\nu_\mu\rangle + U_{\tau 1}^{*}|\nu_\tau\rangle\right)e^{-i\phi_1} \\ + U_{e2}\left(U_{e2}^{*}|\nu_e\rangle + U_{\mu 2}^{*}|\nu_\mu\rangle + U_{\tau 2}^{*}|\nu_\tau\rangle\right)e^{-i\phi_2} \\ + U_{e3}\left(U_{e3}^{*}|\nu_e\rangle + U_{\mu 3}^{*}|\nu_\mu\rangle + U_{\tau 3}^{*}|\nu_\tau\rangle\right)e^{-i\phi_3}\end{aligned}$$

which can be rearranged to give

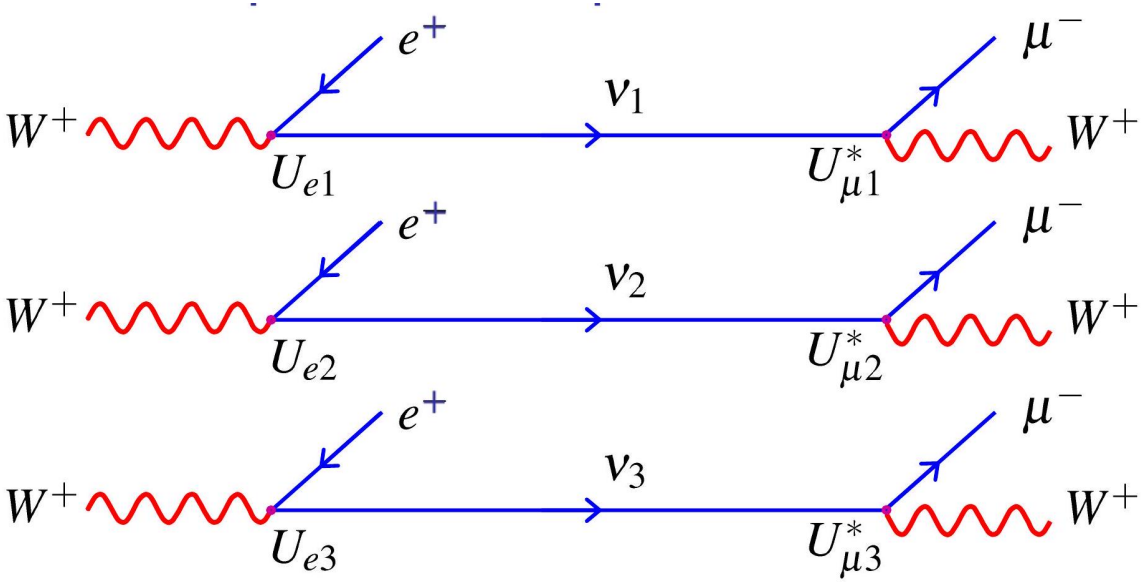
$$\begin{aligned}|\psi(L)\rangle = \left(U_{e1}U_{e1}^{*}e^{-i\phi_1} + U_{e2}U_{e2}^{*}e^{-i\phi_2} + U_{e3}U_{e3}^{*}e^{-i\phi_3}\right)|\nu_e\rangle \\ + \left(U_{e1}U_{\mu 1}^{*}e^{-i\phi_1} + U_{e2}U_{\mu 2}^{*}e^{-i\phi_2} + U_{e3}U_{\mu 3}^{*}e^{-i\phi_3}\right)|\nu_\mu\rangle \\ + \left(U_{e1}U_{\tau 1}^{*}e^{-i\phi_1} + U_{e2}U_{\tau 2}^{*}e^{-i\phi_2} + U_{e3}U_{\tau 3}^{*}e^{-i\phi_3}\right)|\nu_\tau\rangle\,. \qquad (131)\end{aligned}$$

-Therefore

$$\begin{aligned}P\left(\nu_e \to \nu_\mu\right) &= |\langle\nu_\mu \mid \psi(L)\rangle|^2 \\ &= \left|U_{e1}U_{\mu 1}^{*}e^{-i\phi_1} + U_{e2}U_{\mu 2}^{*}e^{-i\phi_2} + U_{e3}U_{\mu 3}^{*}e^{-i\phi_3}\right|^2. \qquad (132)\end{aligned}$$

Three flavour oscillations (cont.) III

-The terms in the last expression (132) can be depicted as follows:



-Because of the unitarity of the PMNS matrix we have (128):

$$U_{e1}U^*_{\mu 1} + U_{e2}U^*_{\mu 2} + U_{e3}U^*_{\mu 3} = 0.$$

Consequently, unless the phases of the different components are different, the sum of these three diagrams (i.e. the expression (132) for $P(\nu_e \to \nu_\mu)$) is zero.

Conclusion (as before)

We require different neutrino masses for oscillations!

-Simplify

$$P\left(\nu_e \to \nu_\mu\right) = \left| U_{e1}U_{\mu 1}^* e^{-i\phi_1} + U_{e2}U_{\mu 2}^* e^{-i\phi_2} + U_{e3}U_{\mu 3}^* e^{-i\phi_3} \right|^2$$

using the identity

$$\boxed{\left|z_1 + z_2 + z_3\right|^2 \equiv \left|z_1\right|^2 + \left|z_2\right|^2 + \left|z_3\right|^2 + 2\Re\left(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*\right)} \tag{133}$$

giving:

$$P\left(\nu_e \to \nu_\mu\right) = \left|U_{e1}U_{\mu 1}^*\right|^2 + \left|U_{e2}U_{\mu 2}^*\right|^2 + \left|U_{e3}U_{\mu 3}^*\right|^2 + \tag{134}$$
$$2\Re\left(U_{e1}U_{\mu 1}^* U_{e2}^* U_{\mu 2} e^{-i(\phi_1-\phi_2)} + U_{e1}U_{\mu 1}^* U_{e3}^* U_{\mu 3} e^{-i(\phi_1-\phi_3)} + U_{e2}U_{\mu 2}^* U_{e3}^* U_{\mu 3} e^{-i(\phi_2-\phi_3)}\right).$$

-But by applying identity (133) to $\left|(128)\right|^2$ we see that:

$$\left|U_{e1}U_{\mu 1}^* + U_{e2}U_{\mu 2}^* + U_{e3}U_{\mu 3}^*\right|^2 = 0$$

and so

$$\left|U_{e1}U_{\mu 1}^*\right|^2 + \left|U_{e2}U_{\mu 2}^*\right|^2 + \left|U_{e3}U_{\mu 3}^*\right|^2 =$$
$$-2\Re\left(U_{e1}U_{\mu 1}^* U_{e2}^* U_{\mu 2} + U_{e1}U_{\mu 1}^* U_{e3}^* U_{\mu 3} + U_{e2}U_{\mu 2}^* U_{e3}^* U_{\mu 3}\right)$$

which can be substituted into equation (134) to give:

$$\begin{aligned} P\left(\nu_e \to \nu_\mu\right) = & 2\Re\left\{U_{e1}U_{\mu 1}^* U_{e2}^* U_{\mu 2}\left[e^{-i(\phi_1-\phi_2)}-1\right]\right\} \\ & + 2\Re\left\{U_{e1}U_{\mu 1}^* U_{e3}^* U_{\mu 3}\left[e^{-i(\phi_1-\phi_3)}-1\right]\right\} \\ & + 2\Re\left\{U_{e2}U_{\mu 2}^* U_{e3}^* U_{\mu 3}\left[e^{-i(\phi_2-\phi_3)}-1\right]\right\} \end{aligned} . \tag{135}$$

$\star$ The corresponding expression for the electron survival probability (below) may be obtained from the coefficient for $|\nu_e\rangle$ in (131):

$$\begin{aligned} P(\nu_e \to \nu_e) &= |\langle \nu_e \mid \psi(L)\rangle|^2 \\ &= \left| U_{e1}U_{e1}^* e^{-i\phi_1} + U_{e2}U_{e2}^* e^{-i\phi_2} + U_{e3}U_{e3}^* e^{-i\phi_3} \right|^2 . \end{aligned}$$

which using the unitarity relation (125)

$$|U_{e1}U_{e1}^* + U_{e2}U_{e2}^* + U_{e3}U_{e3}^*|^2 = 1.$$

can be written

$$\begin{aligned} P(\nu_e \to \nu_e) = 1 &+ 2\,|U_{e1}|^2\,|U_{e2}|^2\,\Re\{[e^{-i(\phi_1-\phi_2)} - 1]\} \\ &+ 2\,|U_{e1}|^2\,|U_{e3}|^2\,\Re\{[e^{-i(\phi_1-\phi_3)} - 1]\}. \\ &+ 2\,|U_{e2}|^2\,|U_{e3}|^2\,\Re\{[e^{-i(\phi_2-\phi_3)} - 1]\} \end{aligned} \tag{136}$$

The expression (136) can be simplified using

$$\begin{aligned}\Re\left\{e^{-i(\phi_1-\phi_2)}-1\right\} &= \cos\left(\phi_2-\phi_1\right)-1 \\ &= -2\sin^2\left(\frac{\phi_2-\phi_1}{2}\right) \quad \text{with } \phi_i \approx \frac{m_i^2}{2E}L \\ &= -2\sin^2\left(\frac{\left(m_2^2-m_1^2\right)L}{4E}\right) \qquad (137) \\ &= -2\sin^2\left(\Delta_{21}\right). \qquad (138)\end{aligned}$$

where we have defined the dimensionless phase difference:

$$\boxed{\Delta_{ij} = \frac{\left(m_i^2-m_j^2\right)L}{4E} = \frac{\Delta m_{ij}^2 L}{4E}} \qquad (139)$$

with $\Delta m_{21}^2 = m_2^2 - m_1^2$ as before.
-Using the above the electron neutrino survival probability becomes:

$$\boxed{P\left(\nu_e \to \nu_e\right) = 1 - 4\left|U_{e1}\right|^2\left|U_{e2}\right|^2\sin^2\Delta_{21} - 4\left|U_{e1}\right|^2\left|U_{e3}\right|^2\sin^2\Delta_{31} - 4\left|U_{e2}\right|^2\left|U_{e3}\right|^2\sin^2\Delta_{32}}.$$

Similar expressions can be obtained for the muon and tau neutrino survival probabilities for muon and tau neutrinos.

Note that since we only have three neutrino generations there are only two independent mass-squared differences, i.e.:

$$m_3^2 - m_1^2 = \left(m_3^2 - m_2^2\right) + \left(m_2^2 - m_1^2\right)$$

so in

$$\boxed{P\left(\nu_e \to \nu_e\right) = 1 - 4\left|U_{e1}\right|^2\left|U_{e2}\right|^2 \sin^2\Delta_{21} - 4\left|U_{e1}\right|^2\left|U_{e3}\right|^2 \sin^2\Delta_{31} - 4\left|U_{e2}\right|^2\left|U_{e3}\right|^2 \sin^2\Delta_{32}}. \tag{140}$$

only two of the Δ_{ij} are independent.

Aside

Conversion to more units commonly used in neutrino physics gives:

$$\Delta_{21} = 1.27\frac{\Delta m_{21}^2\left(\mathrm{eV}^2\right)\ L(\mathrm{km})}{E(\mathrm{GeV})}$$

and

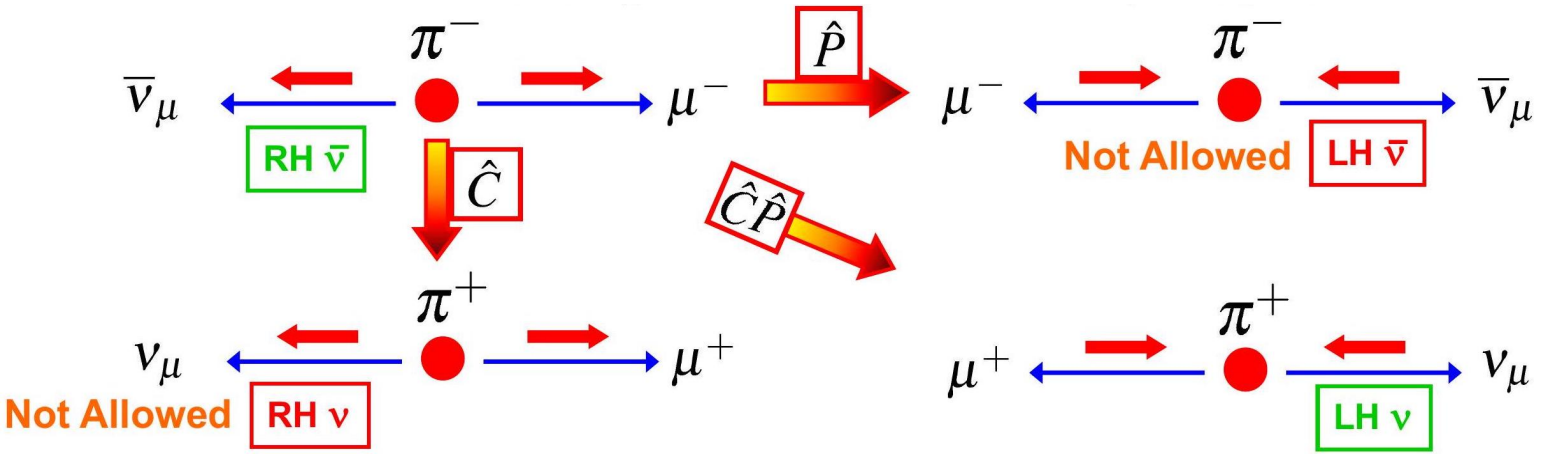
$$\lambda_{\mathrm{osc}}(\mathrm{km}) = 2.47\frac{E(\mathrm{GeV})}{\Delta m^2\left(\mathrm{eV}^2\right)}.$$

CP and CPT in the Weak Interaction

⋆ In addition to parity there are two other important discrete symmetries:

Parity	$\hat{P}$:	$\vec{r} \rightarrow -\vec{r}$
Time Reversal	$\hat{T}$:	$t \rightarrow -t$
Charge Conjugation	$\hat{C}$:	Particle $\longleftrightarrow$ Anti-particle

- The weak interaction violates parity conservation, but what about $\hat{C}$? Consider pion decay remembering that the neutrino is ultra-relativistic and only left-handed neutrinos and right-handed anti-neutrinos participate in weak interaction.



- Hence weak interaction also violates charge conjugation symmetry but appears to be invariant under combined effect of $\hat{C}$ and $\hat{P}$.

- *CP* transforms:

$$\boxed{\begin{array}{ccc} \text{RH Particles} & \longleftrightarrow & \text{LH Anti-particles} \\ \text{LH Particles} & \longleftrightarrow & \text{RH Anti-particles} \end{array}}.$$

- If the weak interaction were invariant under *CP* expect

$$\boxed{\Gamma\left(\pi^+ \to \mu^+ \nu_\mu\right) = \Gamma\left(\pi^- \to \mu^- \bar{\nu}_\mu\right)}.$$

- The *CPT*-Theorem says that all Lorentz invariant Quantum Field Theories make invariant predictions under (charge conjugation) + (parity) + (time reversal).
- The *CPT*-Theorem says that particles/anti-particles have identical mass, lifetime, magnetic moments, ...
- The best current experimental test of the *CPT*-Theorem is perhaps the measurement of $m_{K^0} - m_{\bar{K}^0} < 6 \times 10^{-19} m_{K^0}$ so we currently believe that *CPT* is a good symmetry.
- Therefore, *T*-violation implies *CP*-violation!
- We are interested in *CP* violation as the small excess of matter over anti-matter in the universe may have been generated by it!
- *CP* violation can arise in the weak interaction (see also Handout 12).

CP and T Violation in Neutrino Oscillations

-Previously derived the oscillation probability for $\nu_e \to \nu_\mu$:

$$\begin{aligned} P(\nu_e \to \nu_\mu) = & \, 2\,\mathrm{Re}\left\{ U_{e1} U_{\mu 1}^* U_{e2}^* U_{\mu 2} \left[e^{-i(\phi_1 - \phi_2)} - 1 \right] \right\} \\ & + 2\,\mathrm{Re}\left\{ U_{e1} U_{\mu 1}^* U_{e3}^* U_{\mu 3} \left[e^{-i(\phi_1 - \phi_3)} - 1 \right] \right\} \\ & + 2\,\mathrm{Re}\left\{ U_{e2} U_{\mu 2}^* U_{e3}^* U_{\mu 3} \left[e^{-i(\phi_2 - \phi_3)} - 1 \right] \right\} \end{aligned}$$

-The oscillation probability for $\nu_\mu \to \nu_e$ can be obtained in the same manner or by simply exchanging the labels $(e) \leftrightarrow (\mu)$:

$$\begin{aligned} P(\nu_\mu \to \nu_e) = & \, 2\,\mathrm{Re}\left\{ U_{\mu 1} U_{e1}^* U_{\mu 2}^* U_{e2} \left[e^{-i(\phi_1 - \phi_2)} - 1 \right] \right\} \\ & + 2\,\mathrm{Re}\left\{ U_{\mu 1} U_{e1}^* U_{\mu 3}^* U_{e3} \left[e^{-i(\phi_1 - \phi_3)} - 1 \right] \right\} \\ & + 2\,\mathrm{Re}\left\{ U_{\mu 2} U_{e2}^* U_{\mu 3}^* U_{e3} \left[e^{-i(\phi_2 - \phi_3)} - 1 \right] \right\} \end{aligned} \tag{141}$$

$\star$ Unless the elements of the PMNS matrix are real (ignore any overall complex phase)

$$P(\nu_e \to \nu_\mu) \neq P(\nu_\mu \to \nu_e)\,. \tag{142}$$

so **three-flavour** neutrino oscillations are not invariant under time reversal $t \to -t$.

- Consider the effects of T, CP and CPT on neutrino oscillations:

$$\boxed{T} \quad \nu_\mu \to \nu_e \quad \overset{\hat{T}}{\longleftrightarrow} \quad \nu_e \to \nu_\mu$$

$$\boxed{CP} \quad \nu_\mu \to \nu_e \quad \overset{\hat{C}\hat{P}}{\longleftrightarrow} \quad \bar{\nu}_\mu \to \bar{\nu}_e$$

$$\boxed{CPT} \quad \nu_\mu \to \nu_e \quad \overset{\hat{C}\hat{P}\hat{T}}{\longleftrightarrow} \quad \bar{\nu}_e \to \bar{\nu}_\mu$$

(C alone is not sufficient as it transforms LH neutrinos into LH anti-neutrinos (not involved in Weak Interaction).)

-Thus if the weak interactions is invariant under CPT then

$$P(\nu_\mu \to \nu_e) = P(\bar{\nu}_e \to \bar{\nu}_\mu)$$

which, if combined with (142) tells us that if the PMNS matrix is not purely real, then

$$\boxed{P(\nu_e \to \nu_\mu) \neq P(\bar{\nu}_e \to \bar{\nu}_\mu)}.$$

Hence: unless the PMNS matrix is real, CP will be violated in neutrino oscillations!

Future experiments, e.g. 'a neutrino factory', are being considered as a way to investigate CP violation in neutrino oscillations. However, CP violating effects are well below the current experimental sensitivity. In the following discussion we will take the PMNS matrix to be real.

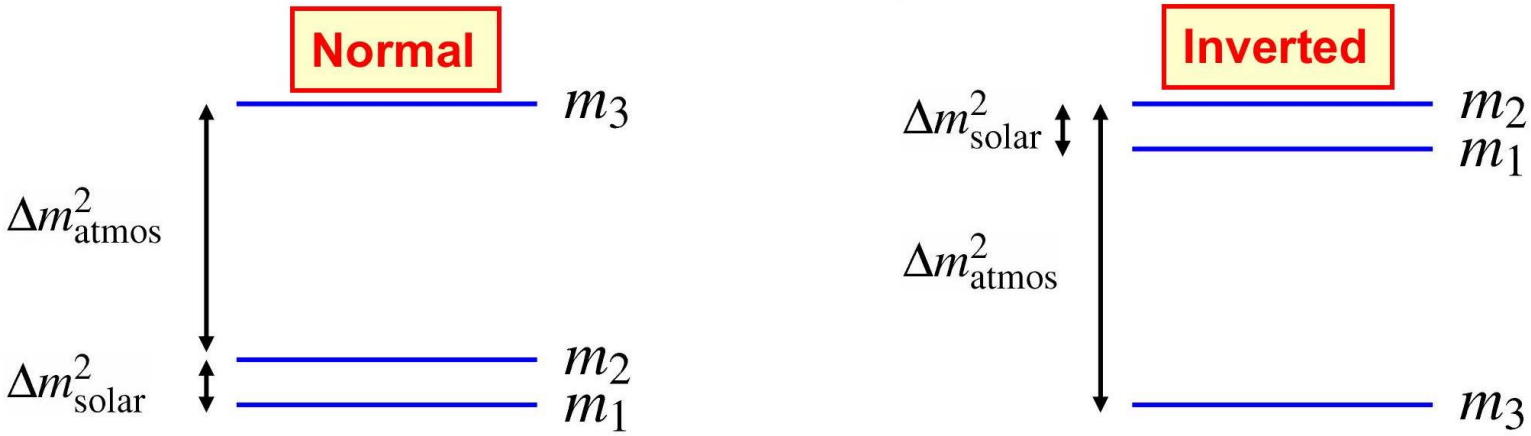
(Now try question 22.)

Neutrino Mass Hierarchy

- Neutrino oscillations to date have found two distinct mass squared differences at different scales:

$$\left|\Delta m^2_{ji}\right| = \left|m^2_j - m^2_i\right| .$$

  - Atmospheric neutrino oscillations : $\left|\Delta m^2\right|_{\text{atmos}} \sim 2.5 \times 10^{-3}\ \text{eV}^2$
  - Solar neutrino oscillations: $\left|\Delta m^2\right|_{\text{solar}} \sim 8 \times 10^{-5}\ \text{eV}^2$
- **Two neutrino mass hierarchies are** still possible (although results since 2024 are starting to favour the 'Normal' hierarchy and the new NOvA experiment might resolve the question soon):



In both cases: $\Delta m^2_{21} \sim 8 \times 10^{-5}\ \text{eV}^2$ (solar)

$$\left|\Delta m^2_{31}\right| \approx \left|\Delta m^2_{32}\right| \sim 2.5 \times 10^{-3}\ \text{eV}^2 \quad \text{(atmospheric)}$$

hence using (139) we approximate $\left|\Delta m^2_{31}\right| \approx \left|\Delta m^2_{32}\right|$ and $\left|\Delta_{31}\right| \approx \left|\Delta_{32}\right|$. (143)

Approx/Simplified Three Flavour Oscillations, neglecting CP Violation

- Taking the PMNS matrix to be real (i.e. neglecting CP violation) considerably simplifies the algebra of three flavour oscillations: E.g. osc. prob. (135) becomes:

$$P(\nu_e \to \nu_\mu) = -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} - 4U_{e1}U_{\mu 1}U_{e3}U_{\mu 3}\sin^2\Delta_{31} - 4U_{e2}U_{\mu 2}U_{e3}U_{\mu 3}\sin^2\Delta_{32}$$

Then using (143) ($|\Delta_{31}| \approx |\Delta_{32}|$) we can further simplify the ν_e **oscillation probablility**:

$$P(\nu_e \to \nu_\mu) \approx -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} - 4\left(U_{e1}U_{\mu 1} + U_{e2}U_{\mu 2}\right)U_{e3}U_{\mu 3}\sin^2\Delta_{32}$$

which can be simplified again using (128) ($U_{e1}U_{\mu 1}^* + U_{e2}U_{\mu 2}^* + U_{e3}U_{\mu 3}^* = 0$) to give

$$\boxed{P(\nu_e \to \nu_\mu) \approx -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} + 4U_{e3}^2U_{\mu 3}^2\sin^2\Delta_{32}}.$$

- Can apply same approximations to the ν_e **survival probability** (140):

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - 4U_{e1}^2U_{e2}^2\sin^2\Delta_{21} - 4U_{e1}^2U_{e3}^2\sin^2\Delta_{31} - 4U_{e2}^2U_{e3}^2\sin^2\Delta_{32} \\ &\approx 1 - 4U_{e1}^2U_{e2}^2\sin^2\Delta_{21} - 4\left(U_{e1}^2 + U_{e2}^2\right)U_{e3}^2\sin^2\Delta_{32} \end{aligned}$$

which can be simplified using (125) $U_{e1}^2 + U_{e2}^2 + U_{e3}^2 = 1$ to give:

$$\boxed{P(\nu_e \to \nu_e) \approx 1 - 4U_{e1}^2U_{e2}^2\sin^2\Delta_{21} - 4\left(1 - U_{e3}^2\right)U_{e3}^2\sin^2\Delta_{32}}.$$

Summary of approx three flavour oscillation probabilities neglecting *CP*-violation and using $\left|\Delta m^2_{31}\right| \approx \left|\Delta m^2_{32}\right|$

Survival probabilities

$$P\left(\nu_e \to \nu_e\right) \approx 1 - 4U^2_{e1}U^2_{e2}\sin^2\Delta_{21} - 4\left(1 - U^2_{e3}\right)U^2_{e3}\sin^2\Delta_{32} \tag{144}$$

$$P\left(\nu_\mu \to \nu_\mu\right) \approx 1 - 4U^2_{\mu1}U^2_{\mu2}\sin^2\Delta_{21} - 4\left(1 - U^2_{\mu3}\right)U^2_{\mu3}\sin^2\Delta_{32} \tag{145}$$

$$P\left(\nu_\tau \to \nu_\tau\right) \approx 1 - 4U^2_{\tau1}U^2_{\tau2}\sin^2\Delta_{21} - 4\left(1 - U^2_{\tau3}\right)U^2_{\tau3}\sin^2\Delta_{32} \tag{146}$$

Oscillation probabilities

$$P\left(\nu_e \to \nu_\mu\right) = P\left(\nu_\mu \to \nu_e\right) \approx -4U_{e1}U_{\mu1}U_{e2}U_{\mu2}\sin^2\Delta_{21} + 4U^2_{e3}U^2_{\mu3}\sin^2\Delta_{32} \tag{147}$$

$$P\left(\nu_e \to \nu_\tau\right) = P\left(\nu_\tau \to \nu_e\right) \approx -4U_{e1}U_{\tau1}U_{e2}U_{\tau2}\sin^2\Delta_{21} + 4U^2_{e3}U^2_{\tau3}\sin^2\Delta_{32} \tag{148}$$

$$P\left(\nu_\mu \to \nu_\tau\right) = P\left(\nu_\tau \to \nu_\mu\right) \approx -4U_{\mu1}U_{\tau1}U_{\mu2}U_{\tau2}\sin^2\Delta_{21} + 4U^2_{\mu3}U^2_{\tau3}\sin^2\Delta_{32} \tag{149}$$

The wavelengths associated with $\sin^2\Delta_{21}$ and $\sin^2\Delta_{32}$ are:

"ATMOSPHERIC" $\lambda_{32} = \frac{4\pi E}{\Delta m^2_{32}}$ "Short"-Wavelength

and

"SOLAR" $\lambda_{21} = \frac{4\pi E}{\Delta m^2_{21}}$ "Long"-Wavelength.

PMNS Matrix

- The PMNS matrix is usually expressed in terms of 3 rotation angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a complex phase δ, using the notation $s_{ij} = \sin\theta_{ij}, \quad c_{ij} = \cos\theta_{ij}$

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}} \times \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{+i\delta} & 0 & c_{13} \end{pmatrix} \times \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Dominates: **"Atmospheric"** **"Solar"**

- Writing this out in full:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

- So there are six SM parameters that can be measured in ν oscillation experiments:

$\lvert\Delta m_{21}\rvert^2 = \lvert m_2^2 - m_1^2\rvert$	θ_{12}	Solar and reactor neutrino experiments
$\lvert\Delta m_{32}\rvert^2 = \lvert m_3^2 - m_2^2\rvert$	θ_{23}	Atmospheric and beam neutrino experiments
	θ_{13}	Reactor neutrino experiments + future beam
	δ	Future beam experiments

Neutrino Experiments

Before discussing current experimental data, need to consider how neutrinos interact in matter (i.e. our detectors)

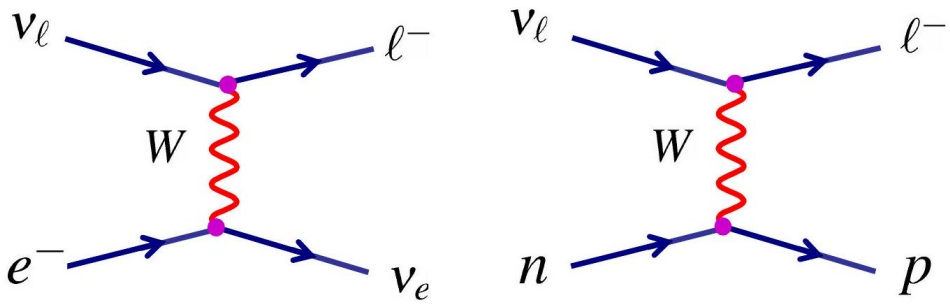
Two processes:

- Charged current (CC) interactions (via a W-boson) $\Rightarrow$ charged lepton
- Neutral current (NC) interactions (via a Z-boson)

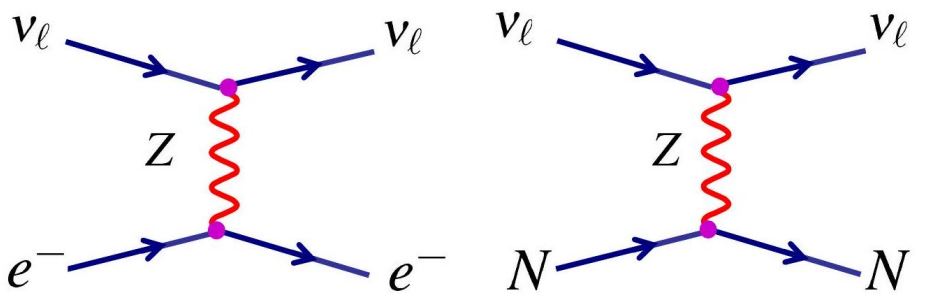
Two possible "targets": can have neutrino interactions with:

- atomic electrons
- nucleons within the nucleus

CHARGED CURRENT



NEUTRAL CURRENT



Neutrino Interaction Thresholds I

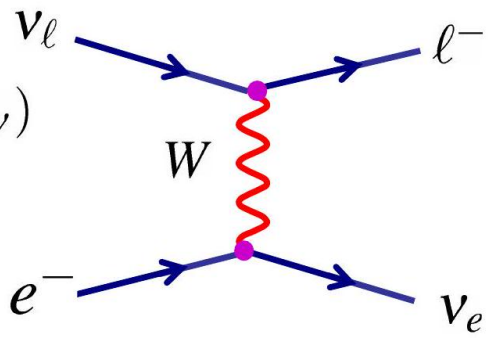
Neutrino detection method depends on the neutrino energy and (weak) flavour

- Neutrinos from the sun and nuclear reactions have $E_V \sim 1\ \mathrm{MeV}$.
- Atmospheric neutrinos have $E_\nu \sim 1\ \mathrm{GeV}$.

These energies are relatively low and not all interactions are kinematically allowed, i.e. there is a threshold energy before an interaction can occur. Require sufficient energy in the centre-of-mass frame to produce the final state particles

1. Charged current interactions **on atomic electrons** (in laboratory frame)

$p_\nu = (E_\nu, 0, 0, E_\nu)$
$p_e = (m_e, 0, 0, 0)$



$$s = (p_\nu + p_e)^2 = (E_\nu + m_e)^2 - E_\nu^2$$

Interaction requires $s > m_\ell^2$ so

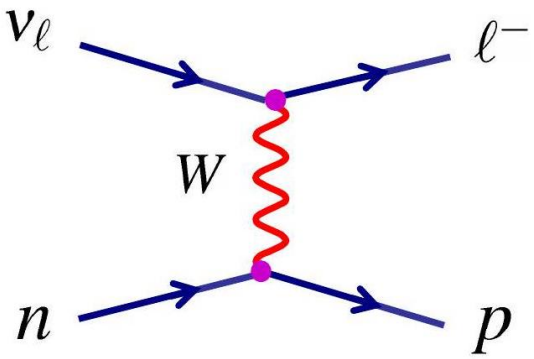
$$\boxed{E_\nu > \left[\left(\frac{m_\ell}{m_e}\right)^2 - 1\right]\frac{m_e}{2}}.$$

Putting in the numbers, *CC* interactions **with atomic electrons** require:

$$E_{\nu_e} > 0 \qquad \underbrace{E_{\nu_\mu} > 11\ \mathrm{GeV} \quad E_{\nu_\tau} > 3090\ \mathrm{GeV}}_{\text{These are high energy thresholds compared to most neutrino energies we will consider.}} .$$

Neutrino Interaction Thresholds II

2. Charged Current interactions **on nucleons** (in laboratory frame):



$$s = (p_\nu + p_n)^2 = (E_\nu + m_n)^2 - E_\nu^2$$

Interaction requires $s > (m_\ell + m_p)^2$ so

$$\boxed{E_\nu > \frac{(m_p^2 - m_n^2) + m_\ell^2 + 2m_p m_\ell}{2m_n}}$$

Therefore, *CC* interactions **with neutrons** require:

$$E_{\nu_e} > 0 \quad E_{\nu_\mu} > 110\ \mathrm{MeV} \quad E_{\nu_\tau} > 3.5\ \mathrm{GeV}.$$

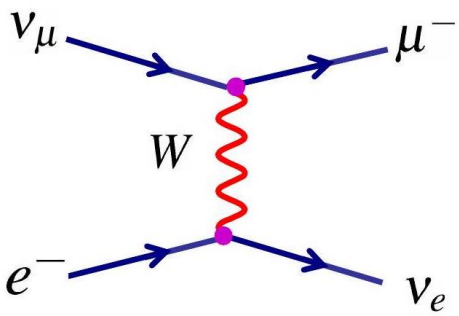
Some neutrinos therefore effectively 'disappear'.
From point of view of CC, these cannot interact:

Electron neutrinos from the sun and nuclear reactors $E_\nu \sim 1\ \mathrm{MeV}$ which oscillate **into muon or tau** neutrinos.
Atmospheric muon neutrinos $E_\nu \sim 1\ \mathrm{GeV}$ which oscillate **into tau** neutrinos.

Therefore, many experimental signatures for neutrino oscillation are **deficits** of neutrino interactions (a notable exception is SNO (pg 350), which makes use of the Neutral Current) because most neutrinos are below threshold for production of leptons of different flavour to the original neutrino!

In Handout 10 we derived expressions for CC neutrino-quark cross sections in ultra-relativistic limit (neglecting masses of neutrinos/quarks). Therefore ...

- for high energy **muon neutrinos** can directly use the result (116) namely $\sigma_{\nu_\mu e^-} = \frac{G_F^2 s}{\pi}$

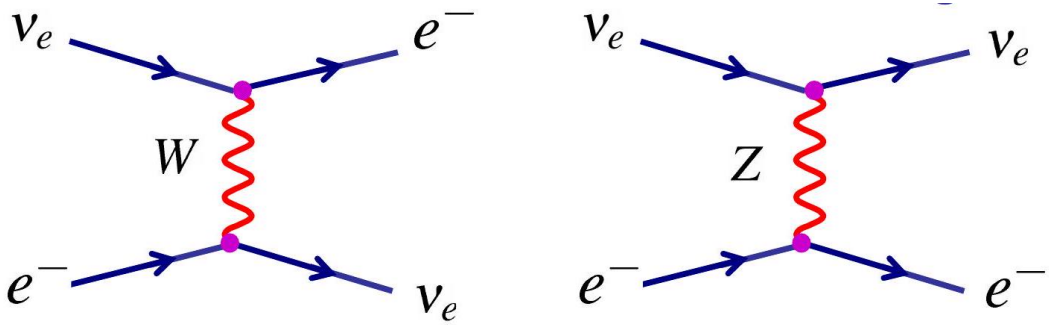


and so using $s = (E_\nu + m_e)^2 - E_\nu^2 \approx 2 m_e E_\nu$ one obtains:

$$\boxed{\sigma_{\nu_\mu e^-} = \frac{2 m_e G_F^2 E_\nu}{\pi}}.$$

Note that the neutrino scattering coss section increases linearly with lab. frame neutrino energy!

- For **electron neutrinos** there is another lowest order diagram with the same final state

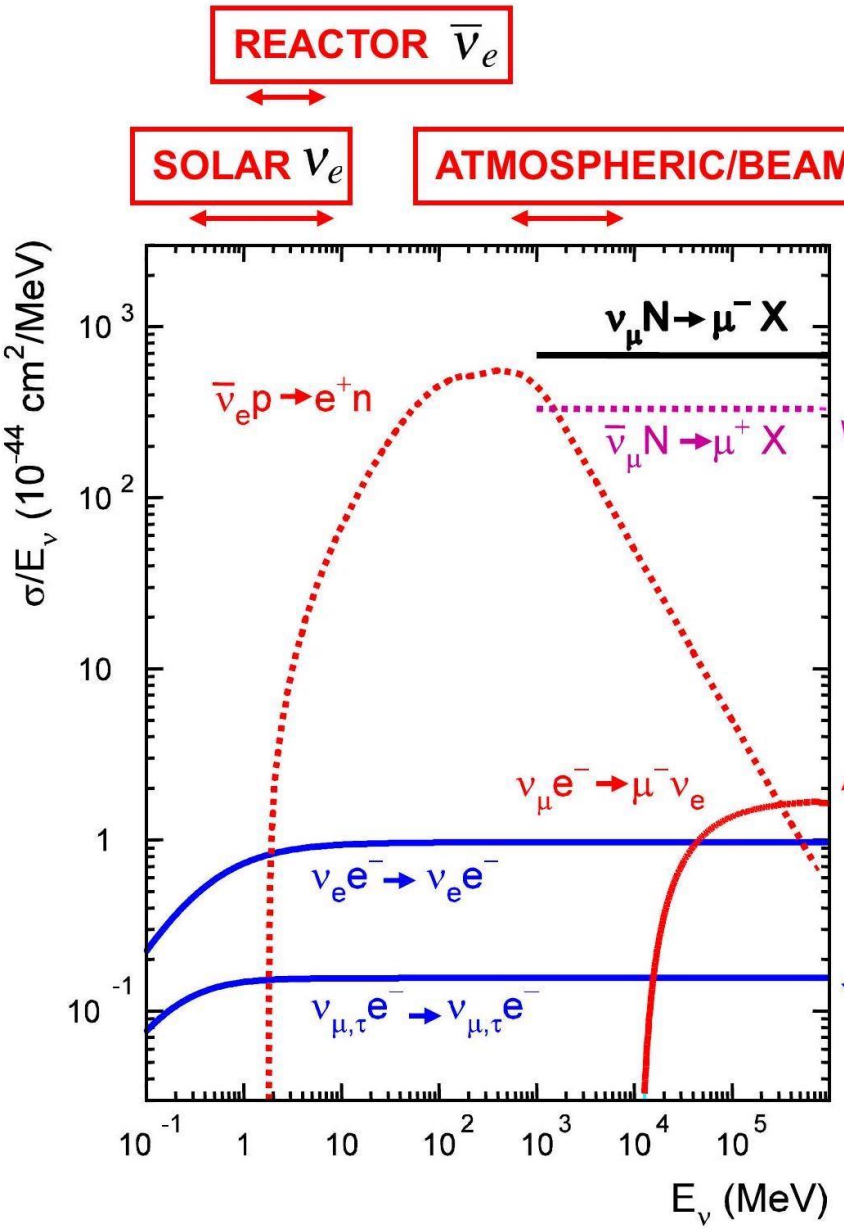


It turns out that the cross section is lower than the pure CC cross section due to negative interference when summing matrix elements $|M_{CC} + M_{NC}|^2 < |M_{CC}|^2$ so

$$\boxed{\sigma_{\nu_e e} \approx 0.6 \sigma^{CC}_{\nu_e e}}.$$

- In the high energy limit the CC neutrino-nucleon cross sections are larger due to the higher centre-of-mass energy: $s = (E_\nu + m_n)^2 - E_\nu^2 \approx 2 m_n E_\nu$.

Neutrino scattering cross sections compared



$$\sigma_{\nu_\mu N} = 0.22 \frac{2 m_p G_F^2 E_\nu}{\pi}$$

$$\sigma_{\overline{\nu}_\mu N} = 0.33 \frac{2 m_p G_F^2 E_\nu}{3\pi}$$

Deep Inelastic Scattering

$$\sigma_{\nu_\mu e^-} = \frac{2 m_e G_F^2 E_\nu}{\pi}$$

CC only
Threshold 11 GeV

$$\sigma_{\nu_e e^-} \approx 0.6 \frac{2 m_e G_F^2 E_\nu}{\pi}$$

NC + CC

NC only

Detector technology depends on process and neutrino type/energy

Atmospheric/Beam Neutrinos $\nu_e, \nu_\mu, \overline{\nu}_e, \overline{\nu}_\mu : E_\nu > 1\,\mathrm{GeV}$

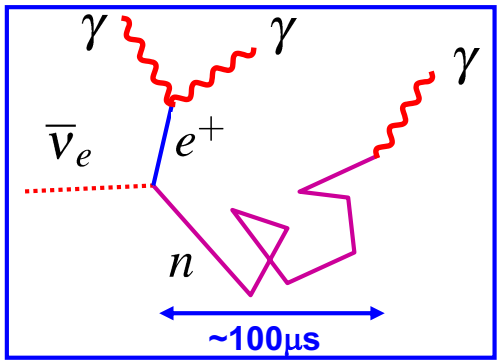
1. **Water Čerenkov:** e.g. Super Kamiokande
2. **Iron Calorimeters:** e.g. MINOS, CDHS
   - Produce high energy charged lepton – relatively easy to detect

Solar Neutrinos $\nu_e \;:\; E_\nu < 20\,\mathrm{MeV}$

1. **Water Čerenkov:** e.g. Super Kamiokande
   - Detect Čerenkov light from electron produced in $\nu_e + e^- \rightarrow \nu_e + e^-$
   - Because of background from natural radioactivity limited to $E_\nu > 5\,\mathrm{MeV}$
   - Because Oxygen is a doubly magic nucleus don't get $\nu_e + n \rightarrow e^- + p$
2. **Radio-Chemical:** e.g. Homestake, SAGE, GALLEX
   - Use inverse beta decay process, e.g. $\nu_e + {}^{71}\mathrm{Ga} \rightarrow e^- + {}^{71}\mathrm{Ge}$
   - Chemically extract produced isotope and count decays (only gives a rate)

Reactor Neutrinos $\overline{\nu}_e \;:\; E_{\overline{\nu}} < 5\,\mathrm{MeV}$

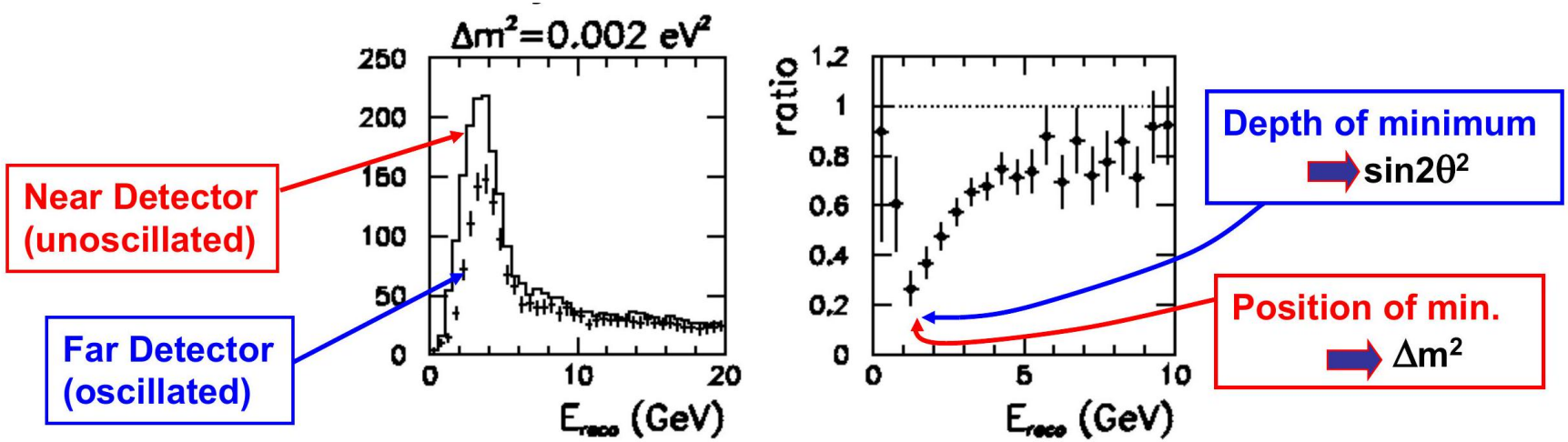
1. **Liquid Scintillator:** e.g. KamLAND
   - Low energies ➡ large radioactive background
   - Dominant interaction: $\overline{\nu}_e + p \rightarrow e^+ + n$
   - **Prompt** positron annihilation signal **+ delayed** signal from ***n*** (space/time correlation reduces background)
   - electrons produced by photons excite scintillator which produces light



1a: Long Baseline Neutrino Experiments

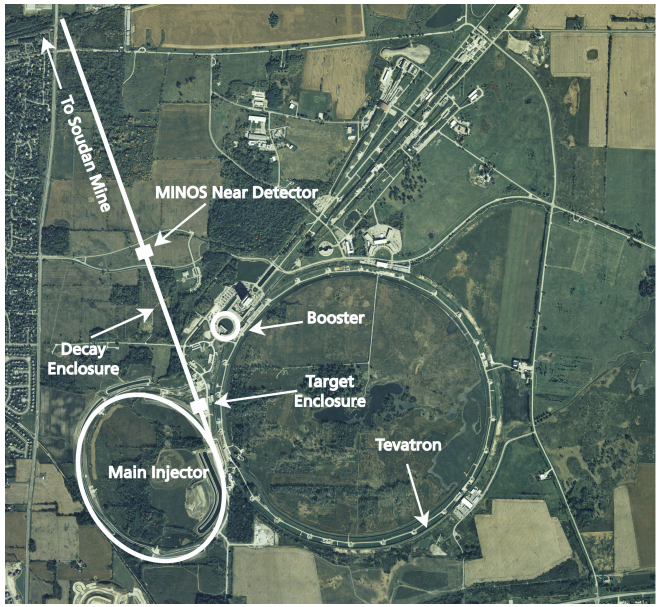
Basic Idea:

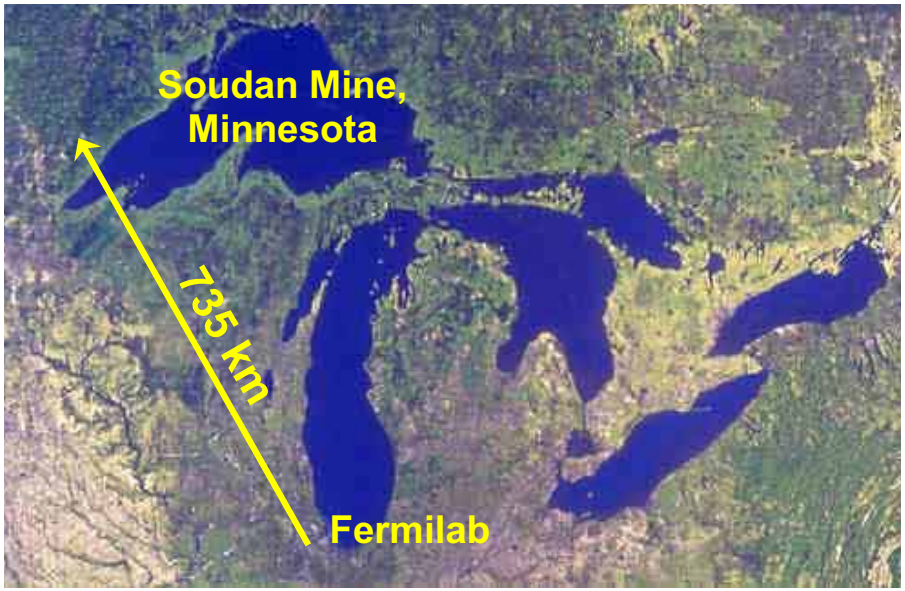
- Create intense neutrino beam.
- Build two "near" and "far" detectors: one close to beam, the other many km away.
- Measure ratio of the neutrino energy spectrum in far detector (oscillated) to that in the near detector (unoscillated)
- Duplication of detectors allows for partial cancellation of systematic biases.



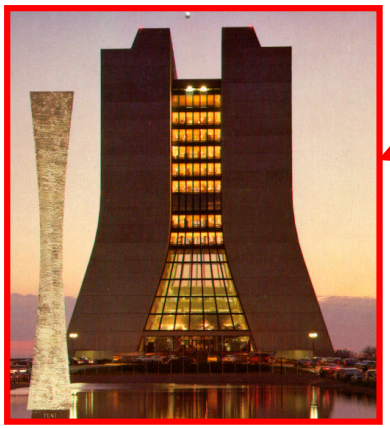
Long baseline example experiment: MINOS (2005-2016)

- 120 GeV proton beam extracted from Fermilab main injector.
- 2.5×10^{13} protons per pulse hit target. Proton beam is 0.3 MW on target!





Two detectors:



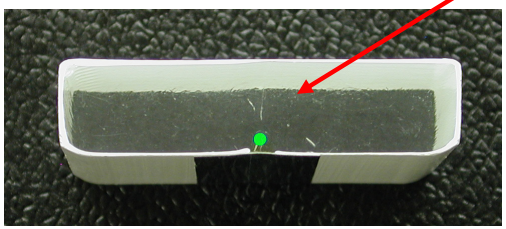
★ **1000 ton, NEAR Detector at Fermilab : 1 km from beam**

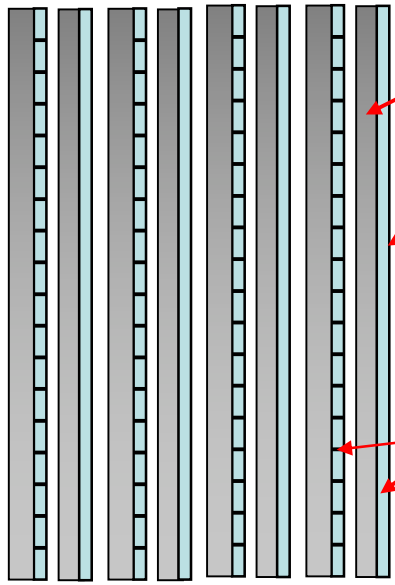
★ **5400 ton FAR Detector, 720m underground in Soudan mine, N. Minnesota: 735 km from beam**



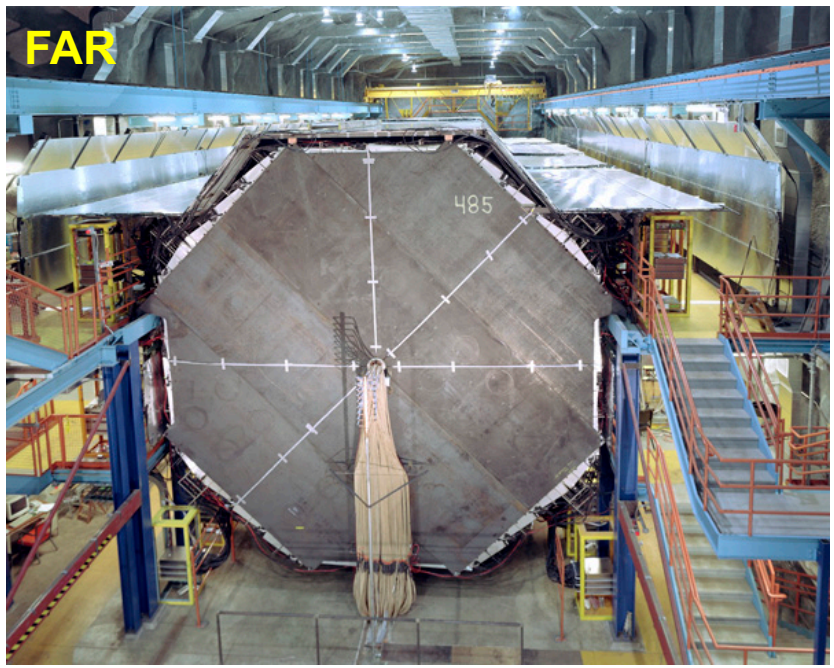
The MINOS Detectors:

- Dealing with high energy neutrinos $E_\nu > 1\,\mathrm{GeV}$
- The muons produced by ν_μ interactions travel several metres
- Steel-Scintillator sampling calorimeter
- Each plane: 2.54 cm steel +1 cm scintillator
- A charged particle crossing the scintillator produces light – detect with PMTs



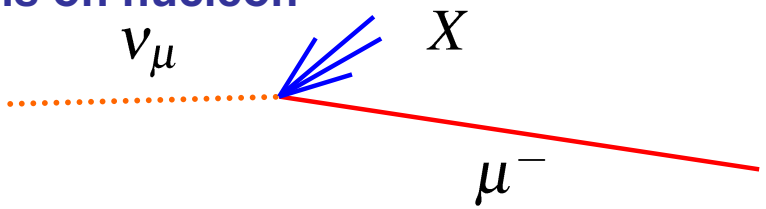




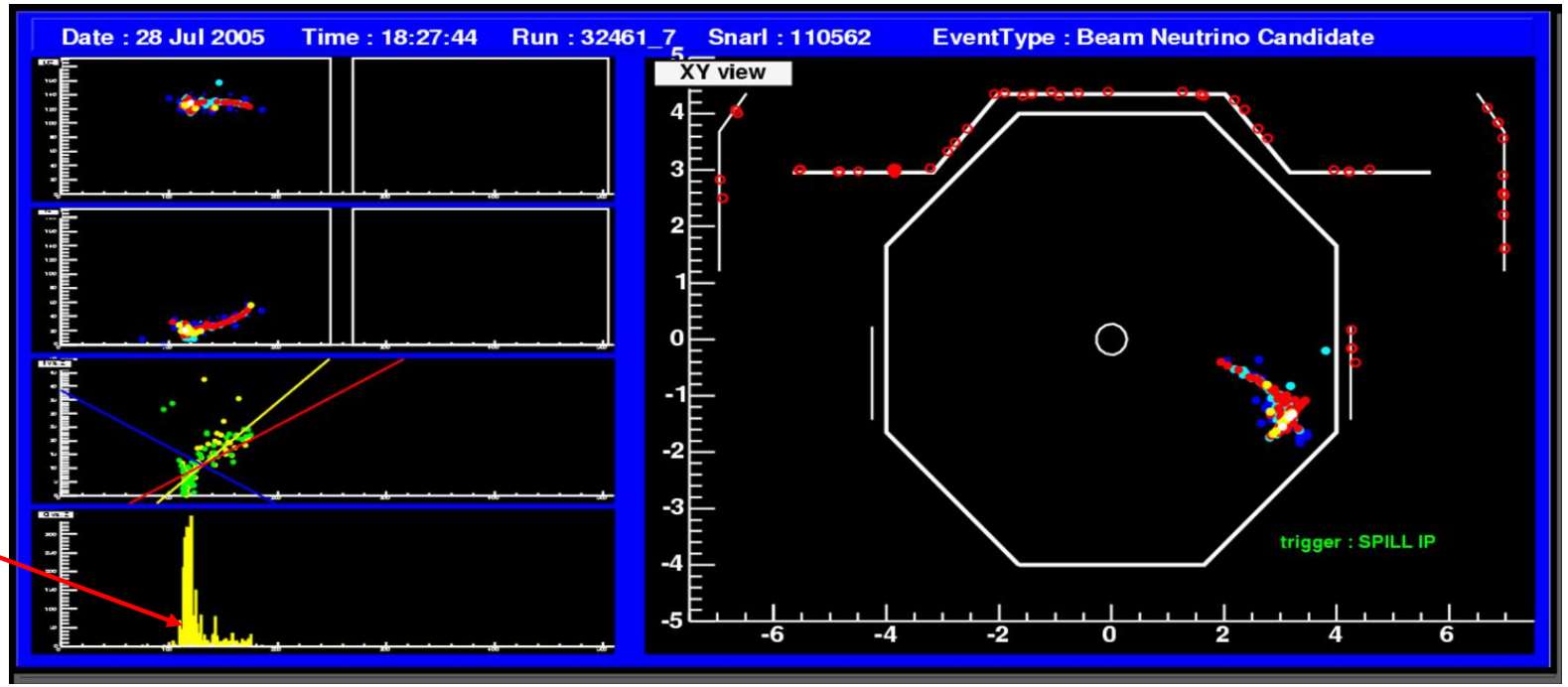


- **Neutrino detection via CC interactions on nucleon**

$$\nu_\mu + N \rightarrow \mu^- + X$$



Example event:



Signal from hadronic shower

- **The main feature of the MINOS detector is the very good neutrino energy resolution**

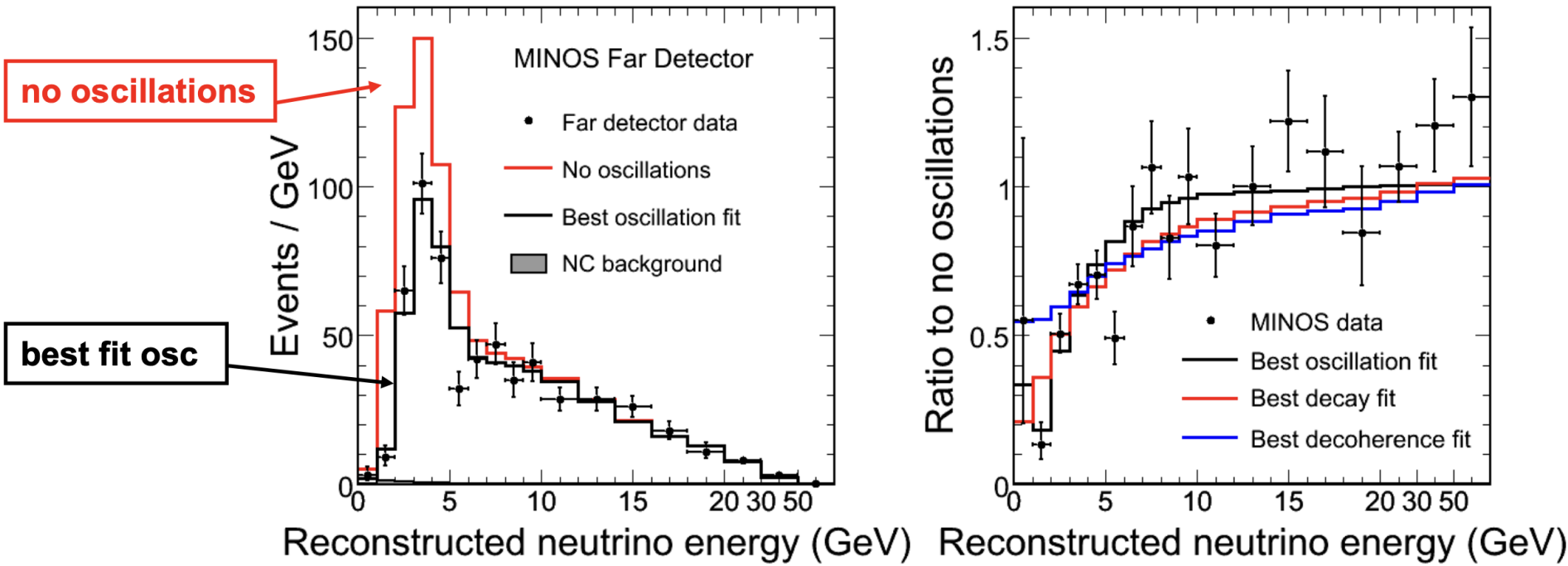
$$E_\nu = E_\mu + E_X$$

- **Muon energy from range/curvature in B-field**
- **Hadronic energy from amount of light observed**

MINOS Results

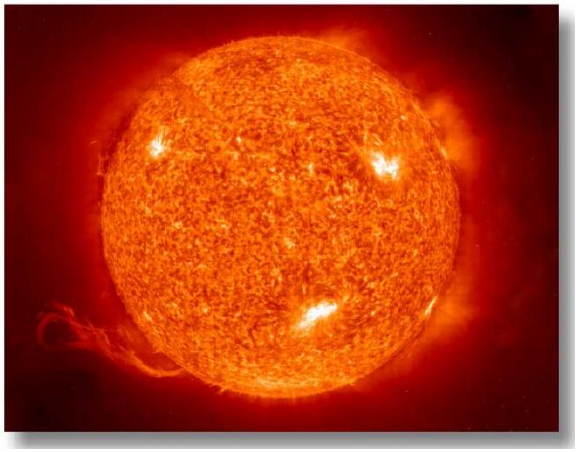
- For the MINOS experiment L is fixed and observe oscillations as function of E_ν.
- For $\left|\Delta m^2_{32}\right| \sim 2.5 \times 10^{-3}\ \mathrm{eV}^2$ first oscillation minimum at $E_\nu = 1.5\ \mathrm{GeV}$.
- To a very good approximation can use two flavour formula as oscillations corresponding to $\left|\Delta m^2_{21}\right| \sim 8 \times 10^{-5}\ \mathrm{eV}^2$ occur at $E_\nu = 50\ \mathrm{MeV}$, beam contains very few neutrinos at this energy + well below detection threshold.

MINOS Collaboration, Phys. Rev. Lett. 101, 131802, 2008

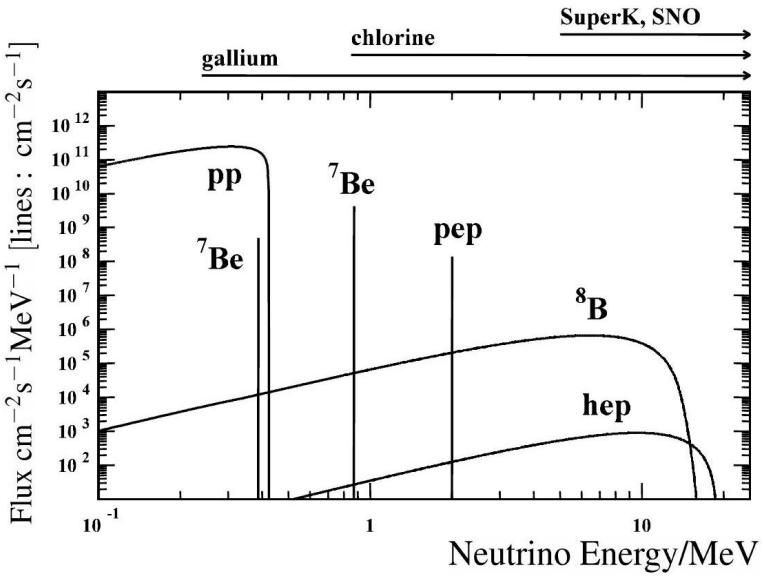


$$\left|\Delta m^2_{32}\right| = (2.43 \pm 0.12) \times 10^{-3}\ \mathrm{eV}^2$$

2: Solar Neutrinos



- The Sun is powered by the weak interaction, and produces a very large flux of electron neutrinos: $\sim 2 \times 10^{38}\ \nu_e \mathrm{s}^{-1}$.
- There are several different nuclear reactions in the sun so neutrino energy spectrum is complex:



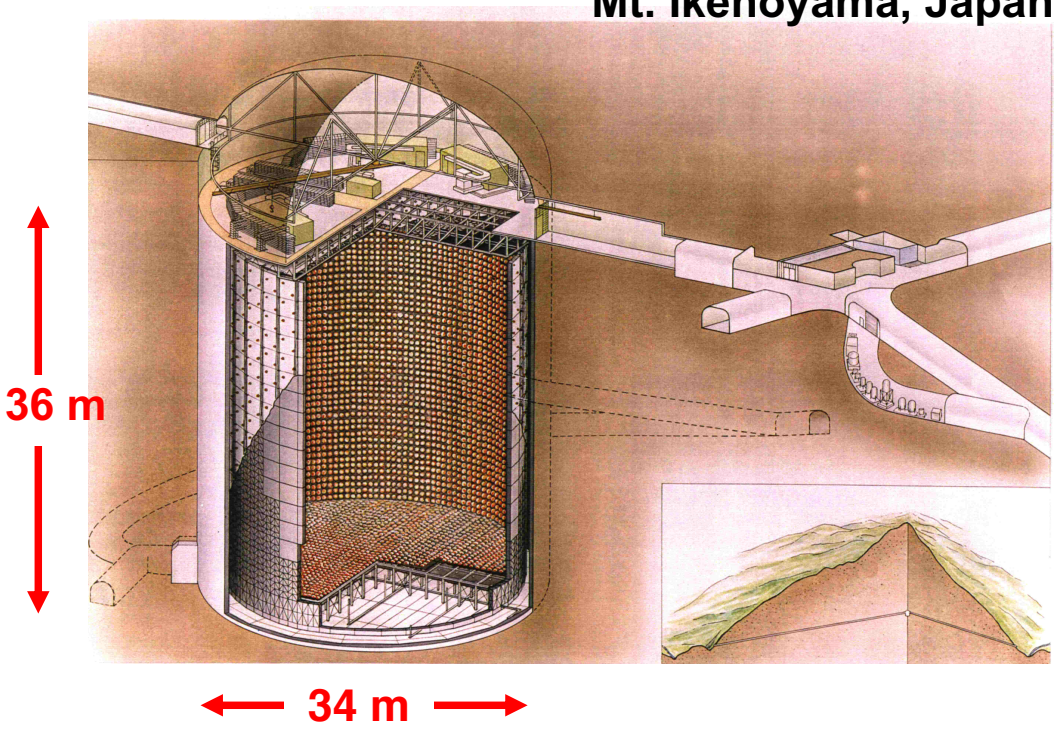
$p + p \to d + e^+ + \nu_e$	$E_\nu < 0.5$ MeV
$^8\mathrm{B} \to {}^8\mathrm{Be}^* + e^+ + \nu_e$	$E_\nu \sim 5$ MeV
$p + e^- + p \to d + \nu_e$	
$^7\mathrm{Be} + e^- \to {}^7\mathrm{Li} + \nu_e$	
$^3\mathrm{He} + p \to {}^4\mathrm{He} + e^+ + \nu_e$	

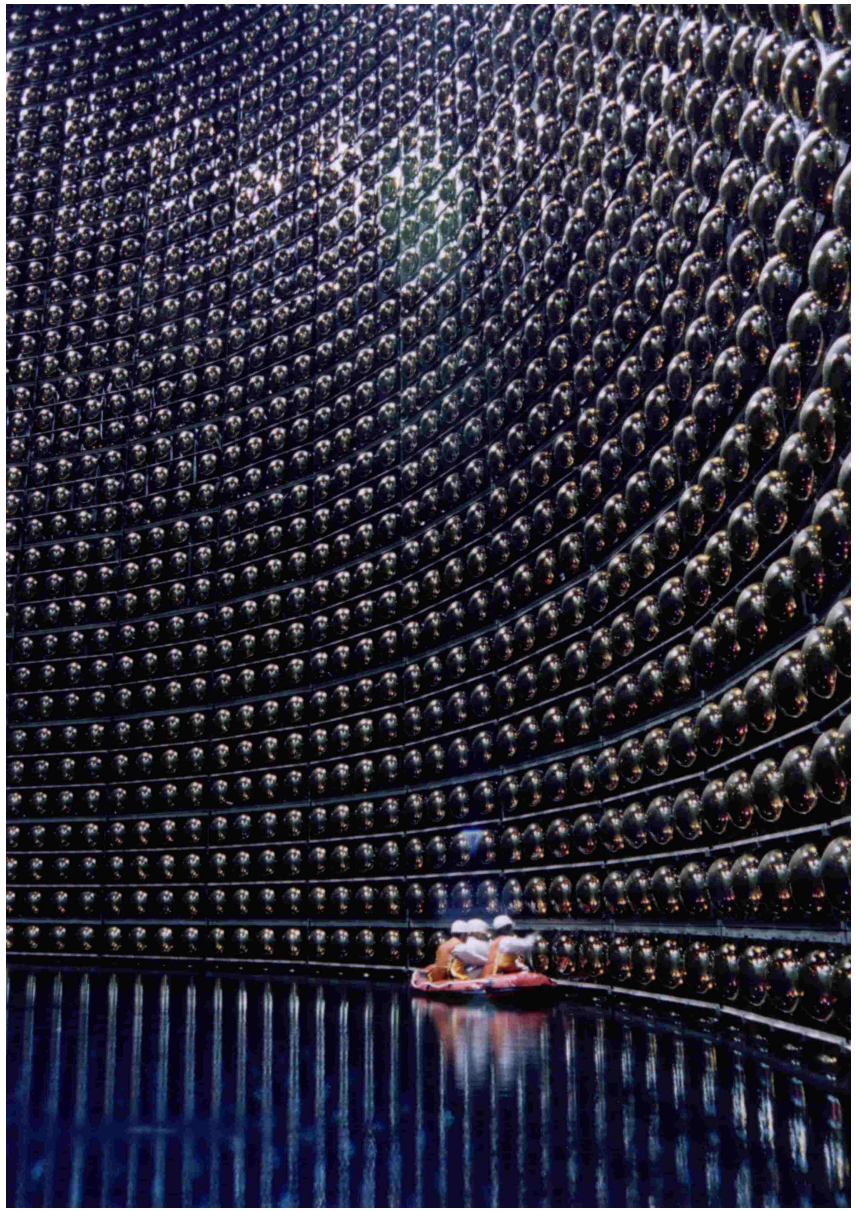
A number of experiments (Homestake Mine, Super Kamiokande, *etc.*) saw a deficit of electron neutrinos compared to experimental prediction.
This was called the SOLAR NEUTRINO PROBLEM!

Solar Neutrinos I: Super Kamiokande

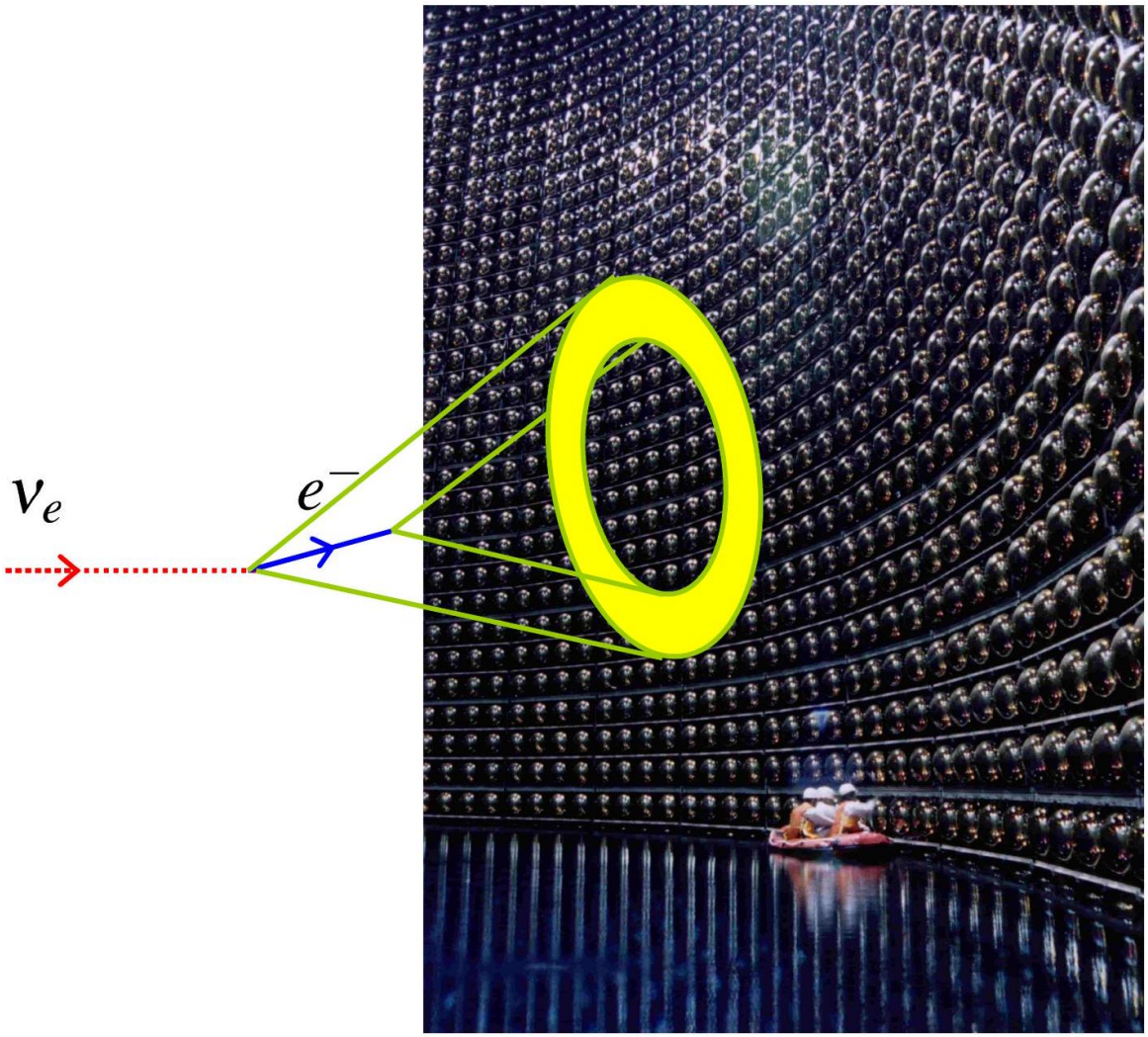
- 50000 ton water Čerenkov detector.
- Water viewed by 11146 photo-multiplier tubes.
- Deep underground to filter out cosmic rays otherwise difficult to detect rare neutrino interactions.

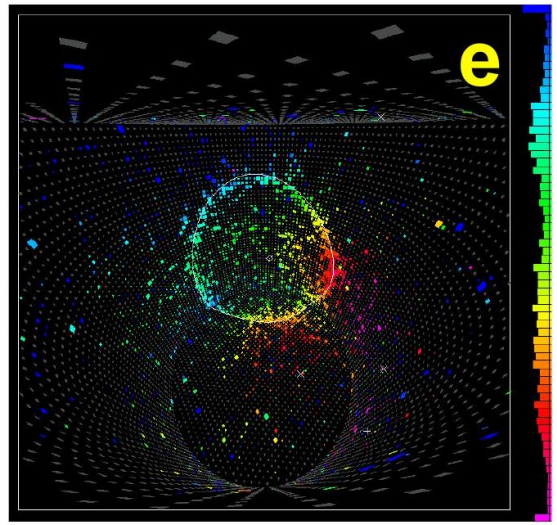
Mt. Ikenoyama, Japan

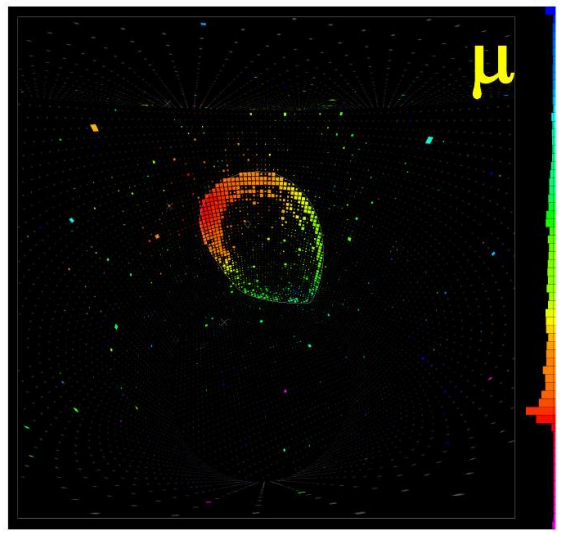




Detect neutrinos by observing Čerenkov radiation from charged particles which travel faster than speed of light in water (which is c/n for refractive index n).



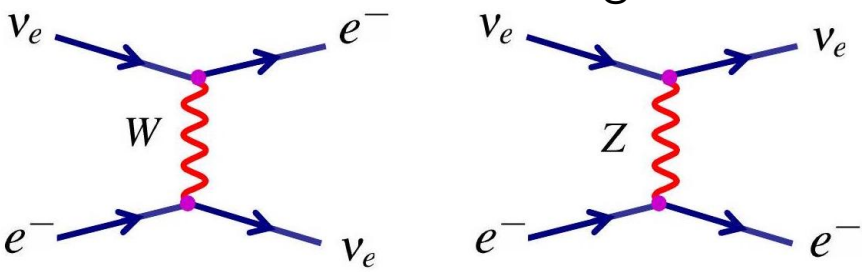




Can distinguish electrons from muons from pattern of light — muons produce clean rings whereas electrons produce more diffuse 'fuzzy' rings.

Super K Sensitivity

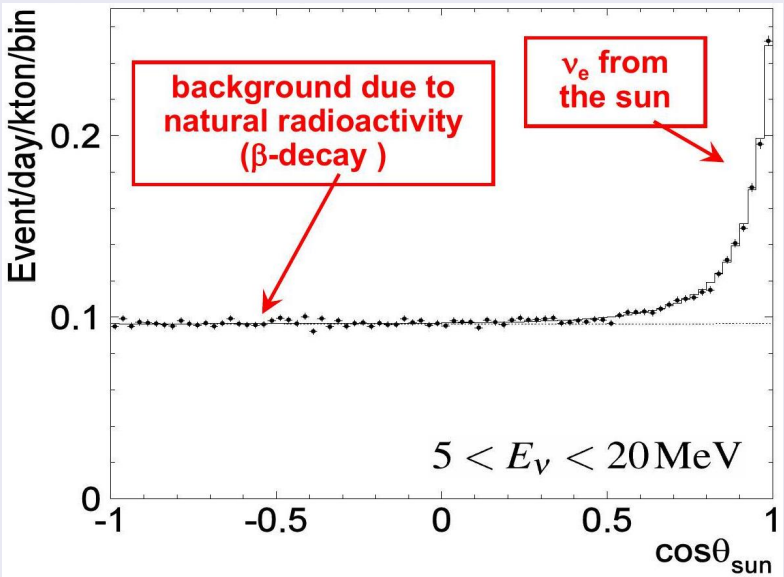
- Detect electron Čerenkov rings from:



- In LAB frame the electron is produced preferentially along the ν_e direction.

- Sensitive to solar neutrinos with $E_\nu > 5$ MeV.
- For lower energies too much background from natural radioactivity (β-decays) Hence detect mostly neutrinos from ${}^8B \rightarrow {}^8Be^* + e^+ + \nu_e$.

Super K Results



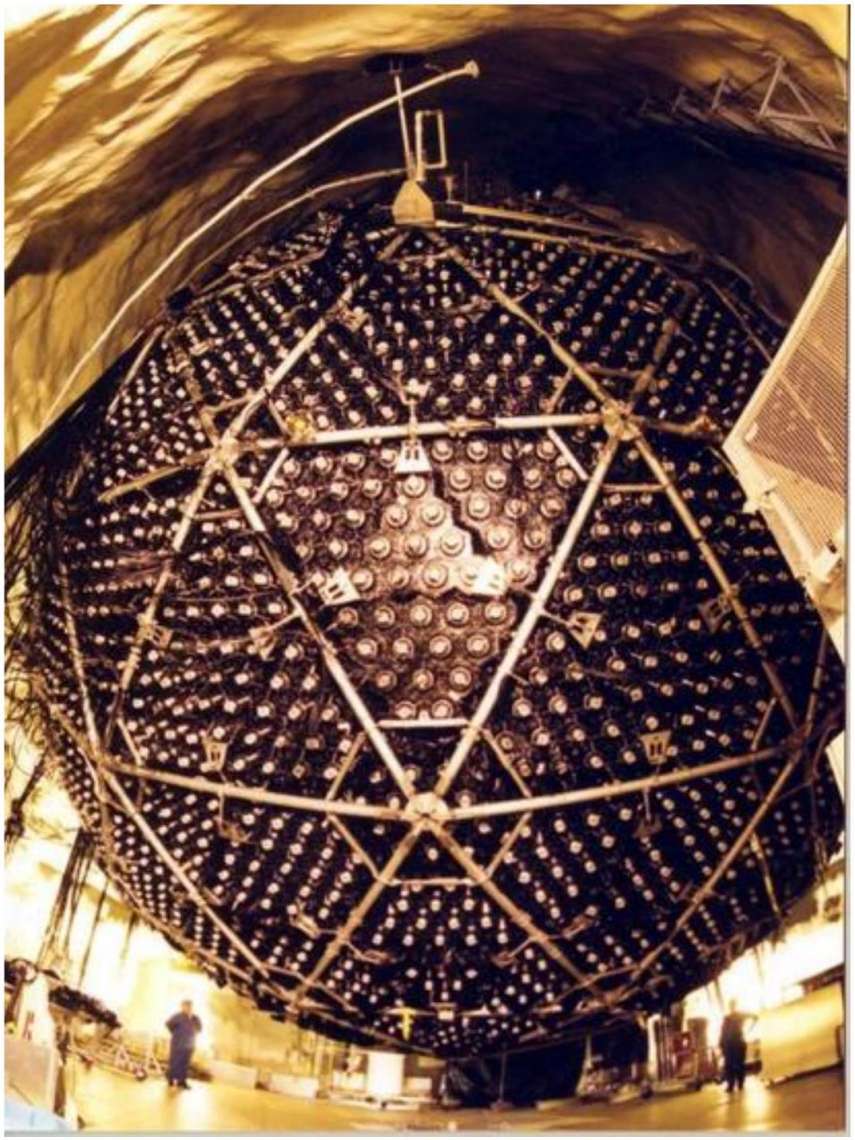
- Plot left from **S.Fukada et al., Phys. Rev. Lett. 86 5651-5655, 2001** [8].
- Clear signal of neutrinos from the sun
- However too few neutrinos:

$$\text{DATA/SSM} = 0.45 \pm 0.02.$$

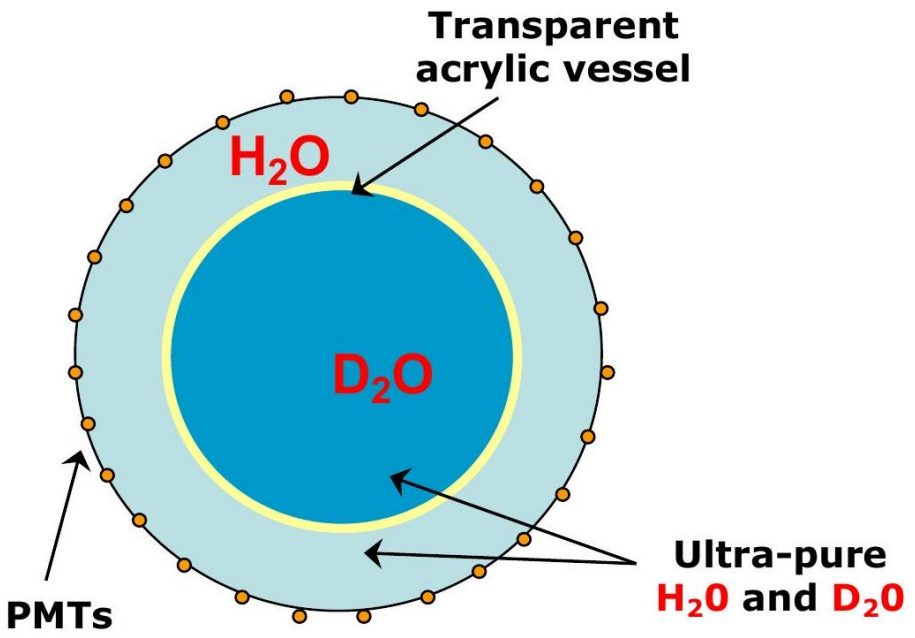
SSM = 'Standard Solar Model' Prediction

Solar Neutrinos II: Sudbury Neutrino Observatory (SNO)

A 1000 ton heavy water (D_2O) Čerenkov detector in a deep mine in Ontario, Canada.



- D_2O inside a 12 m diameter acrylic vessel.
- Surrounded by **3000** tons of normal water.
- Main experimental challenge is the need for very low background from radioactivity.
- Ultra-pure H_2O and D_2O.
- Surrounded by 9546 PMTs.

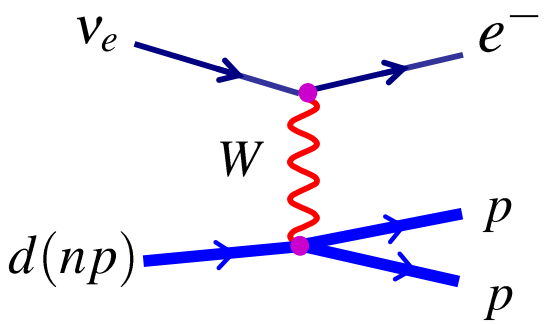


Detect Čerenkov light from three different reactions:

CHARGE CURRENT

- Detect Čerenkov light from electron
- Only sensitive to ν_e (thresholds)
- Gives a measure of ν_e flux

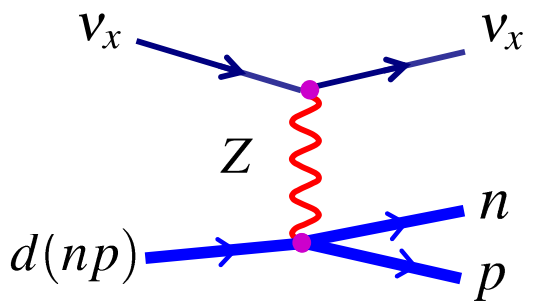
$$\text{CC Rate} \propto \phi(\nu_e)$$



NEUTRAL CURRENT

- The neutral current reaction in heavy water liberates a neutron which is thermalized in the heavy water and observed following capture on other nuclei within the detector.
- MEASURES TOTAL NEUTRINO FLUX.

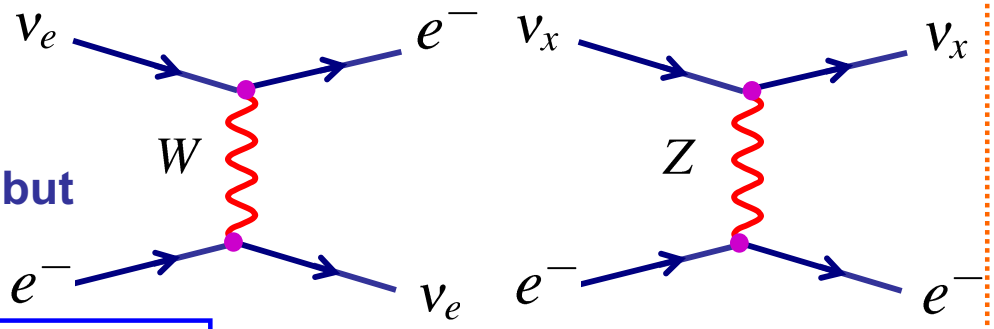
$$\text{NC Rate} \propto \phi(\nu_e) + \phi(\nu_\mu) + \phi(\nu_\tau)$$



ELASTIC SCATTERING

- Detect Čerenkov light from electron
- Sensitive to all neutrinos (NC part) – but larger cross section for ν_e

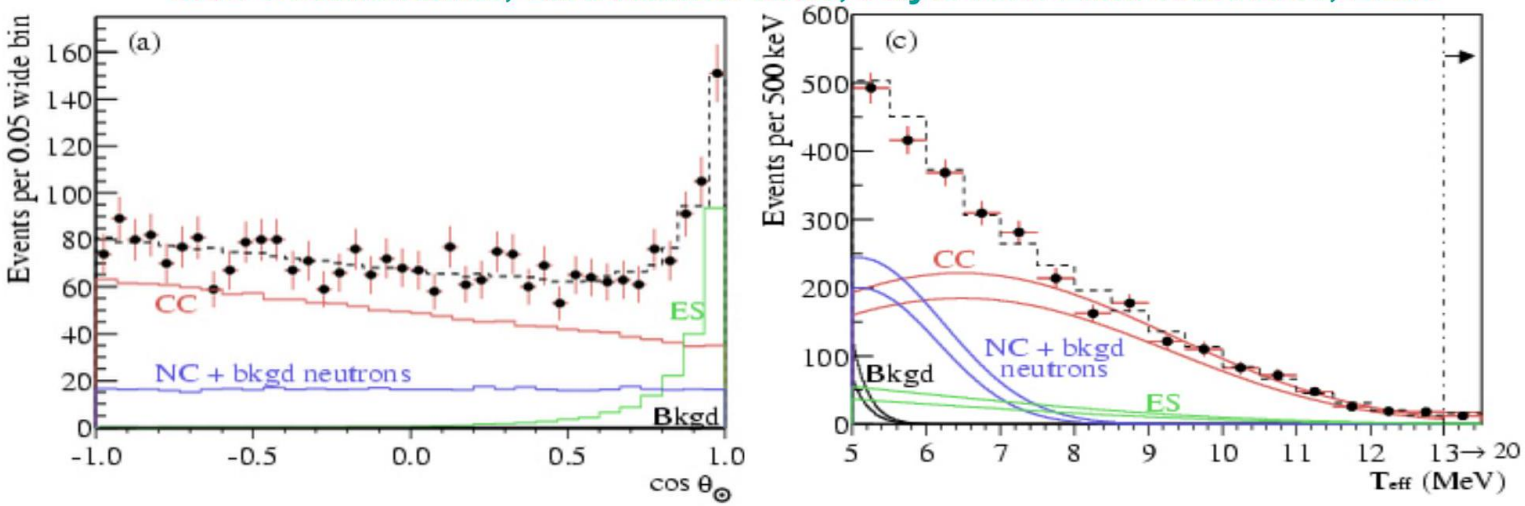
$$\text{ES Rate} \propto \phi(\nu_e) + 0.154(\phi(\nu_\mu) + \phi(\nu_\tau))$$



Experimentally can determine rates for different interactions from:

- Electrons from elastic scattering (ES) point back to sun.
- Energy: NC events have lower energy — 6.25 MeV photon from neutron capture.
- Radius from centre of detector gives a measure of background from neutrons.

SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. 89:011301, 2002



Using different distributions obtain a measure of numbers of events of each type:

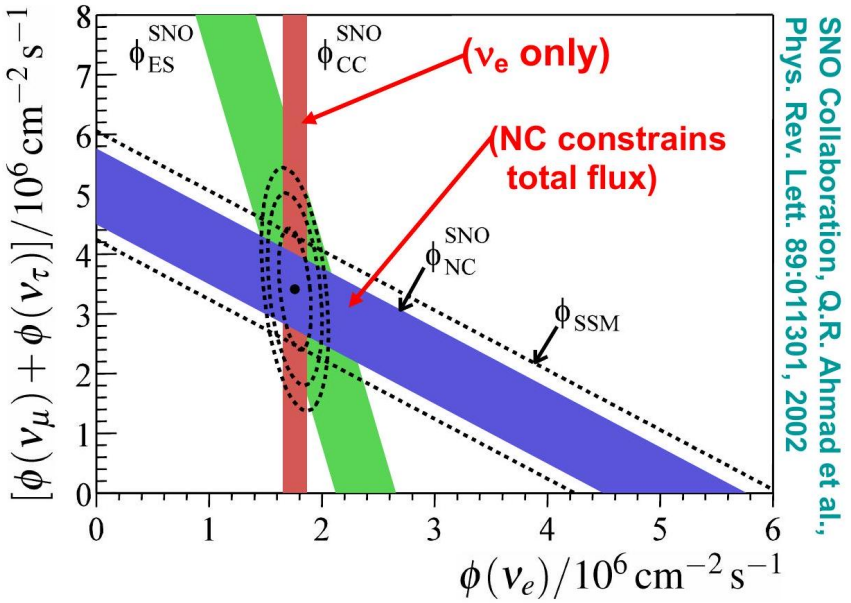
CC : 1968 ± 61	$\propto \phi(\nu_e)$
ES : 264 ± 26	$\propto \phi(\nu_e) + 0.154\,[\phi(\nu_\mu) + \phi(\nu_\tau)]$
NC: 576 ± 50	$\propto \phi(\nu_e) + \phi(\nu_\mu) + \phi(\nu_\tau)$

Uniquely:

SNO is able to masure of electron neutrino flux **and total flux**!

SNO Results

- Using known cross sections can convert observed numbers of events into fluxes
- The different processes impose different constraints
- Where constraints meet gives separate measurements of ν_e and $\nu_\mu + \nu_\tau$ fluxes



SNO Results:
$$\left\{\begin{array}{l}\phi(\nu_e) = (1.8 \pm 0.1) \times 10^{-6}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1} \\ \phi(\nu_\mu) + \phi(\nu_\tau) = (3.4 \pm 0.6) \times 10^{-6}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}\end{array}\right\}.$$

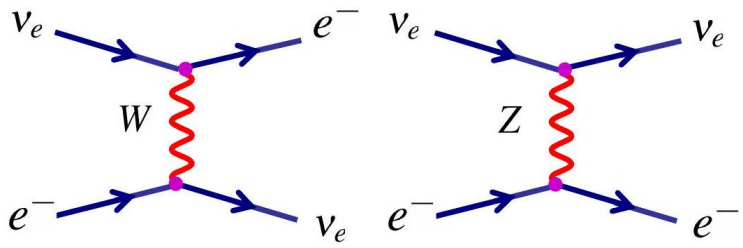
SSM Prediction: $\phi(\nu_e) = 5.1 \times 10^{-6}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}$.

- Clear evidence for a flux of ν_μ and/or ν_τ from the sun.
- Total neutrino flux is consistent with expectation from SSM.
- Clear evidence of $\nu_e \to \nu_\mu$ and/or $\nu_e \to \nu_\tau$ neutrino transitions.

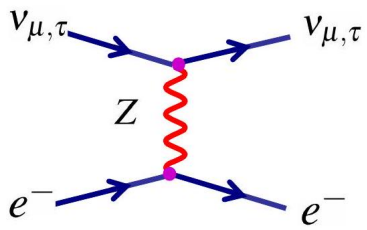
Complication for interpretation of Solar Neutrino Data

The interpretation of the solar neutrino data is complicated by **matter effects**:

- A quantitative treatment is non-trivial and is not given here.
- Basic idea is that as a neutrino leaves the sun it crosses a region of high electron density.
- The coherent forward scattering process ($\nu_e \to \nu_e$) for an electron neutrino



  is different to that for a muon or tau neutrino:



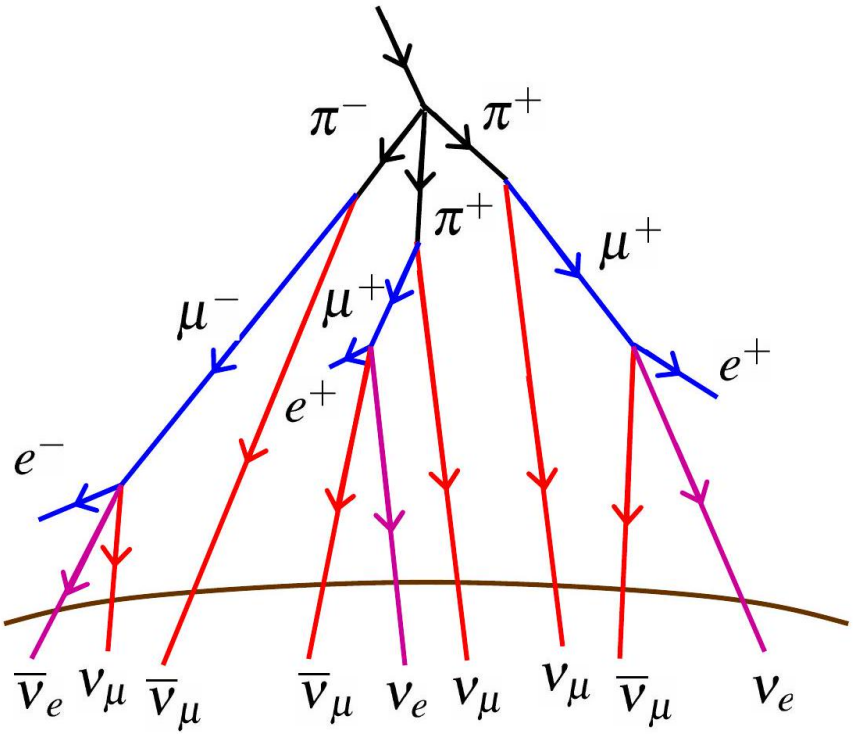
- Effect can enhance oscillations — is called "MSW effect".

A combined analysis of all solar neutrino data gives:

$$\Delta m^2_{\text{solar}} \approx 8 \times 10^{-5}\ \text{eV}^2, \qquad \sin^2 2\theta_{\text{solar}} \approx 0.85.$$

1b : Atmospheric Neutrinos

- High energy cosmic rays (up to 10^{20} eV) interact in the upper part of the Earth's atmosphere.
- The cosmic rays (26% protons, 11% He Nuclei, $\sim$ 1% heavier nuclei, 2% electrons) mostly interact hadronically giving showers of mesons (mainly pions)



- Neutrinos produced by:

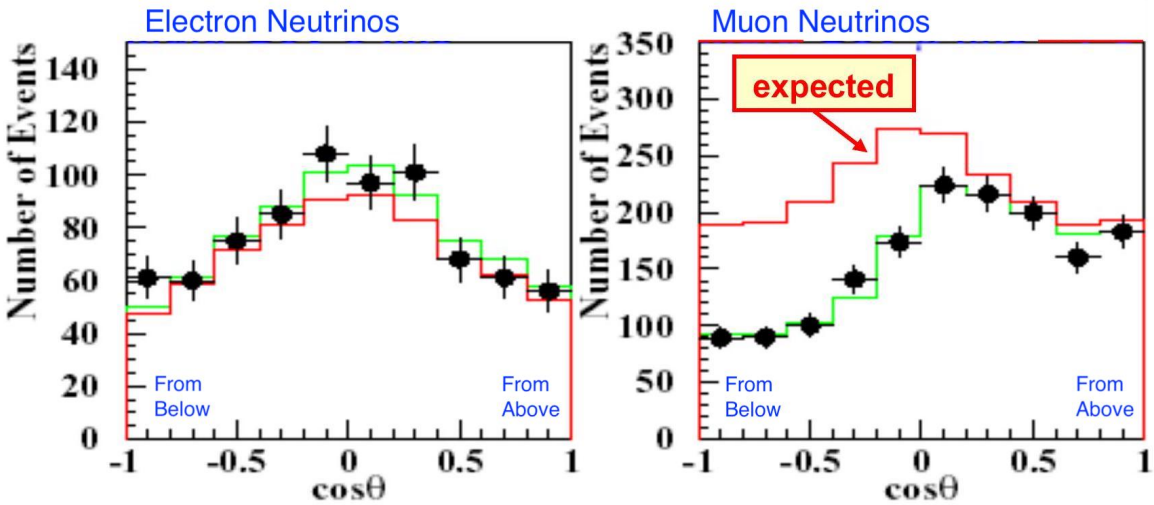
$$\pi^+ \to \mu^+ \nu_\mu \to e^+ \nu_e \bar{\nu}_\mu \nu_\mu$$
$$\pi^- \to \mu^- \bar{\nu}_\mu \to e^- \bar{\nu}_e \nu_\mu \bar{\nu}_\mu$$

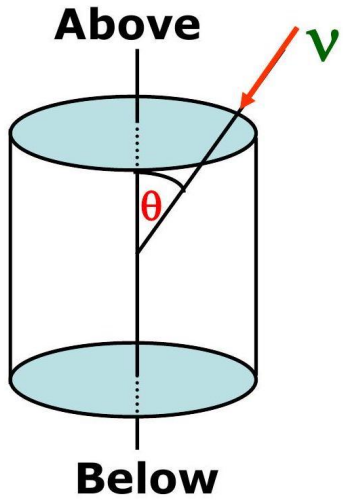
  therefore expect $\dfrac{N(\nu_\mu + \bar{\nu}_\mu)}{N(\nu_e + \bar{\nu}_e)} \approx 2$.

- Neutrino flux $\sim 1\ \mathrm{cm}^{-2}\mathrm{Sr}^{-1}\ \mathrm{s}^{-1}$.
- Typical energy : $\mathrm{E}_\nu \sim 1\ \mathrm{GeV}$.
- Observe a lower ratio with deficit of $\nu_\mu/\bar{\nu}_\mu$ coming from below the horizon, i.e. large distance from production point on other side of the Earth. (See evidence on next slide.)

Super Kamiokande Atmospheric Results

- Typical energy: $\mathrm{E}_\nu \sim 1\ \mathrm{GeV}$ (much greater than solar neutrinos — no confusion).
- Identify ν_e and ν_μ interactions from nature of Čerenkov rings.
- Measure rate as a function of angle with respect to local vertical.
- Neutrinos coming from above travel 20 km.
- Neutrinos coming from below (i.e. other side of the Earth) travel $\sim$ 12800 km.

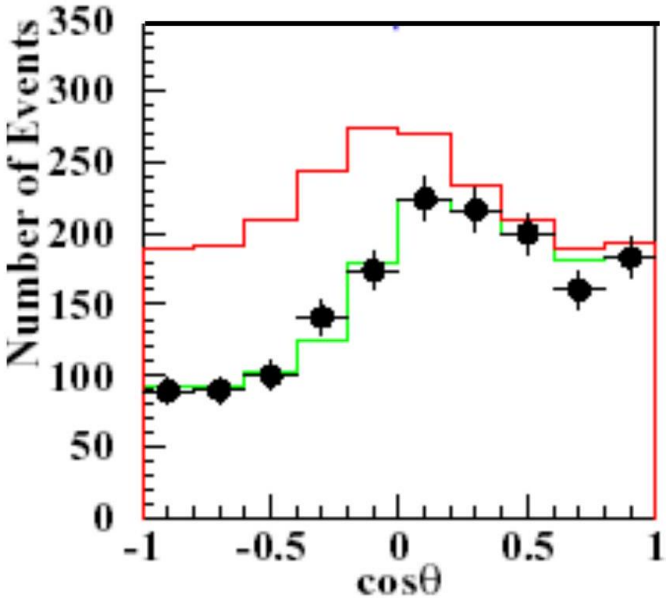


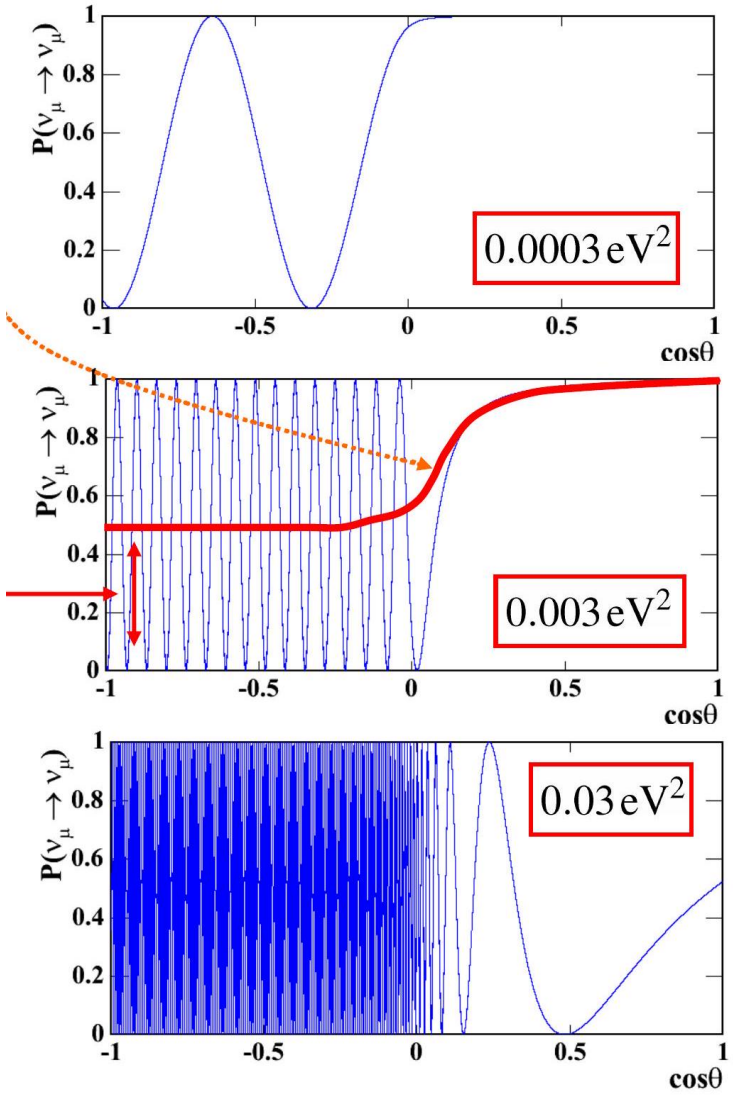


⋆ Prediction for ν_e rate agrees with data.
⋆ Strong evidence for disappearance of ν_μ for large distances.
⋆ Consistent with $\nu_\mu \to \nu_\tau$ oscillations.
⋆ Don't detect the oscillated ν_τ, as typically below interaction threshold of 3.5 GeV.

Interpretation of Atmospheric Neutrino Data

- Measure muon direction and energy not neutrino direction/energy.
- Don't have E or θ resolution to see oscillations.
- Oscillations 'smeared' out in data.
- Compare data to predictions for $\left|\Delta m^2\right|$.





- Data consistent with: $\left|\Delta m^2_{\text{atmos}}\right| \approx 0.0025\ \text{eV}^2$ and $\sin^2 2\theta_{\text{atmos}} \approx 1$.

3: Reactor Experiments

- To explain reactor neutrino experiments we need the full three neutrino expression for the electron neutrino survival probability (144) which depends on U_{e1}, U_{e2}, U_{e3}.
- Substituting these PMNS matrix elements in (144):

$$\begin{aligned} P(\nu_e \to \nu_e) &\approx 1 - 4U_{e1}^2 U_{e2}^2 \sin^2 \Delta_{21} - 4\left(1 - U_{e3}^2\right) U_{e3}^2 \sin^2 \Delta_{32} \\ &= 1 - 4\left(c_{12}c_{13}\right)^2 \left(s_{12}c_{13}\right)^2 \sin^2 \Delta_{21} - 4\left(1 - s_{13}^2\right) s_{13}^2 \sin^2 \Delta_{32} \\ &= 1 - c_{13}^4 \left(2s_{12}c_{12}\right)^2 \sin^2 \Delta_{21} - \left(2c_{13}s_{13}\right)^2 \sin^2 \Delta_{32} \\ &= 1 - \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} - \sin^2 2\theta_{13} \sin^2 \Delta_{32} \end{aligned}$$

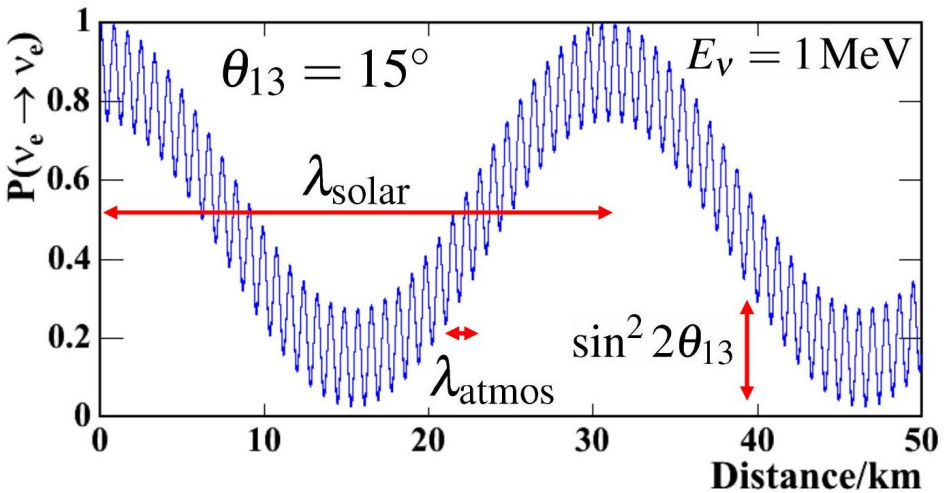
- Contributions with short wavelength (atmospheric) and long wavelength (solar).
- For a 1 MeV neutrino:

$$\lambda_{\text{osc}}(\text{km}) = 2.47 \frac{E(\text{GeV})}{\Delta m^2 \left(\text{eV}^2\right)}$$

$$\Rightarrow \lambda_{21} = 30.0 \text{ km}$$

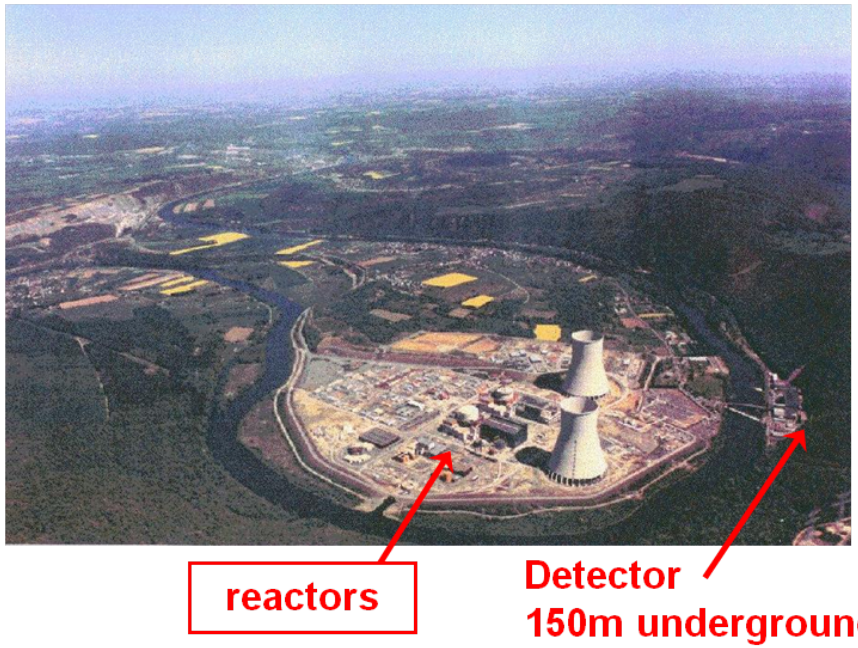
$$\lambda_{32} = 0.8 \text{ km}.$$

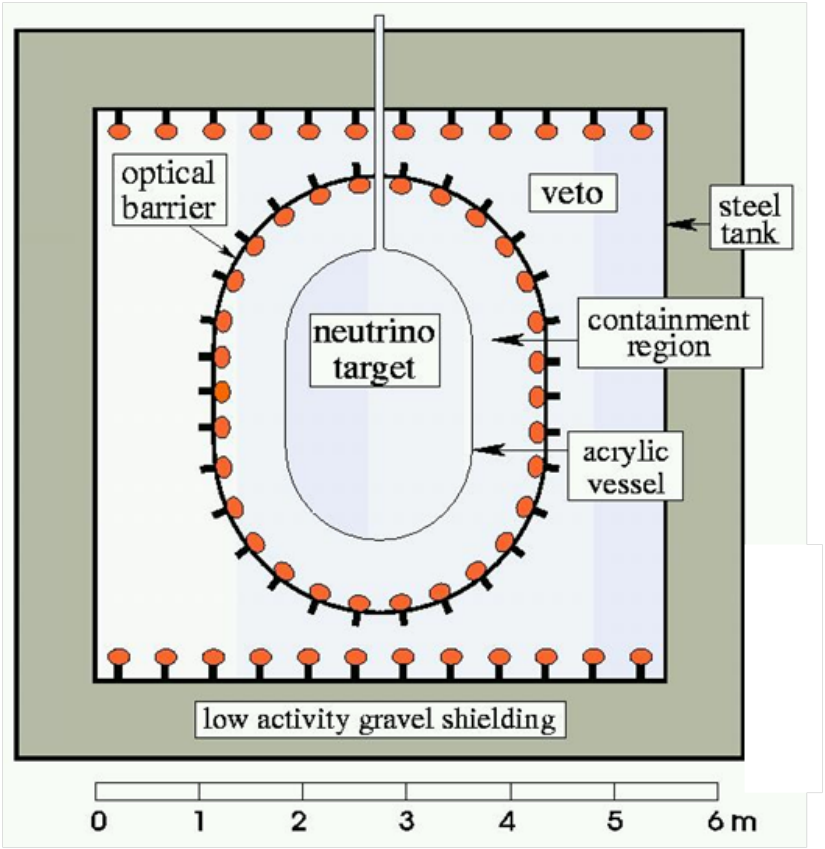
- Amplitude of short wavelength oscillations given by: $\sin^2 2\theta_{13}$.



Reactor Experiments I : CHOOZ (France) near two 4.2 GW nuclear reactors

- Place detector 1 km from reactor cores.
- Reactors produce intense flux of $\bar{\nu}_e$.





- Anti-neutrinos interact via inverse beta decay $\bar{\nu}_e + p \rightarrow e^+ + n$.
- Detector is liquid scintillator loaded with Gadolinium (large n capture cross section).
- Detect photons from positron annihilation and a delayed signal from photons from neutron capture on Gadolinium:

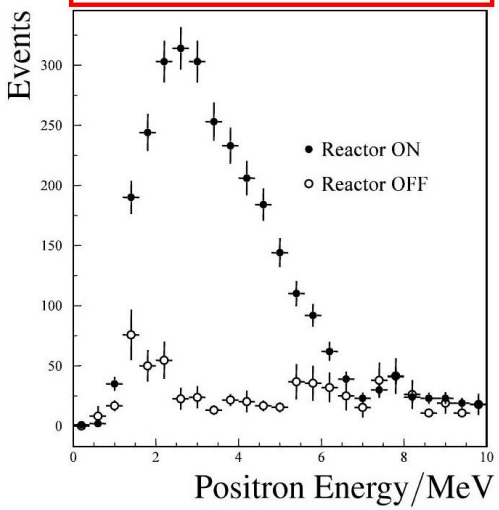
$$e^+ + e^- \rightarrow \gamma + \gamma$$

$$n + \mathrm{Gd} \rightarrow \mathrm{Gd}^* \rightarrow \mathrm{Gd} + \gamma + \gamma + \ldots$$

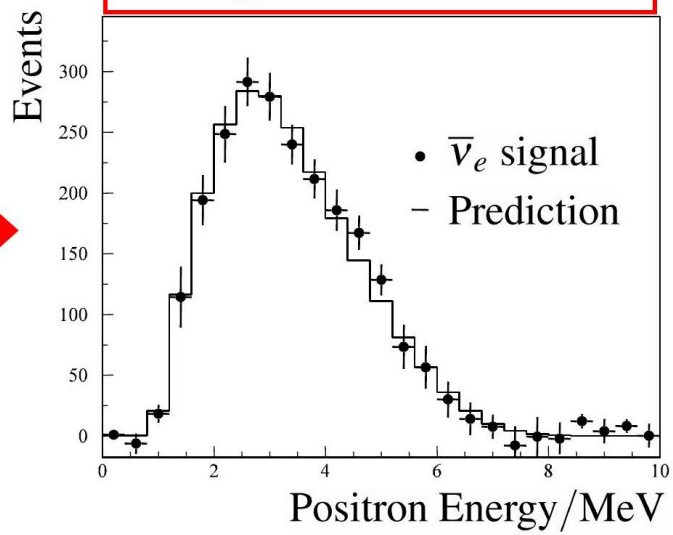
At 1 km and energies > 1 MeV, only the short wavelength component matters

$$P(\nu_e \to \nu_e) = 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2\Delta_{32} \approx 1 - \sin^2 2\theta_{13}\sin^2\Delta_{32}.$$

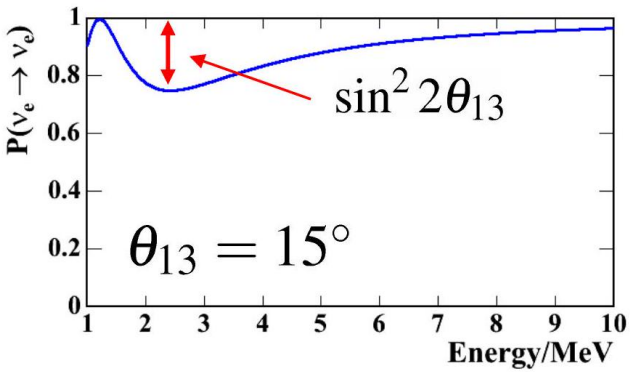
CHOOZ Raw Data



Background subtracted



Compare to effect of oscillations



- Data agree with unoscillated prediction both in terms of rate and energy spectrum

$$\text{Data in [9]} \quad \Longrightarrow \quad N_{\text{data}}/N_{\text{expect}} = 1.01 \pm 0.04.$$

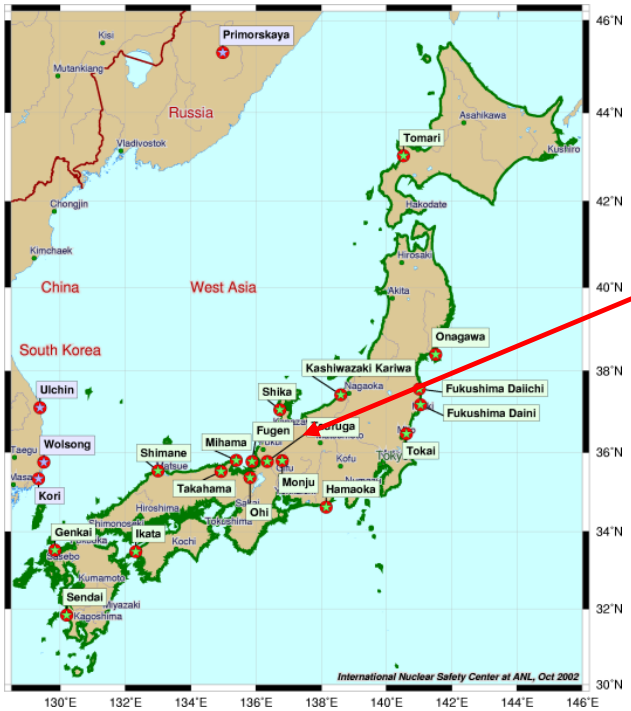
- Hence $\sin^2 2\theta_{13}$ must be small !

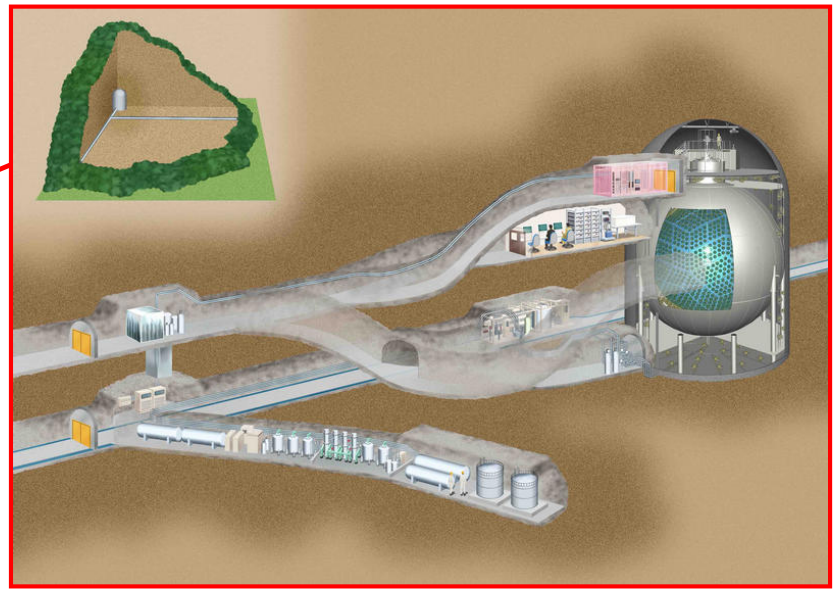
$$\Rightarrow \sin^2 2\theta_{13} < 0.12 - 0.2.$$

- Atmospheric neutrino results (elsewhere) can exclude $\theta_{13} \sim \frac{\pi}{2}$.
- Hence the CHOOZ limit: $\sin^2 2\theta_{13} < 0.2$ can be interpreted as $\sin^2\theta_{13} < 0.05$.

Reactor Experiments II : KamLAND

- Detector located in same mine as Super Kamiokande





- 70 GW from nuclear power (7% of World total) from reactors within $130 - 240$ km
- Liquid scintillator detector, 1789 PMTs
- Detection via inverse beta decay: $\bar{\nu}_e + p \rightarrow e^+ + n$ followed promptly by

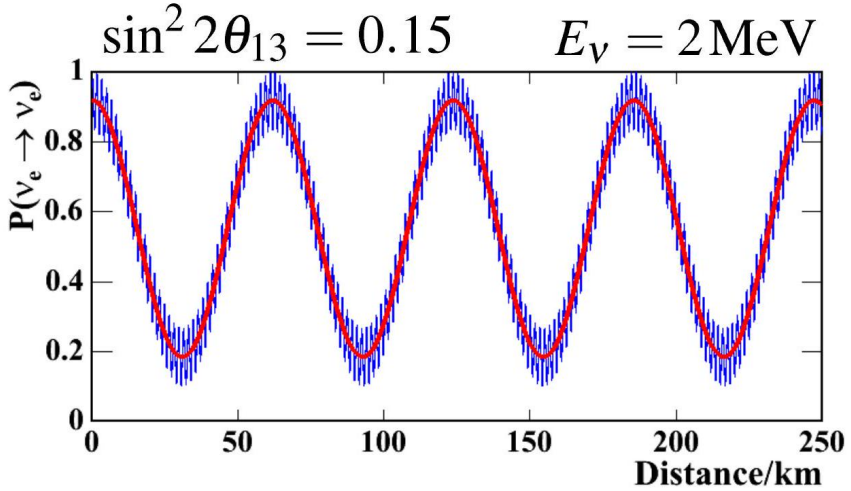
$$e^+ + e^- \rightarrow \gamma + \gamma$$

and then, after a (very helpful) characteristic delay by

$$n + p \rightarrow d + \gamma_{(2.2\ \mathrm{MeV})}.$$

- For MeV neutrinos at a distance of 130-240 km oscillations due to Δm^2_{32} are very rapid.
- Experimentally, only see average effect

$$\left\langle \sin^2 \Delta_{32} \right\rangle = 0.5.$$



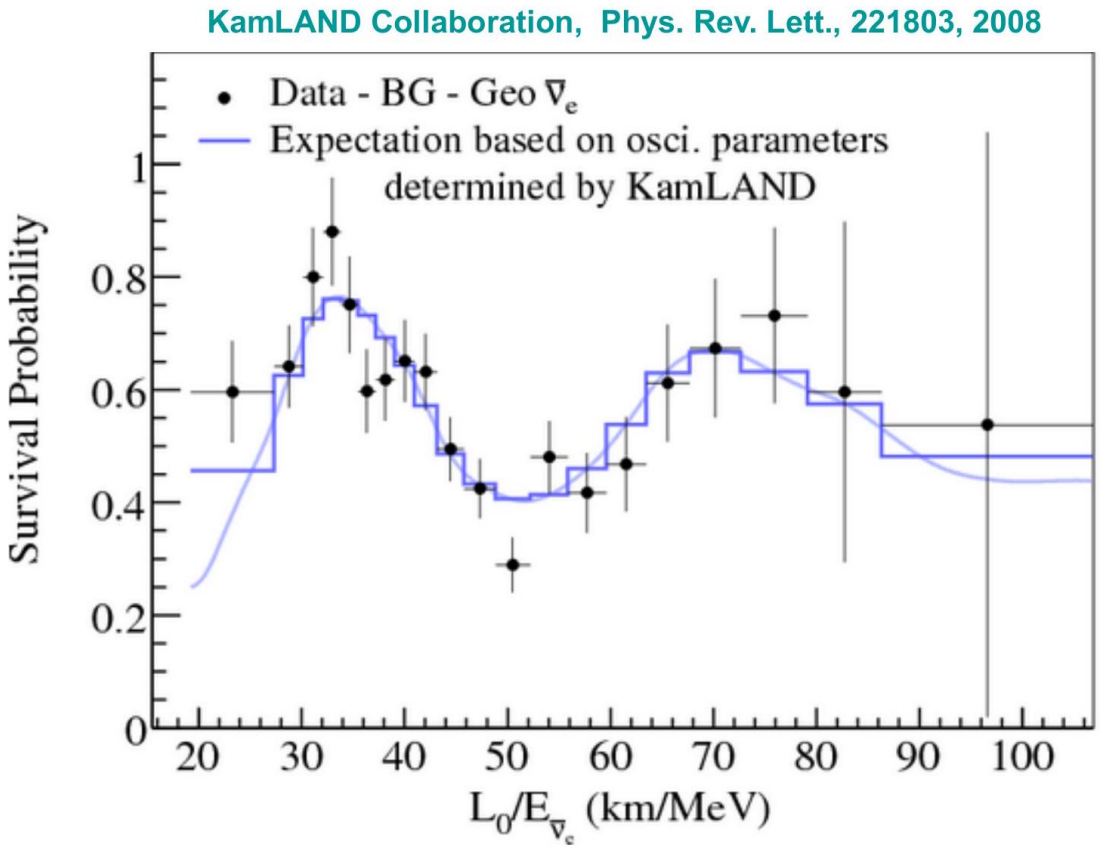
- Here:

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} - \sin^2 2\theta_{13} \sin^2 \Delta_{32} \\ &\approx 1 - \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} - \frac{1}{2} \sin^2 2\theta_{13} \qquad \text{(averaging over rapid oscillations)} \\ &= \cos^4\theta_{13} + \sin^4\theta_{13} - \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} \\ &\approx \cos^4\theta_{13} \left(1 - \sin^2 2\theta_{12} \sin^2 \Delta_{21}\right) \qquad \text{(neglect } \sin^4\theta_{13}\text{)} \end{aligned}$$

- Obtain two-flavour oscillation formula multiplied by $\cos^4\theta_{13}$ (which CHOOZ has already shown has $\cos^4\theta_{13} > 0.9$).

KamLAND Results

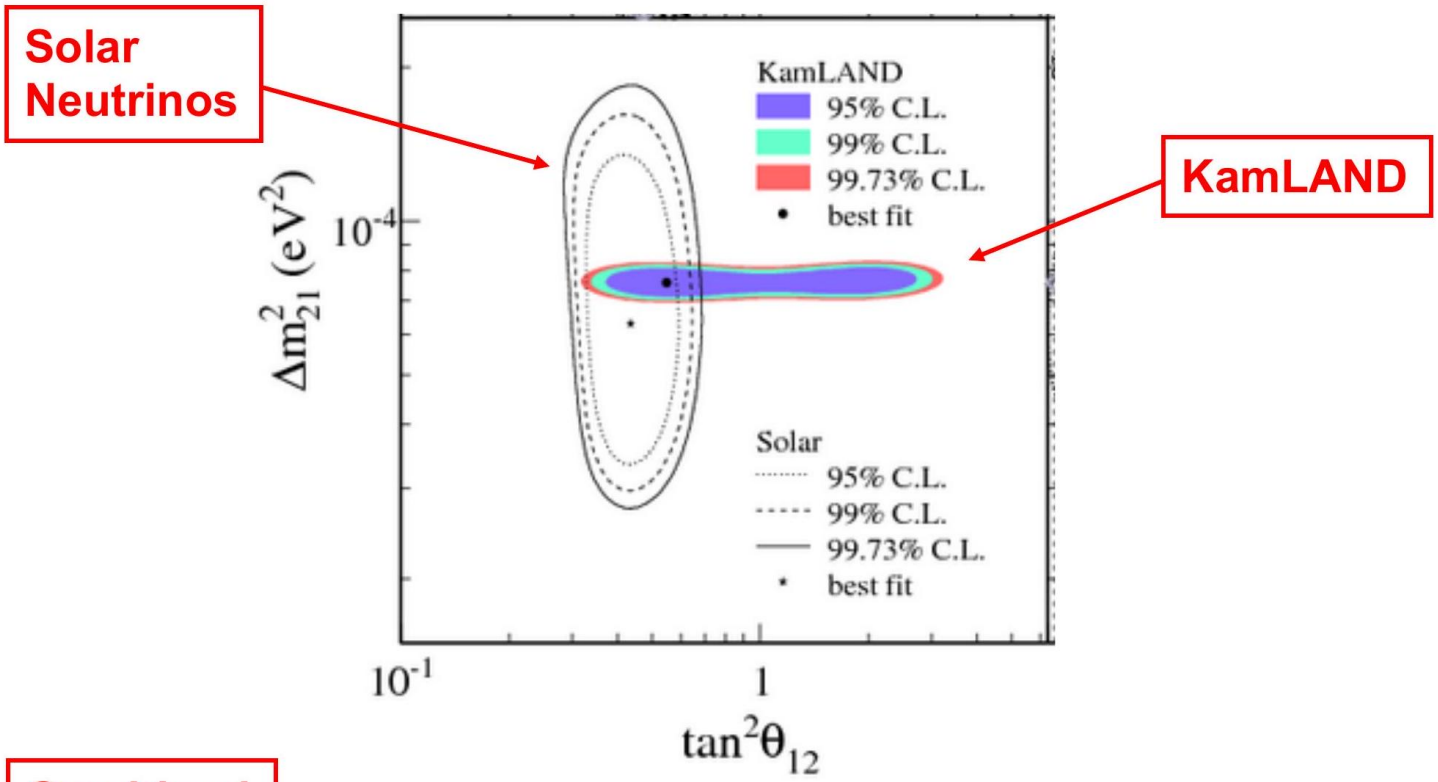
Observe: 1609 events.
Expect: 2179 ± 89 events (if no oscillations).



- Clear evidence of electron anti-neutrino oscillations consistent with the results from solar neutrinos.
- Oscillatory structure clearly visible.
- Compare data with expectations for different osc. parameters and perform χ^2 fit to extract measurment.

Combined Solar Neutrino and KamLAND Results

- KamLAND data provides strong constraints on $\left|\Delta m^2_{21}\right|$.
- Solar neutrino data (especially SNO) provides a strong constraint on θ_{12}.



Combined

$$|\Delta m^2_{21}| = (7.59 \pm 0.21) \times 10^{-5}\,\mathrm{eV}^2$$

$$\tan^2\theta_{12} = 0.47^{+0.06}_{-0.05}$$

Other more recent work ...

$\nu_\mu \to \nu_e$ appearance:

- T2K: $\nu_\mu \to \nu_e$ appearance (3.1σ) [arXiv:1304.0841].
- MINOS: $\nu_\mu \to \nu_e$ appearance (2σ).

$\bar{\nu}_e$ disappearance:

- Double-CHOOZ (2012): $\bar{\nu}_e$ disappearance (2σ) (doi = 10.1103/PhysRevLett.108.131801)

$$\sin^2 2\theta_{13} \approx 0.086 \pm 0.06.$$

- RENO (2012): 4.9σ evidence for $\bar{\nu}_e$ disappearance [arXiv:1204.0626], and

$$\sin^2 2\theta_{13} \approx 0.113 \pm 0.003$$

Throughout lifetime of course, evidence for non-zero value of θ_{13} has grown from 'none' to 'very clear':

- In Daya Bay experiment (see Question 21) measured:

$$\sin^2 2\theta_{13} = 0.090 \pm 0.009 \quad [\text{ arXiv:1310.6732 }] \text{ (2013), and then}$$

$$\sin^2 2\theta_{13} = 0.0851 \pm 0.0024 \quad [\text{ arXiv:2211.14988 }] \text{ (2022).}$$

Summary of Current Knowledge (from Review of Particle Physics 2024)

	Ref. [181] w/o SK-ATM		Ref. [181] w SK-ATM		Ref. [182] w SK-ATM		Ref. [183] w SK-ATM	
NO	Best Fit Ordering		Best Fit Ordering		Best Fit Ordering		Best Fit Ordering	
Param	bfp $\pm1\sigma$	3σ range	bfp $\pm1\sigma$	3σ range	bfp $\pm1\sigma$	3σ range	bfp $\pm1\sigma$	3σ range
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.03^{+0.12}_{-0.11}$	$2.70 \to 3.41$	$3.03^{+0.12}_{-0.12}$	$2.70 \to 3.41$	$3.03^{+0.13}_{-0.13}$	$2.63 \to 3.45$	$3.18^{+0.16}_{-0.16}$	$2.71 \to 3.69$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.40^{+0.80}_{-0.82}$	$30.85 \to 35.97$	$34.3^{+1.0}_{-1.0}$	$31.4 \to 37.4$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.72^{+0.18}_{-0.23}$	$4.06 \to 6.20$	$4.51^{+0.19}_{-0.16}$	$4.08 \to 6.03$	$4.55^{+0.18}_{-0.15}$	$4.16 \to 5.99$	$5.74^{+0.14}_{-0.14}$	$4.34 \to 6.10$
$\theta_{23}/^\circ$	$49.1^{+1.0}_{-1.3}$	$39.6 \to 51.9$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$42.4^{+1.0}_{-0.9}$	$40.2 \to 50.7$	$49.3^{+0.8}_{-0.8}$	$41.2 \to 51.3$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.203^{+0.056}_{-0.059}$	$2.029 \to 2.391$	$2.225^{+0.056}_{-0.059}$	$2.052 \to 2.398$	$2.23^{+0.07}_{-0.06}$	$2.04 \to 2.44$	$2.200^{+0.069}_{-0.062}$	$2.00 \to 2.405$
$\theta_{13}/^\circ$	$8.54^{+0.11}_{-0.12}$	$8.19 \to 8.89$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.59^{+0.13}_{-0.12}$	$8.21 \to 8.99$	$8.53^{+0.13}_{-0.12}$	$8.13 \to 8.92$
$\delta_{CP}/^\circ$	197^{+42}_{-25}	$108 \to 404$	232^{+36}_{-26}	$144 \to 350$	223^{+32}_{-23}	$139 \to 355$	194^{+24}_{-22}	$128 \to 359$
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.36^{+0.16}_{-0.15}$	$6.93 \to 7.93$	$7.50^{+0.22}_{-0.20}$	$6.94 \to 8.14$
$\frac{\Delta m^2_{32}}{10^{-3}\ \mathrm{eV}^2}$	$2.437^{+0.028}_{-0.027}$	$2.354 \to 2.523$	$2.433^{+0.026}_{-0.027}$	$2.353 \to 2.516$	$2.448^{+0.023}_{-0.031}$	$2.367 \to 2.521$	$2.47^{+0.02}_{-0.03}$	$2.40 \to 2.46$
IO	$\Delta\chi^2 = 2.3$		$\Delta\chi^2 = 6.4$		$\Delta\chi^2 = 6.5$		$\Delta\chi^2 = 6.4$	
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.03^{+0.12}_{-0.11}$	$2.70 \to 3.41$	$3.03^{+0.12}_{-0.11}$	$2.70 \to 3.41$	$3.03^{+0.13}_{-0.13}$	$2.63 \to 3.45$	$3.18^{+0.16}_{-0.16}$	$2.71 \to 3.69$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.40^{+0.80}_{-0.82}$	$30.85 \to 35.97$	$34.3^{+1.0}_{-1.0}$	$31.4 \to 37.4$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.78^{+0.16}_{-0.21}$	$4.12 \to 6.23$	$5.69^{+0.16}_{-0.21}$	$4.12 \to 6.13$	$5.69^{+0.13}_{-0.21}$	$4.17 \to 6.06$	$5.78^{-0.10}_{+0.17}$	$4.33 \to 6.08$
$\theta_{23}/^\circ$	$49.5^{+0.9}_{-1.2}$	$39.9 \to 52.1$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$	$49.0^{+0.7}_{-1.4}$	$40.2 \to 51.1$	$49.5^{+0.6}_{-1.0}$	$41.2 \to 51.2$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.219^{+0.060}_{-0.057}$	$2.047 \to 2.396$	$2.223^{+0.058}_{-0.058}$	$2.048 \to 2.416$	$2.23^{+0.06}_{-0.06}$	$2.03 \to 2.45$	$2.225^{+0.064}_{-0.070}$	$2.02 \to 2.42$
$\theta_{13}/^\circ$	$8.57^{+0.12}_{-0.11}$	$8.23 \to 8.90$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$	$8.59^{+0.13}_{-0.12}$	$8.19 \to 9.00$	$8.58^{+0.12}_{-0.14}$	$8.17 \to 8.96$
$\delta_{CP}/^\circ$	286^{+27}_{-32}	$192 \to 360$	276^{+22}_{-29}	$194 \to 344$	274^{+25}_{-27}	$193 \to 342$	284^{+26}_{-28}	$200 \to 353$
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.36^{+0.16}_{-0.15}$	$6.93 \to 7.93$	$7.50^{+0.22}_{-0.20}$	$6.94 \to 8.14$
$\frac{\Delta m^2_{32}}{10^{-3}\ \mathrm{eV}^2}$	$-2.498^{+0.032}_{-0.025}$	$-2.581 \to -2.408$	$-2.486^{+0.028}_{-0.025}$	$-2.570 \to -2.406$	$-2.492^{+0.025}_{-0.030}$	$-2.578 \to -2.413$	$-2.52\pm^{+0.03}_{-0.02}$	$-2.60 \to -2.44$

Experiment	Dominant	Important
Solar Experiments	θ_{12}	Δm^2_{21} , θ_{13}
Reactor LBL (KamLAND)	Δm^2_{21}	θ_{12} , θ_{13}
Reactor MBL (Daya-Bay, Reno, D-Chooz)	θ_{13}, $\lvert\Delta m^2_{31,32}\rvert$	
Atmospheric Experiments (SK, IC-DC)		θ_{23},$\lvert\Delta m^2_{31,32}\rvert$, θ_{13},δ_{CP}
Accel LBL ν_μ,$\bar\nu_\mu$, Disapp (K2K, MINOS, T2K, NOνA)	$\lvert\Delta m^2_{31,32}\rvert$, θ_{23}	
Accel LBL ν_e,$\bar\nu_e$ App (MINOS, T2K, NOνA)	δ_{CP}	θ_{13} , θ_{23}

Not examinable
Not examinable

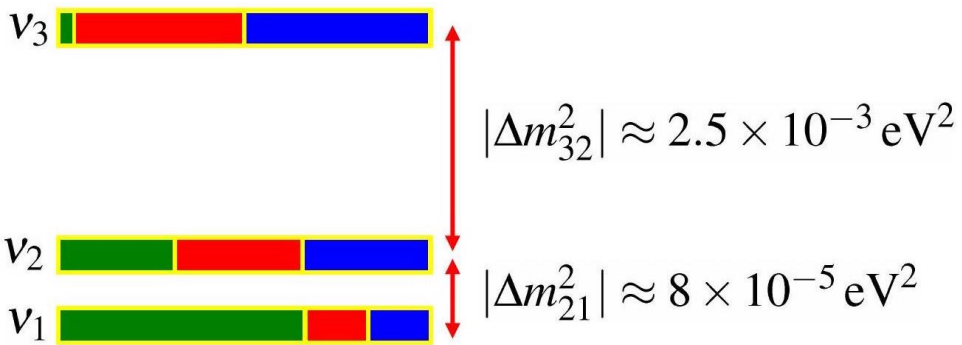
Approximate interpretation of neutrino mixing parameters:

- For the central values of the mixing angles on the previous page obtain:

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \approx \begin{pmatrix} 0.82 & 0.55 & 0.15e^{-i\delta} \\ -0.38 & 0.55 & 0.74 \\ 0.42 & -0.62 & 0.66 \end{pmatrix}.$$

- i.e. approximate expressions for mass eigenstates in terms of weak eigenstates are:

$$|\nu_3\rangle \approx \frac{1}{\sqrt{2}}\left(|\nu_\mu\rangle + |\nu_\tau\rangle\right),$$
$$|\nu_2\rangle \approx 0.56\left(|\nu_e\rangle + |\nu_\mu\rangle - |\nu_\tau\rangle\right),$$
$$|\nu_1\rangle \approx 0.82\,|\nu_e\rangle - 0.4\left(|\nu_\mu\rangle - |\nu_\tau\rangle\right).$$



Final words on Neutrino Masses

- Neutrino oscillations require non-zero neutrino masses.
- Oscillation measuremments only determine mass-squared differences — not the masses themselves.
- We have no direct measure of neutrino mass — only mass limits:

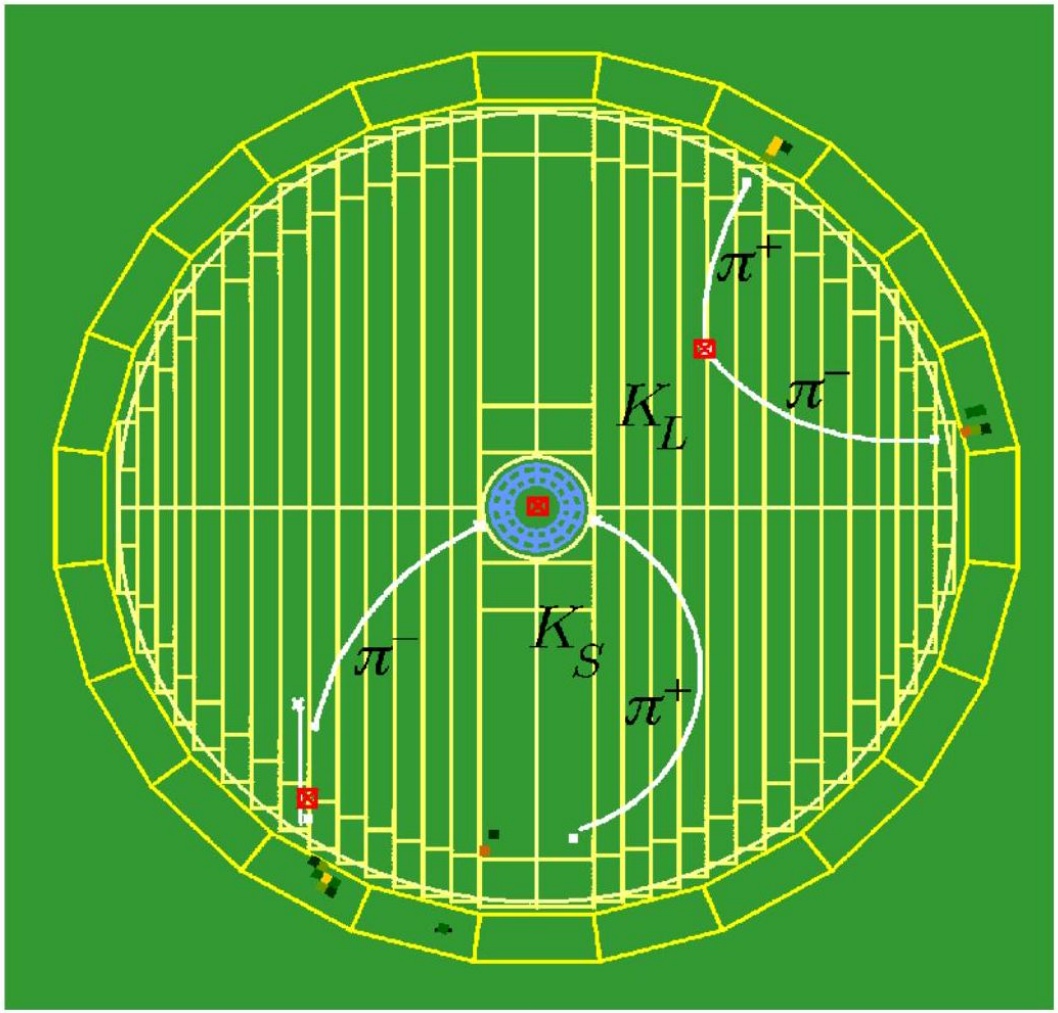
$$m_\nu(e) < 2 \text{ eV}; \quad m_\nu(\mu) < 0.17 \text{ MeV}; \quad m_\nu(\tau) < 18.2 \text{ MeV}.$$

  Note the e, μ, τ refer to charged lepton flavour in the experiment, e.g. $m_\nu(e) < 2$ eV refers to the limit from tritium beta-decay.
- Also from cosmological evolution infer that the sum

$$\sum_i m_{\nu_i} < \text{ few eV } .$$

- Before ∼2008 neutrinos were assumed to be massless, but there were hints that neutrinos might oscillate.
- Now, know a great deal about massive neutrinos, but there are still many unknowns. including: δ, the mass hierarchy, and the absolute values of neutrino masses.
- Measurements of these SM parameters is the focus of the next generation of neutrino expts (e.g. DUNE).

Prof. Alex Mitov, 2026



H12: The CKM Matrix and CP Violation

CP Violation in the Early Universe

- Today the universe is matter dominated. There is no evidence for anti-galaxies, *etc*. The matter/anti-matter asymmetry is estimated to be

$$\xi = \frac{n_B - n_{\bar{B}}}{n_\gamma} \approx \frac{n_B}{n_\gamma} \approx 10^{-9}$$

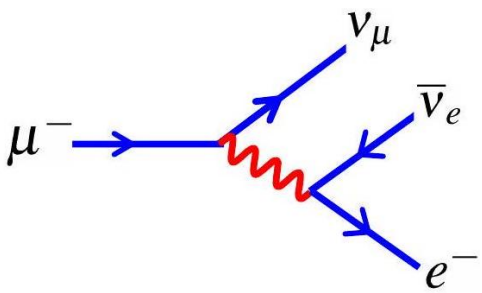
  i.e. for every baryon in the universe today there are 10^9 photons and no anti-baryons.
- One possible explanation is that for every 10^9 anti-baryons in the early universe there were $10^9 + 1$ baryons, and that these annihilated to 1 baryon + $\sim 10^9$ photons + no anti-baryons.
- To generate such an asymmetry from a symmetric precursor state, **three conditions** must be met (Sakharov, 1967) [10]:
  1. **Baryon number violation**: i.e. $n_B - n_{\bar{B}}$ is not constant,
  2. **C and CP violation**: if CP is conserved for a reaction which generates a net number of baryons over anti-baryons there would be a CP conjugate reaction generating a net number of anti-baryons.
  3. **Departure from thermal equilibrium**: In thermal equilibrium any baryon number violating process will be balanced by the inverse reaction.

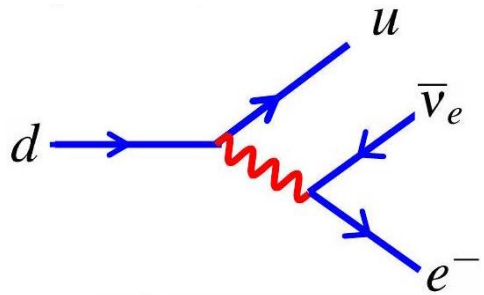
We would like to know if the Standard Model of particle physics provides enough CP violation to generate the observed matter antimatter asymmetry of the universe from a symmetric starting point.

- There are two places in the SM where CP violation enters: complex phases in the **PMNS matrix** (neutrinos) and the **CKM matrix** (quarks).
- To date CP violation has been confirmed only in the quark sector.
- We will approach quark-sector oscillations it in two stages first without and then with CP violation.
- We will see many features in common with neutrino oscillations — except that the oscillating particles (mesons) will have finite rather than infinite lifetimes (neutrinos).

The Weak Interaction of Quarks

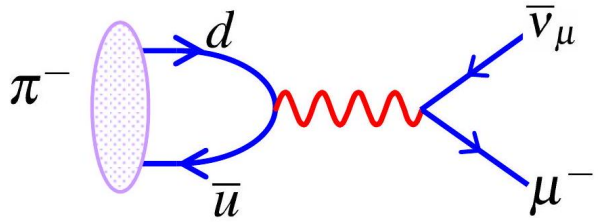
- A slightly different values of G_F is measured in μ decay and nuclear β decay:

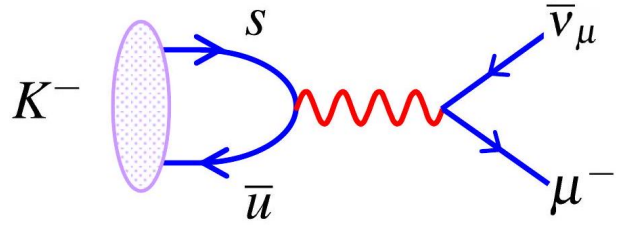




$G_F^\mu = (1.16632 \pm 0.00002) \times 10^{-5}\ \mathrm{GeV}^{-2}$ $\quad$ $G_F^\beta = (1.136 \pm 0.003) \times 10^{-5}\ \mathrm{GeV}^{-2}$

- In addition, certain hadronic decay modes are observed to be suppressed, e.g. compare $K^- \to \mu^- \bar{\nu}_\mu$ and $\pi^- \to \mu^- \bar{\nu}_\mu$. Kaon decay rate suppressed factor 20 compared to the expectation assuming a universal weak interaction for quarks.

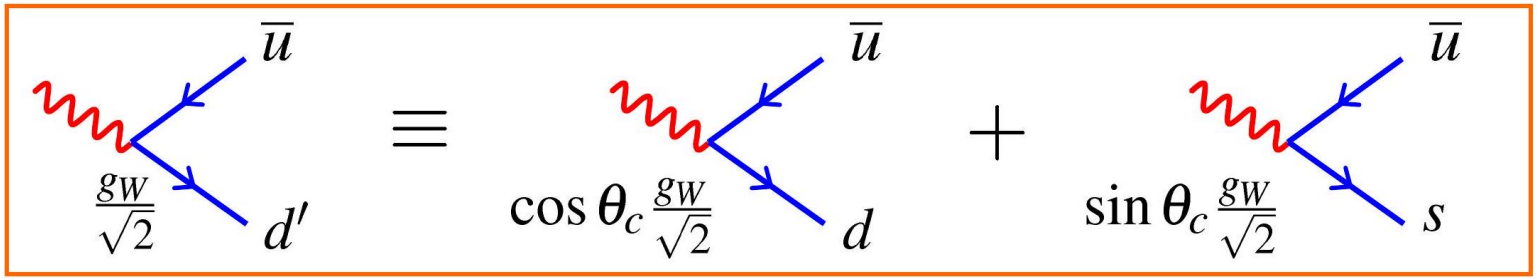




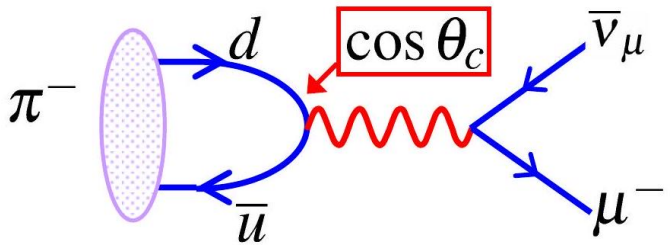
- Both observations explained by **Cabibbo hypothesis** (1963) [11]: weak eigenstates are different from mass eigenstates, i.e. weak interactions of quarks have same strength as for leptons but a u-quark couples to a linear combination of s and d.

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

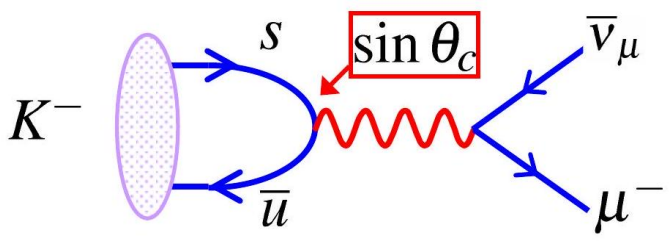
- i.e. can consider weak interaction to couple u exclusively to d':



- Can explain the observations on the previous pages with $\theta_c = 13.1°$.
  Kaon decay is suppressed by a factor of $\tan^2\theta_c \approx 0.05$ relative to pion decay.

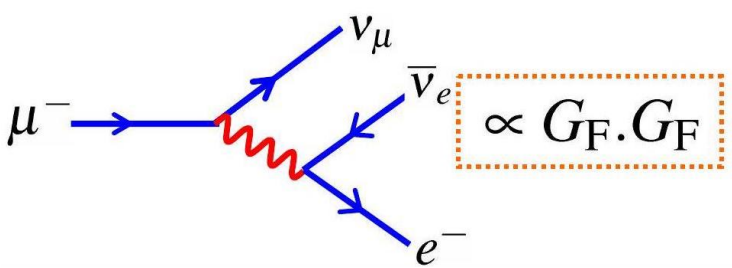


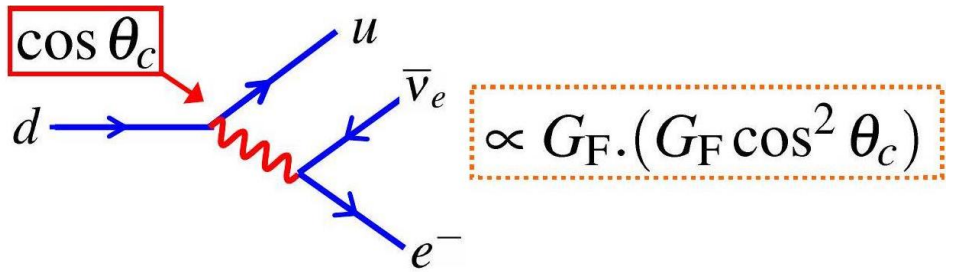
$$\Gamma\left(\pi^- \to \mu^- \bar{\nu}_\mu\right) \propto |M|^2 \propto \cos^2\theta_c$$



$$\Gamma\left(K^- \to \mu^- \bar{\nu}_\mu\right) \propto |M|^2 \propto \sin^2\theta_c$$

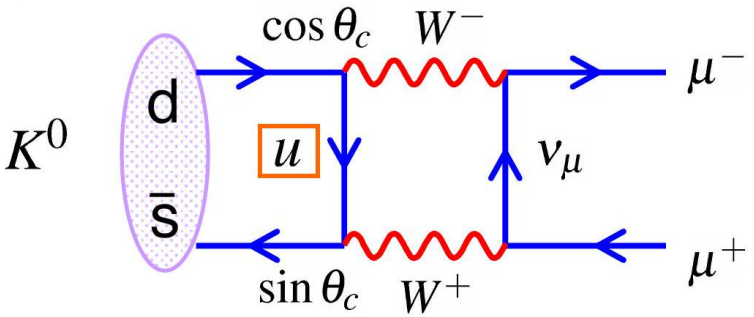
- Hence expect $G_F^\beta = G_F^\mu \cos\theta_c$





GIM Mechanism

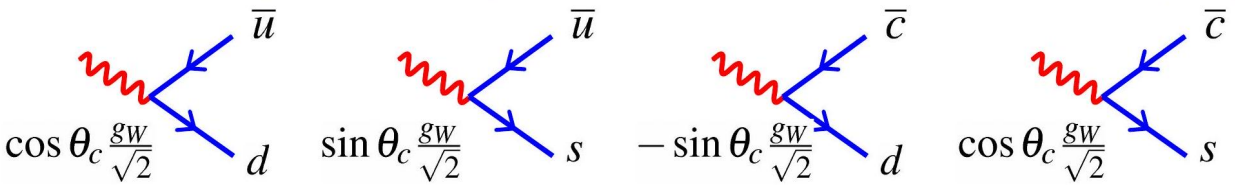
- In the weak interaction have couplings between both *ud* and *us* which implies that neutral mesons can decay via box diagrams, e.g.



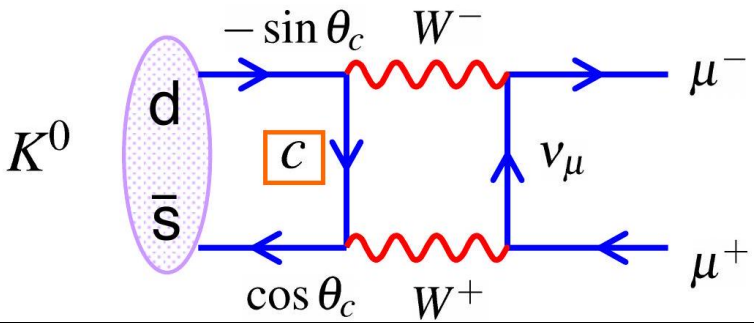
$$\boxed{M_1 \propto g_W^4 \cos\theta_c \sin\theta_c}.$$

Historically, the observed branching was much smaller than predicted.

- Led Glashow, Illiopoulos and Maiani to postulate existence of an extra quark - before discovery of charm quark in 1974. If c couples exclusively to s' then Weak interaction couplings become:



- Gives another box diagram for $K^0 \to \mu^+\mu^-$ with $\boxed{M_2 \propto -g_W^4 \cos\theta_c \sin\theta_c}$:



- Same final state so sum amplitudes:

$$\boxed{|M|^2 = |M_1 + M_2|^2 \approx 0}.$$

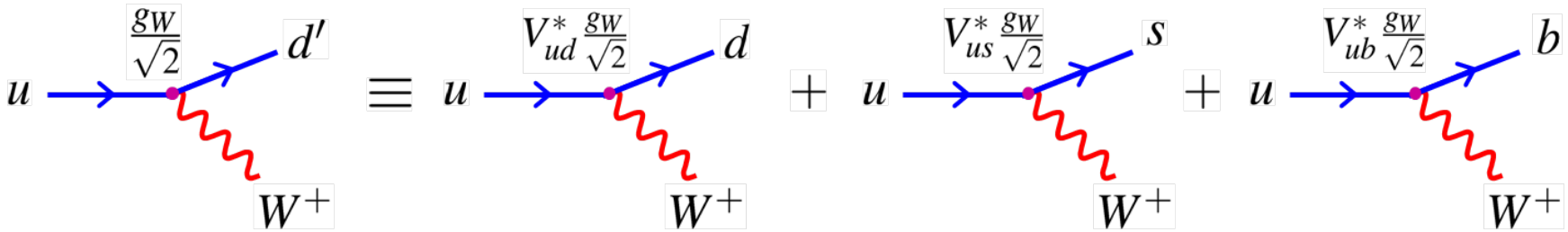
- Cancellation not exact because $m_u \neq m_c$.

Cabibbo–Kobayashi–Maskawa (CKM) Matrix

- Extend ideas to three quark flavours (analogue of three flavour neutrino treatment)

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Weak eigenstates, CKM Matrix , Mass Eigenstates
(Cabibbo, Kobayashi, Maskawa)
e.g. Weak eigenstate d' is produced in weak decay of an up quark:

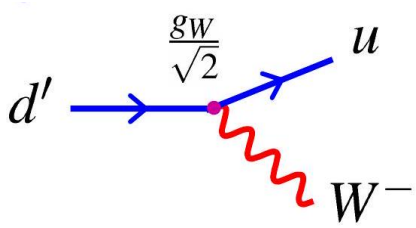


- The CKM matrix elements V_{ij} are complex constants.
- The CKM matrix is unitary.
- The V_{ij} are not predicted by the SM — have to be determined from experiment.

Depending on the order of the interaction $u \to d$ or $d \to u$, the CKM matrix enters as either V_{ud} or V_{ud}^*.

[Spoiler: the happy quark index on V always comes first, and no conjugation is needed if a walk backwards up the fermion line also encounters the happy quark first. See below!]

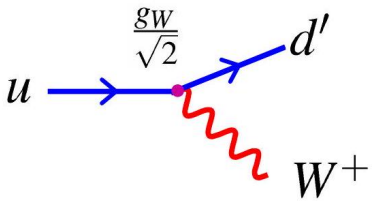
- For $d' \to u$ the weak current is:



$$j_{d'u} = \bar{u}\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu\frac{1}{2}\left(1-\gamma^5\right)\right]d'$$

giving the $d \to u$ weak current: $\boxed{j_{du} = \bar{u}\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu\frac{1}{2}\left(1-\gamma^5\right)\right]V_{ud}\,d}$.

- For $u \to d'$ the weak current is:



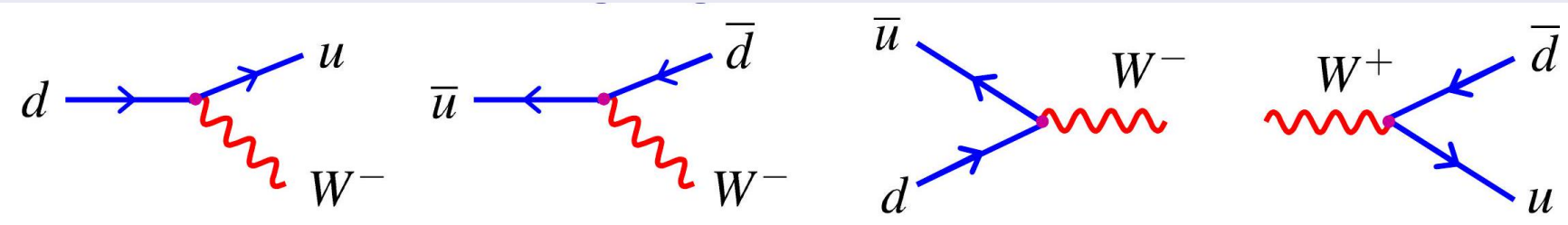
$$j_{ud'} = \bar{d}'\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu\frac{1}{2}\left(1-\gamma^5\right)\right]u.$$

This time $\bar{d}' = d'^\dagger\gamma^0 \to \left(V_{ud}\,d\right)^\dagger\gamma^0 = V_{ud}^* d^\dagger\gamma^0 = V_{ud}^*\bar{d}$

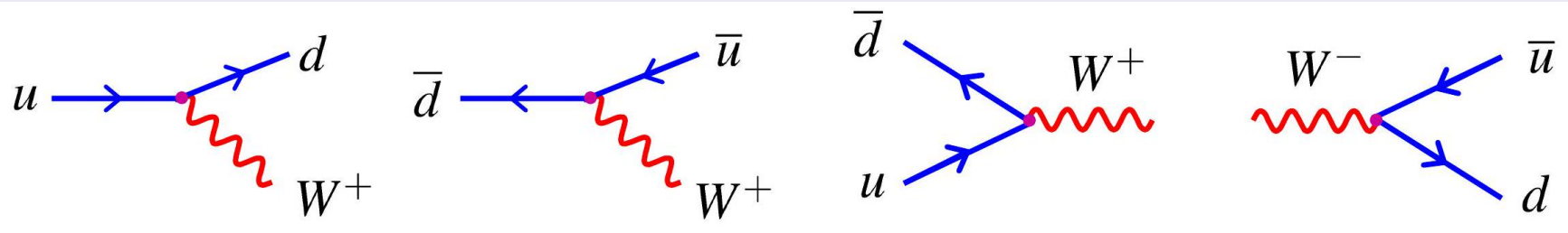
giving the $u \to d$ weak current: $\boxed{j_{ud} = \bar{d}\,V_{ud}^*\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu\frac{1}{2}\left(1-\gamma^5\right)\right]u}$.

Examples showing use of what was explained on the last slide:

$-i\frac{g_W}{\sqrt{2}}V_{ud}\gamma^\mu\frac{1}{2}\left(1-\gamma^5\right)$ is the vertex factor the following diagrams:



$-i\frac{g_W}{\sqrt{2}}V_{ud}^*\gamma^\mu\frac{1}{2}\left(1-\gamma^5\right)$ is the vertex factor for the following diagrams:



CKM summary

- Experimentally (see Appendix XX) determine:

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \approx \begin{pmatrix} 0.97373 & 0.2243 & 0.0038 \\ 0.221 & 0.975 & 0.041 \\ 0.009 & 0.042 & 1.01 \end{pmatrix}.$$

- Assuming unitarity of CKM matrix, e.g. $|V_{ub}|^2 + |V_{cb}|^2 + |V_{tb}|^2 = 1$, get (2022 PDG):

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.97435 \pm 0.00016 & 0.22500 \pm 0.00067 & 0.00369 \pm 0.00011 \\ 0.22486 \pm 0.00067 & 0.97349 \pm 0.00016 & 0.04182^{+0.00085}_{-0.00074} \\ 0.00857^{+0.00020}_{-0.00018} & 0.04110^{+0.00083}_{-0.00072} & 0.999118^{+0.000031}_{-0.000036} \end{pmatrix}$$

which is **fairly diagonal — very different from PMNS!**

Notes:

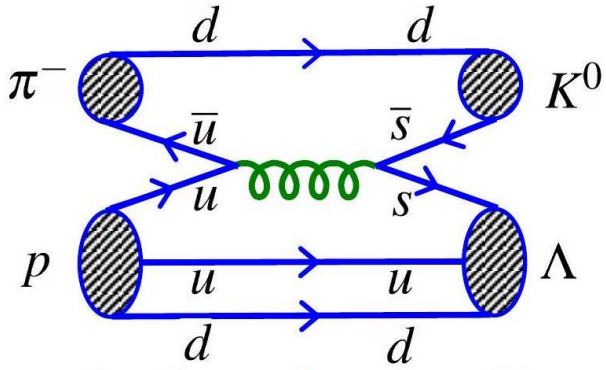
- Weak Charged Current (i.e. $W^{\pm}$) is only SM interaction that changes flavour.
- As off-diagonal elements are small, flavour changes are 'discouraged'.
- Weak interactions are largest between quarks of the same generation.
- Coupling between first and third generation quarks is very small!
- Just as for the PMNS matrix — the CKM matrix allows CP violation in the SM.

The Neutral Kaon System

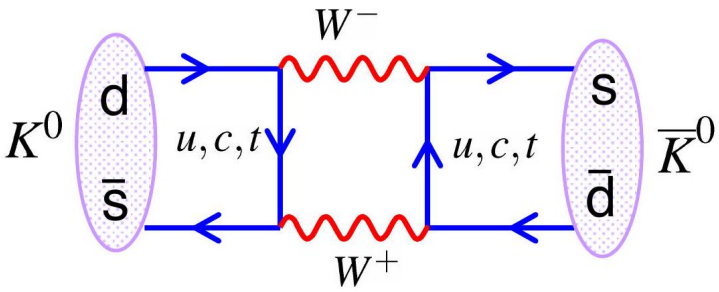
- Neutral Kaons are produced in strong interaction, e.g.:

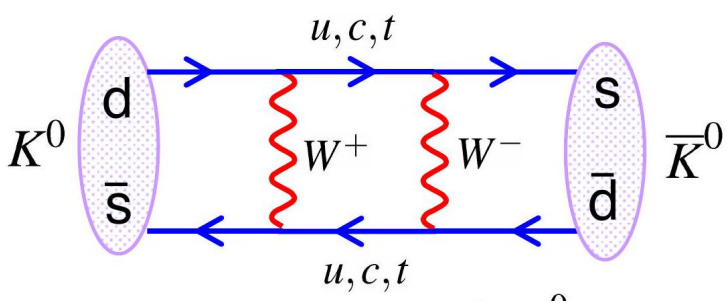
$$\pi^-(d\bar{u}) + p(uud) \to \Lambda(uds) + K^0(d\bar{s})$$
$$\pi^+(u\bar{d}) + p(uud) \to K^+(u\bar{s}) + \bar{K}^0(s\bar{d}) + p(uud)$$

but decay via the weak interaction.



- The Weak Interaction also allows mixing of neutral kaons via "**box diagrams**"





- This allows transitions between the strong eigenstates states $K^0, \bar{K}^0$.
- Consequently, the neutral kaons propagate as eigenstates of the overall strong + weak interaction (Appendix XXI); i.e. as linear combinations of $K^0, \bar{K}^0$

These neutral kaon mass states are called the "K-short", K_S, and the "K-long', K_L. These states have approximately the same mass, $m(K_S) \approx m(K_L) \approx 498\ \mathrm{MeV}$, but have very different lifetimes: $\tau(K_S) = 0.9 \times 10^{-10}\ \mathrm{s}$ versus $\tau(K_L) = 0.5 \times 10^{-7}\ \mathrm{s}$.

CP Eigenstates

The K_S and K_L are closely related to eigenstates of the combined charge conjugation and parity operators: CP.

- The strong eigenstates $K^0(d\bar{s})$ and $\bar{K}^0(s\bar{d})$ have $J^P = 0^-$ and

$$\hat{P}\left|K^0\right\rangle = -\left|K^0\right\rangle, \quad \hat{P}\left|\overline{K^0}\right\rangle = -\left|\bar{K}^0\right\rangle.$$

- The charge conjugation operator changes particle into anti-particle and vice versa:

$$\hat{C}\left|K^0\right\rangle = \hat{C}|d\bar{s}\rangle = +|s\bar{d}\rangle = \left|\bar{K}^0\right\rangle. \quad \text{Likewise:} \quad \hat{C}\left|\bar{K}^0\right\rangle = \left|K^0\right\rangle.$$

  (The plus sign is purely conventional. We could have used a minus sign with no physical consequences.)

- Consequently:

$$\hat{C}\hat{P}\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle \quad \hat{C}\hat{P}\left|\bar{K}^0\right\rangle = -\left|K^0\right\rangle$$

  i.e. neither K^0 or $\bar{K}^0$ are eigenstates of CP.

- Form CP eigenstates $|K_1\rangle$ and $|K_2\rangle$ from linear combs. of $\left|K^0\right\rangle$ and $\left|\bar{K}^0\right\rangle$ as follows:

$$\boxed{\begin{aligned} |K_1\rangle &= \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right) \\ |K_2\rangle &= \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right) \end{aligned}} \quad \text{so that} \quad \boxed{\begin{aligned} \hat{C}\hat{P}\,|K_1\rangle &= +|K_1\rangle \\ \hat{C}\hat{P}\,|K_2\rangle &= -|K_2\rangle \end{aligned}}.$$

Neutral Kaon decays

Neutral kaons often decay to two or three pions.

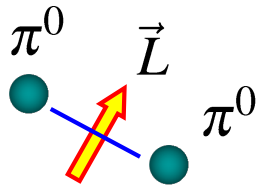
This is because:

- Pions are the lightest hadrons.
- The kaon masses are approximately 498 MeV and the pion masses are approximately 140 MeV, and so there is (just!) room to make three pions.

Neutral Kaon decays to Two Pions

Decays to Two Pions:

★ $K^0 \to \pi^0\pi^0 \qquad J^P: \quad 0^- \to 0^- + 0^-$



- **Conservation of angular momentum** ➡ $\vec{L} = 0$

$$\Rightarrow \quad \hat{P}(\pi^0\pi^0) = -1.-1.(-1)^L = +1$$

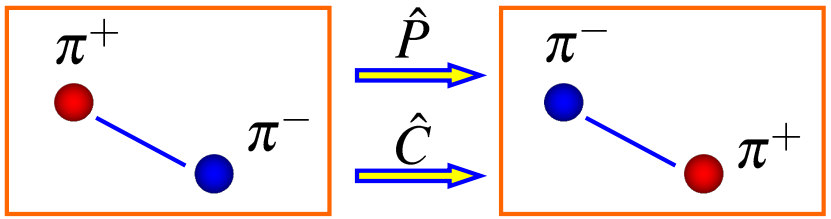
- **The** $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ **is an eigenstate of** $\hat{C}$

$$C(\pi^0\pi^0) = C\pi^0.C\pi^0 = +1.+1 = +1$$

$$\Rightarrow \quad \boxed{CP(\pi^0\pi^0) = +1}$$

★ $K^0 \to \pi^+\pi^-$ **as before** $\hat{P}(\pi^+\pi^-) = +1$

★**Here the C and P operations have the identical effect**



Hence the combined effect of $\hat{C}\hat{P}$ **is to leave the system unchanged**

$$\hat{C}\hat{P}(\pi^+\pi^-) = +1$$

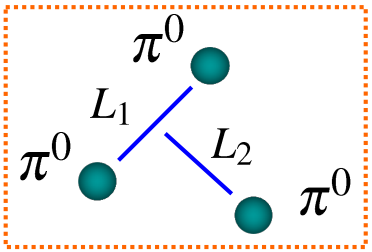
Conclusion:

Neutral kaon decays to two pions occur in CP-even (i.e. $CP = +1$) eigenstates.

Neutral Kaon decays to Three Pions

Decays to Three Pions:

★ $K^0 \to \pi^0\pi^0\pi^0$ $\quad J^P: \quad 0^- \to 0^- + 0^- + 0^-$



•**Conservation of angular momentum:**

$$L_1 \oplus L_2 = 0 \quad \Rightarrow \quad L_1 = L_2$$

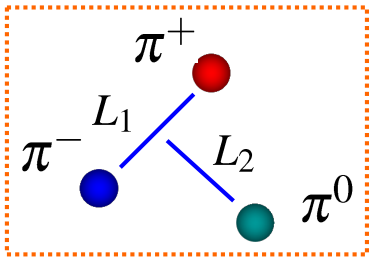
Remember L is magnitude of angular momentum vector

$$P(\pi^0\pi^0\pi^0) = -1.-1.-1.(-1)^{L_1}.(-1)^{L_2} = -1$$

$$C(\pi^0\pi^0\pi^0) = +1.+1.+1$$

$$\Rightarrow \quad CP(\pi^0\pi^0\pi^0) = -1$$

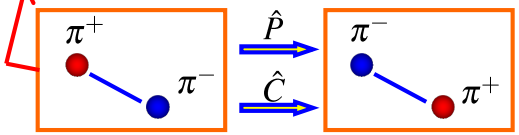
★ $K^0 \to \pi^+\pi^-\pi^0$



•**Again** $L_1 = L_2$

$$P(\pi^+\pi^-\pi^0) = -1.-1.-1.(-1)^{L_1}.(-1)^{L_2} = -1$$

$$C(\pi^+\pi^-\pi^0) = +1.C(\pi^+\pi^-) = P(\pi^+\pi^-) = (-1)^{L_1}$$



Hence: $CP(\pi^+\pi^-\pi^0) = -1.(-1)^{L_1}$

•**The small amount of energy available in the decay,** $m(K) - 3m(\pi) \approx 70\,\text{MeV}$ **means that the L>0 decays are strongly suppressed by the angular momentum barrier effects (recall QM tunnelling in alpha decay)**

Conclusion:

Neutral kaon decays to three pions **mostly** occur in *CP*-odd (i.e. $CP = -1$) eigenstates.

If *CP* were conserved in the weak decays of neutral kaons ...

- we would expect decays to pions to occur from states of definite *CP* (i.e. K_1 and K_2):

$$\boxed{\begin{aligned}|K_1\rangle &= \tfrac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle)\\ |K_2\rangle &= \tfrac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle)\end{aligned}} \qquad \boxed{\begin{aligned}\hat{C}\hat{P}\,|K_1\rangle &= +\,|K_1\rangle\\ \hat{C}\hat{P}\,|K_2\rangle &= -\,|K_2\rangle\end{aligned}} \qquad \boxed{\begin{aligned}K_1 &\to \pi\pi\\ K_2 &\to \pi\pi\pi\end{aligned}} \qquad \boxed{\begin{aligned}&\text{CP EVEN}\\ &\text{CP ODD}\end{aligned}};$$

- we would expect lifetimes of *CP*-eigenstates to be very different given that:

  energy available in **two** pion decay is $m_K - 2m_\pi \approx 220\ \mathrm{MeV}$ and
  energy available in **three** pion decay is $m_K - 3m_\pi \approx 80\ \mathrm{MeV}$;

- and we would expect decays to two pions to be more rapid than decays to three pions due to increased phase space.

This is exactly what is observed:

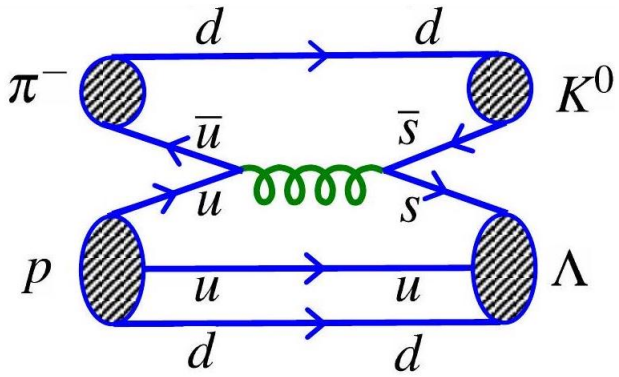
A short-lived state **"K-short"** which decays to (mainly) to two pions and a long-lived state **"K-long"** which decays (only) to three pions.
Therefore: in the absence of *CP*-violation we can identify:

$$\begin{aligned}|K_S\rangle = |K_1\rangle &\equiv \tfrac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right) \quad \text{with decays:} \quad K_S \to \pi\pi, \quad \text{and}\\ |K_L\rangle = |K_2\rangle &\equiv \tfrac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right) \quad \text{with decays:} \quad K_L \to \pi\pi\pi.\end{aligned}$$

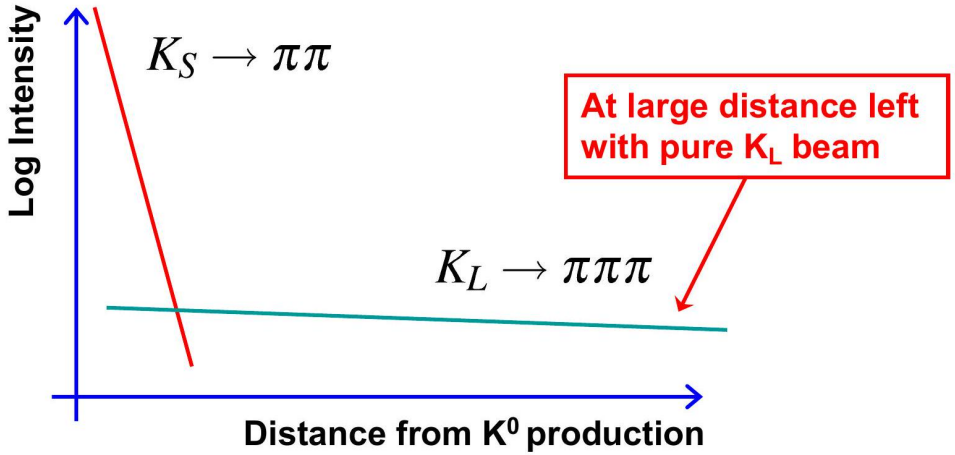
Neutral Kaon Decays to pions

- Consider the decays of a beam of K^0. The decays to pions occur in states of definite CP.
- If CP is conserved in the decay, need to express K^0 in terms of K_S and K_L:

$$|K_0\rangle = \frac{1}{\sqrt{2}} \left(|K_S\rangle + |K_L\rangle \right).$$



- Hence from the point of view of decays to pions, a K^0 beam is a linear combination of CP eigenstates containing a rapidly decaying CP-even component and a long-lived CP-odd component.
- Therefore, expect to see predominantly two-pion decays near start of beam and predominantly three pion decays further downstream:



- Suppose we have a beam of pure K^0 at time $t = 0$: $\boxed{|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|K_S\rangle + |K_L\rangle\right)}$.
- Put in the time dependence: $\boxed{|K_S(t)\rangle = |K_S\rangle\, e^{-im_S t - \Gamma_S t/2}}$ with m_S being the mass of the K-short and $\Gamma_S = 1/\tau_S$ being its decay rate.

Aside: The term $e^{-\Gamma_S t/2}$ ensures the K_S probability density decays exponentially, i.e.:

$$|\psi_S|^2 = \langle K_S(t) \mid K_S(t)\rangle = e^{-\Gamma_S t} = e^{-t/\tau_S}.$$

- Hence the wave-function evolves as:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[|K_S\rangle\, e^{-\left(im_S + \frac{\Gamma_S}{2}\right)t} + |K_L\rangle\, e^{-\left(im_L + \frac{\Gamma_L}{2}\right)t}\right].$$

- Compressing by defining $\theta_S(t) = e^{-(im_S + \Gamma_S/2)t}$ and $\theta_L(t) = e^{-(im_L + \Gamma_L/2)t}$ we have:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\theta_S(t)\,|K_S\rangle + \theta_L(t)\,|K_L\rangle\right).$$

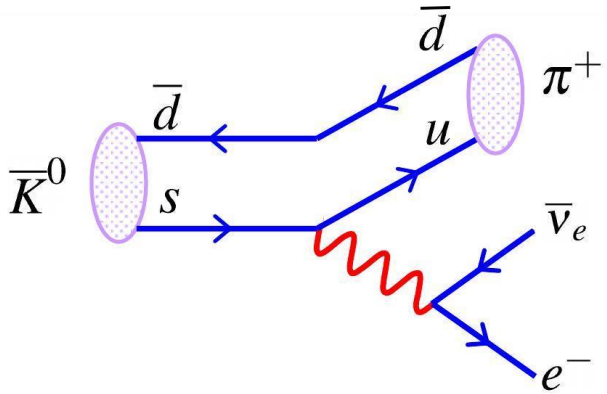
- As anticipated, the decay rate to two pions for a state which was produced as K^0 is:

$$\Gamma\left(K^0_{t=0} \to \pi\pi\right) \propto |\langle K_S \mid \psi(t)\rangle|^2 \propto |\theta_S(t)|^2 = e^{-\Gamma_S t} = e^{-t/\tau_S}.$$

Neutral Kaons decays to Leptons

- Neutral Kaons can also decays to Leptons:

$$\bar{K}^0 \to \pi^+ e^- \bar{\nu}_e \quad \bar{K}^0 \to \pi^+ \mu^- \bar{\nu}_\mu$$
$$K^0 \to \pi^- e^+ \nu_e \quad K^0 \to \pi^- \mu^+ \nu_\mu \quad .$$



- Note: the final states are not CP eigenstates which is why we express these decays in terms of $K^0, \bar{K}^0$.
- Neutral kaons propagate as combined eigenstates of weak + strong interaction i.e. the K_S, K_L.
- The **main** decay modes/branching fractions are:

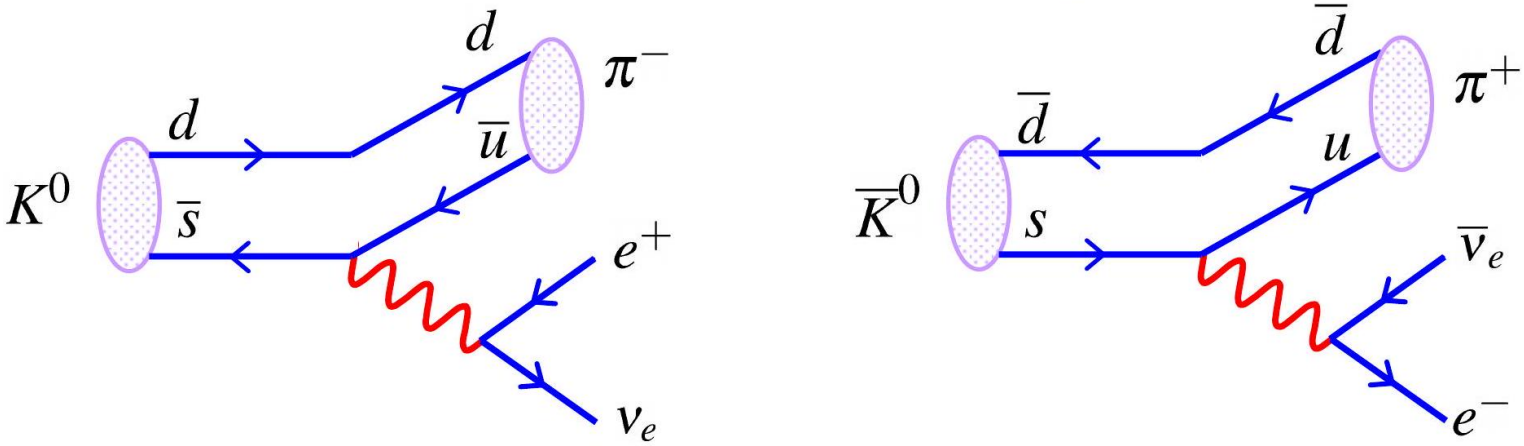
K_S	$\to \pi^+\pi^-$	$BR = 69.2\%$
	$\to \pi^0\pi^0$	$BR = 30.7\%$
	$\to \pi^- e^+ \nu_e$	$BR = 0.03\%$
	$\to \pi^+ e^- \bar{\nu}_e$	$BR = 0.03\%$
	$\to \pi^- \mu^+ \nu_\mu$	$BR = 0.02\%$
	$\to \pi^+ \mu^- \bar{\nu}_\mu$	$BR = 0.02\%$

K_L	$\to \pi^+\pi^-\pi^0$	$BR = 12.6\%$
	$\to \pi^0\pi^0\pi^0$	$BR = 19.6\%$
	$\to \pi^- e^+ \nu_e$	$BR = 20.2\%$
	$\to \pi^+ e^- \bar{\nu}_e$	$BR = 20.2\%$
	$\to \pi^- \mu^+ \nu_\mu$	$BR = 13.5\%$
	$\to \pi^+ \mu^- \bar{\nu}_\mu$	$BR = 13.5\%$

.

- Leptonic decays are more likely for the K-long because the three pion decay modes have a lower decay rate than the two pion modes of the K-short.

What decays **cannot** happen? [and implications for decay tagging]



- The charge of the observed pion (or lepton) tags the decay as from either a $\bar{K}^0$ or K^0 because

$$\begin{array}{l} K^0 \to \pi^- e^+ \nu_e \\ \bar{K}^0 \to \pi^+ e^- \bar{\nu}_e \end{array} \quad \text{but} \quad \begin{array}{l} \bar{K}^0 \not\to \pi^- e^+ \nu_e \\ K^0 \not\to \pi^+ e^- \bar{\nu}_e \end{array} \quad \text{is not allowed (see Question 23).}$$

- So for an initial K^0 beam, observe the decays to both charge combinations:

$$K^0_{t=0} \longrightarrow K^0 \hookrightarrow \pi^- e^+ \nu_e \qquad K^0_{t=0} \longrightarrow \bar{K}^0 \hookrightarrow \pi^+ e^- \bar{\nu}_e$$

which provides a way of measuring strangeness oscillations.

Strangeness Oscillations (neglecting CP violation)

The "semi-leptonic" decay rate to $\pi^- e^+ \nu_e$ occurs from the K^0 state. Hence to calculate the expected decay rate, need to know the K^0 component of the wave-function.
For example, for a beam which was initially K^0 we have:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(\theta_S(t) |K_S\rangle + \theta_L(t) |K_L\rangle\right). \tag{150}$$

Writing K_S and K_L in terms of K^0 and $\bar{K}^0$:

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{2}\left[\theta_S(t)\left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right) + \theta_L(t)\left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right)\right] \\ &= \frac{1}{2}\left(\theta_S + \theta_L\right)\left|K^0\right\rangle + \frac{1}{2}\left(\theta_L - \theta_S\right)\left|\bar{K}^0\right\rangle. \end{aligned}$$

Because $\theta_S(t) \neq \theta_L(t)$ a state that was initially a K^0 evolves with time into a mixture of K^0 and $\bar{K}^0$ — "strangeness oscillations".

The K^0 intensity (i.e. K^0 fraction), whose form we will improve over pages, is thus:

$$\Gamma\left(K^0_{t=0} \to K^0\right) = \left|\left\langle K^0 \mid \psi(t)\right\rangle\right|^2 = \frac{1}{4}\left|\theta_S + \theta_L\right|^2. \tag{151}$$

Similarly:
$$\Gamma\left(K^0_{t=0} \to \bar{K}^0\right) = \left|\left\langle \bar{K}^0 \mid \psi(t)\right\rangle\right|^2 = \frac{1}{4}\left|\theta_S - \theta_L\right|^2. \tag{152}$$

- Using the identity $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}\,(z_1 z_2^*)$ we find:

$$\begin{aligned}
|\theta_S \pm \theta_L|^2 &= \left| e^{-\left(im_S + \frac{1}{2}\Gamma_S\right)t} \pm e^{-\left(im_L + \frac{1}{2}\Gamma_L\right)t} \right|^2 \\
&= e^{-\Gamma_S t} + e^{-\Gamma_L t} \pm 2\,\mathrm{Re}\left\{ e^{-im_S t} e^{-\frac{1}{2}\Gamma_S t} \cdot e^{+im_L t} e^{-\frac{1}{2}\Gamma_L t} \right\} \\
&= e^{-\Gamma_S t} + e^{-\Gamma_L t} \pm 2e^{-\frac{\Gamma_S + \Gamma_L}{2} t}\,\mathrm{Re}\left\{ e^{-i(m_S - m_L)t} \right\} \\
&= e^{-\Gamma_S t} + e^{-\Gamma_L t} \pm 2e^{-\frac{\Gamma_S + \Gamma_L}{2} t} \cos\left(m_S - m_L\right) t \\
&= e^{-\Gamma_S t} + e^{-\Gamma_L t} \pm 2e^{-\frac{\Gamma_S + \Gamma_L}{2} t} \cos \Delta m t.
\end{aligned}$$

Thus we see oscillations between neutral kaon states

The frequency is given by the mass splitting $\Delta m = m\left(K_L\right) - m\left(K_S\right)$.

This is reminiscent of neutrino oscillations! Only this time we have decaying states!

Thus, from (151) and (152) get form of kaon fractions that is nicer than on last page:

$$\Gamma\left(K^0_{t=0} \to K^0\right) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S + \Gamma_L)t/2} \cos \Delta m t \right] \tag{153}$$

$$\Gamma\left(K^0_{t=0} \to \bar{K}^0\right) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S + \Gamma_L)t/2} \cos \Delta m t \right]. \tag{154}$$

Observed strangeness oscillations in the Kaon System

Experimentally it is measured/found that:

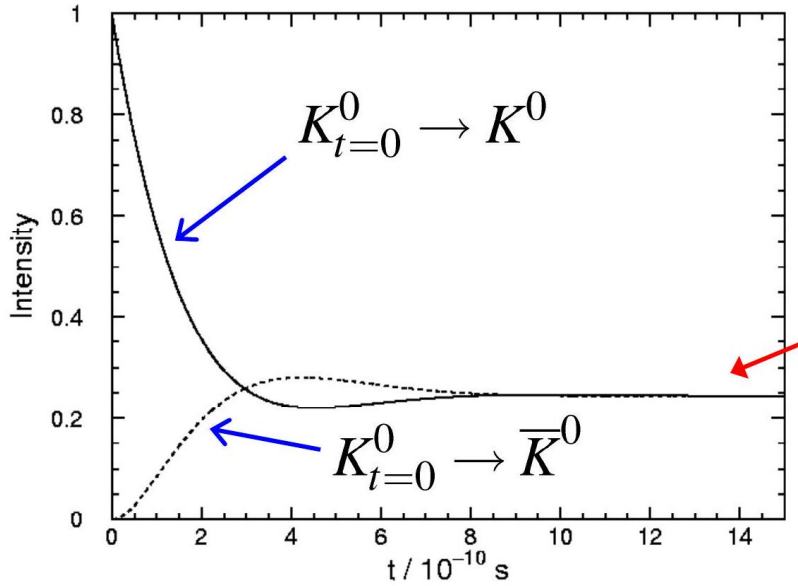
$\boxed{\tau(K_S) = 0.9 \times 10^{-10}\ \text{s}}$, $\boxed{\tau(K_L) = 0.5 \times 10^{-7}\ \text{s}}$ and

$\boxed{\Delta m = (3.506 \pm 0.006) \times 10^{-15}\ \text{GeV}}$

i.e. the K-long mass is greater than the K-short by one part in 10^{16}.

Oscillation period is thus $\boxed{T_{osc} = \dfrac{2\pi\hbar}{\Delta m} \approx 1.2 \times 10^{-9}\ \text{s}}$.

- The oscillation period is relatively long compared to the K_S lifetime. Consequently, we only observe a slight wiggle. (See below!)



$$\Gamma(K^0_{t=0} \to K^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma(K^0_{t=0} \to \overline{K}^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

After a few K_S lifetimes, left with a pure K_L beam which is half K^0 and half $\overline{K}^0$

The CPLEAR Experiment [12]



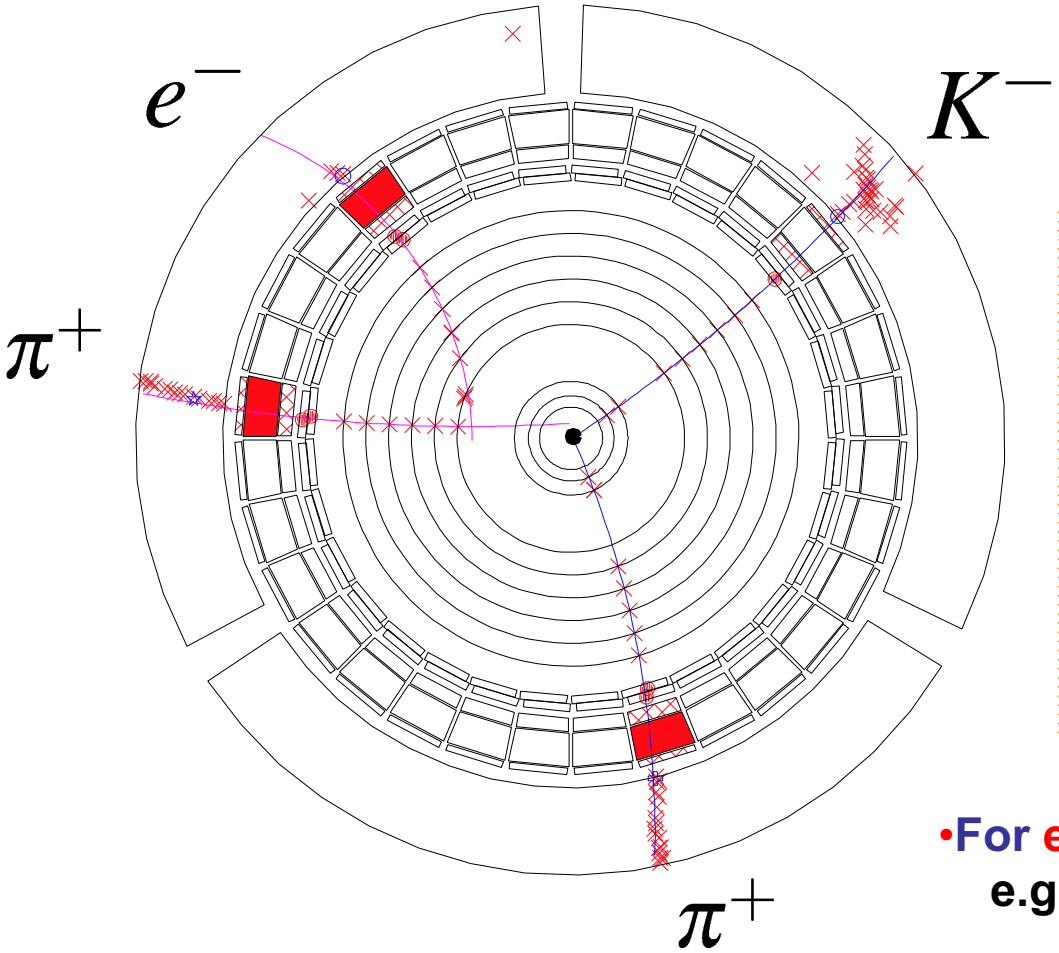
- CPLEAR used a low energy anti-proton beam at CERN (1990-1996).
- Produced neutral kaons in reactions like:

$$\bar{p}p \to K^- \pi^+ K^0,$$
$$\bar{p}p \to K^+ \pi^- \bar{K}^0,$$

- Low energy, so particles produced almost at rest.
- Observe production process and decay in the same detector.

- Charge of $K^{\pm}\pi^{\mp}$ in the production process tags the initial neutral kaon as either K^0 or $\bar{K}^0$.
- Charge of decay products tags the decay as either as being either K^0 or $\bar{K}^0$.
- Provides a direct probe of strangeness oscillations.

An example of a CPLEAR event



$$K^-(s\bar{u})$$
$$K^0(d\bar{s})$$
$$\overline{K}^0(s\bar{d})$$

Production:

$$\bar{p}p \to K^-\pi^+K^0$$

Mixing

Decay:

$$\overline{K}^0 \to \pi^+e^-\overline{\nu}_e$$

- For each event know initial wave-function,
  e.g. here: $|\psi(t=0)\rangle = |K^0\rangle$

Quantifying the asymmetry

Can measure decay rates as a function of time for all combinations:
e.g. $R^+ = \Gamma\left(K^0_{t=0} \to \pi^- e^+ \bar{\nu}_e\right) \propto \Gamma\left(K^0_{t=0} \to K^0\right)$.

From equations (153) and (154) and similar relations:

$$R_+ \equiv \Gamma\left(K^0_{t=0} \to \pi^- e^+ \nu_e\right) = N_{\pi ev}\frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$R_- \equiv \Gamma\left(K^0_{t=0} \to \pi^+ e^- \bar{\nu}_e\right) = N_{\pi ev}\frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\bar{R}_- \equiv \Gamma\left(\bar{K}^0_{t=0} \to \pi^+ e^- \bar{\nu}_e\right) = N_{\pi ev}\frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

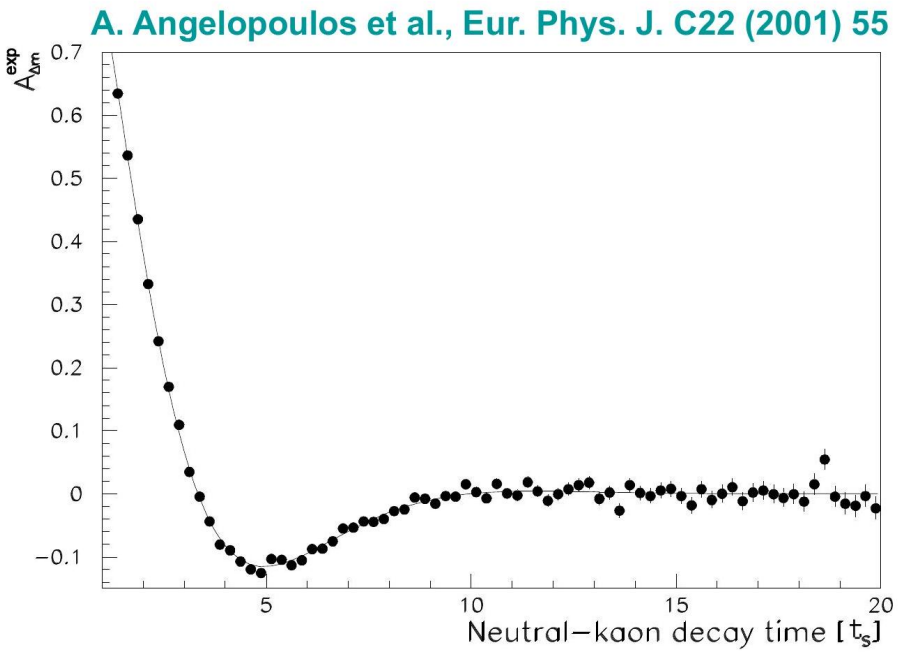
$$\bar{R}_+ \equiv \Gamma\left(\bar{K}^0_{t=0} \to \pi^- e^+ \nu_e\right) = N_{\pi ev}\frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

where $N_{\pi ev}$ is some overall normalisation factor.

Express measurements as an "asymmetry" to remove dependence on $N_{\pi ev}$:

$$A_{\Delta m} = \frac{\left(R_+ + \bar{R}_-\right) - \left(R_- + \bar{R}_+\right)}{\left(R_+ + \bar{R}_-\right) + \left(R_- + \bar{R}_+\right)} = \frac{2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t}{e^{-\Gamma_S t} + e^{-\Gamma_L t}}.$$

CPLEAR's results:



- Points show the data.
- The line shows the theoretical prediction for the value of Δm most consistent with the CPLEAR data:

$$\Delta m = 3.485 \times 10^{-15}\ \mathrm{GeV}$$

using

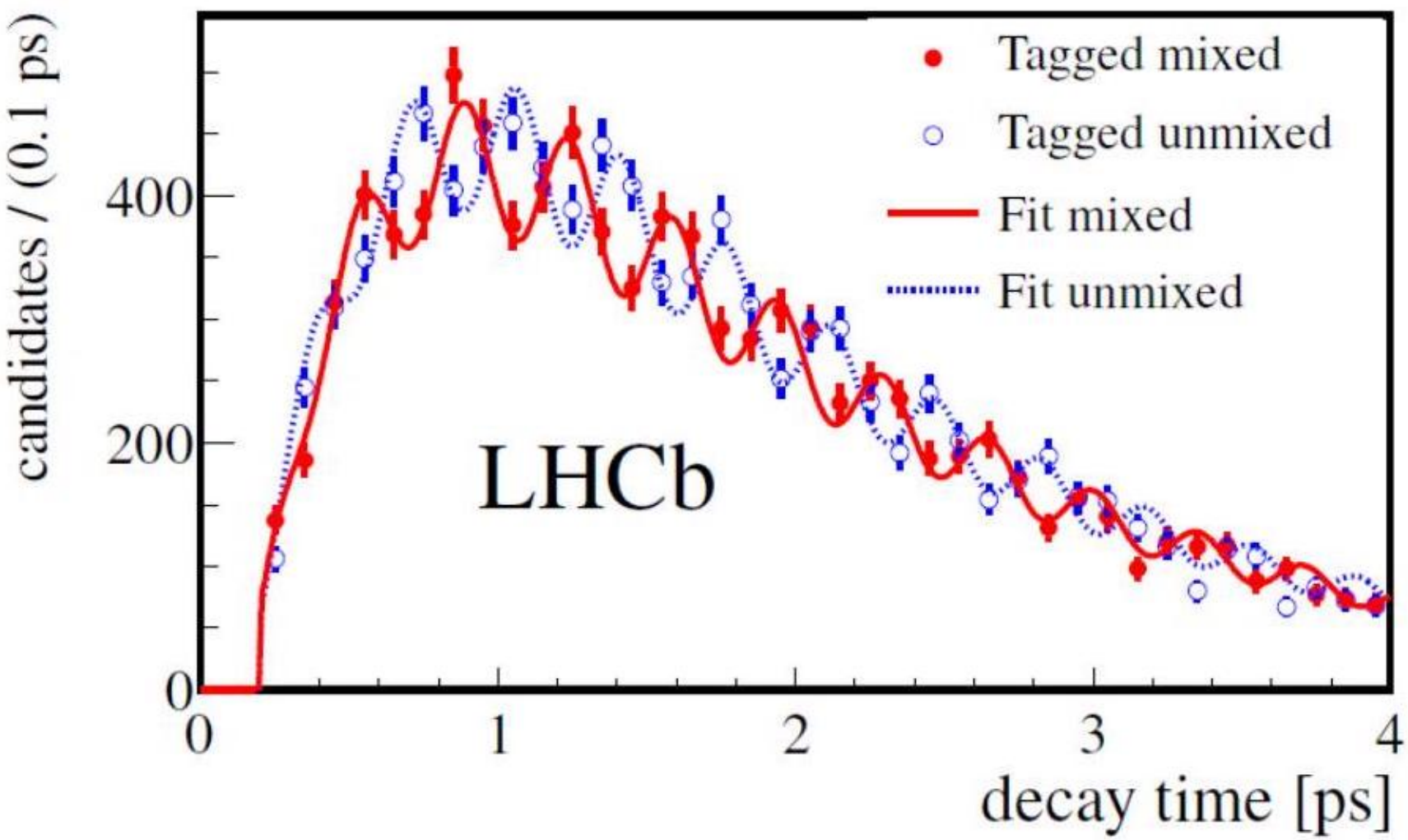
$$A_{\Delta m} = \frac{2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t}{e^{-\Gamma_S t}+e^{-\Gamma_L t}}.$$

- The sign of Δm is not determined here but is known from other experiments.

When the CPLEAR results are combined with experiments at FermiLab obtain:

$$\Delta m = m(K_L) - m(K_S) = (3.506 \pm 0.006) \times 10^{-15}\ \mathrm{GeV}.$$

In 2013 even nicer oscillations were seen in B-hadrons:



[If you are interested, look could up the B_s^0–$\bar{B}_s^0$ mixing evidence in arXiv:1304.4741 [13].]

CP-Violation in the Kaon System

- So far we have ignored *CP*-violation in the neutral kaon system.
- We identified the *K*-short as the *CP*-even state and the *K*-long as the *CP*-odd state:

$$|K_S\rangle = |K_1\rangle \equiv \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right) \quad \text{with decays:} \quad K_S \to \pi\pi \quad \boxed{CP = +1},$$
$$|K_L\rangle = |K_2\rangle \equiv \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right) \quad \text{with decays:} \quad K_L \to \pi\pi\pi \quad \boxed{CP = -1}.$$

- At a long distance from the production point a beam of neutral kaons will be 100% K-long (the K-short component will have decayed away). Hence, if CP is conserved, would expect to see only three-pion decays.

In 1964 Fitch & Cronin (joint Nobel prize) observed 45 $K_L \to \pi^+\pi^-$ decays in a sample of 22700 kaon decays a long distance from the production point.

This implies that weak interactions violate *CP*!

- CP is violated in hadronic weak interactions, but only at the level of 2 parts in 1000.

K_L to pion BRs:

K_L	$\to \pi^+\pi^-\pi^0$	$BR = 12.6\%$	$CP = -1$
K_L	$\to \pi^0\pi^0\pi^0$	$BR = 19.6\%$	$CP = -1$
K_L	$\to \pi^+\pi^-$	$BR = 0.20\%$	$CP = +1$
K_L	$\to \pi^0\pi^0$	$BR = 0.08\%$	$CP = +1$

There are two possible explanations of CP violation in the kaon system:

1. The K_S and K_L do not correspond exactly to the CP-eigenstates K_1 and K_2:

$$|K_S\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_1\rangle + \varepsilon\,|K_2\rangle\right] \qquad |K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_2\rangle + \varepsilon\,|K_1\rangle\right]$$

with $|\varepsilon| \sim 2\times 10^{-3}$.
In this case the observation of $K_L \to \pi\pi$ is accounted for by:

$$|K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_2\rangle + \varepsilon|K_1\rangle\right]$$

$\to \pi\pi$ **CP = +1**

$\to \pi\pi\pi$ **CP = -1**

2. and/or CP is violated in the decay

$$|K_L\rangle = |K_2\rangle$$ **CP = -1**

$\to \pi\pi\pi$ **CP = -1**

$\to \pi\pi$ **CP = +1**

This effect is parameterised by a parameter ϵ'.

- Experimentally both possibilities are found to contribute, but the first source dominates since $\epsilon'/\varepsilon = (1.7 \pm 0.3)\times 10^{-3}$ (NA48(CERN) and KTeV (Fermilab)).
- The dominant mechanism is discussed in Appendix XXII.

CP Violation in Semi-leptonic decays

- If observe a neutral kaon beam a long time after production (i.e. at large distances) it will consist of a pure K_L component:

$$|K_L\rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+|\varepsilon|^2}} \left[(1+\varepsilon)|K_0\rangle + (1-\varepsilon)|\overline{K}^0\rangle \right]$$

$|K_0\rangle \longrightarrow \pi^- e^+ \nu_e$, $|\overline{K}^0\rangle \longrightarrow \pi^+ e^- \overline{\nu}_e$

- Decays to $\pi^- e^+ \nu_e$ must come from the K^0 component, and decays to $\pi^+ e^- \bar{\nu}_e$ must come from the $\bar{K}^0$ component so:

$$\Gamma\left(K_L \to \pi^+ e^- \bar{\nu}_e\right) \propto \left|\left\langle \bar{K}^0 \mid K_L \right\rangle\right|^2 \propto |1-\varepsilon|^2 \approx 1 - 2\Re\{\varepsilon\},$$

$$\Gamma\left(K_L \to \pi^- e^+ \nu_e\right) \propto \left|\left\langle K^0 \mid K_L \right\rangle\right|^2 \propto |1+\varepsilon|^2 \approx 1 + 2\Re\{\varepsilon\}.$$

- This results in a small difference in decay rates:

the K_L decay to $\pi^- e^+ \nu_e$ is 0.7% more likely than the K_L decay to $\pi^+ e^- \bar{\nu}_e$.

- This difference has been observed and thus provides the first direct evidence for an absolute difference between matter and anti-matter.

It also provides an unambiguous definition of matter which could, for example, be transmitted to aliens in a distant galaxy!

"The electrons in our atoms have the same charge as those emitted **least** often in the decays of the long-lived neutral kaon."

Summary of quark sector *CP*-violation

The weak interactions of quarks are controlled by the CKM matrix.

Similar structure to the lepton sector, although unlike the PMNS matrix, the CKM matrix is nearly diagonal.

CP violation enters through via a complex phase in the CKM matrix for same reasons as PMNS matrix.

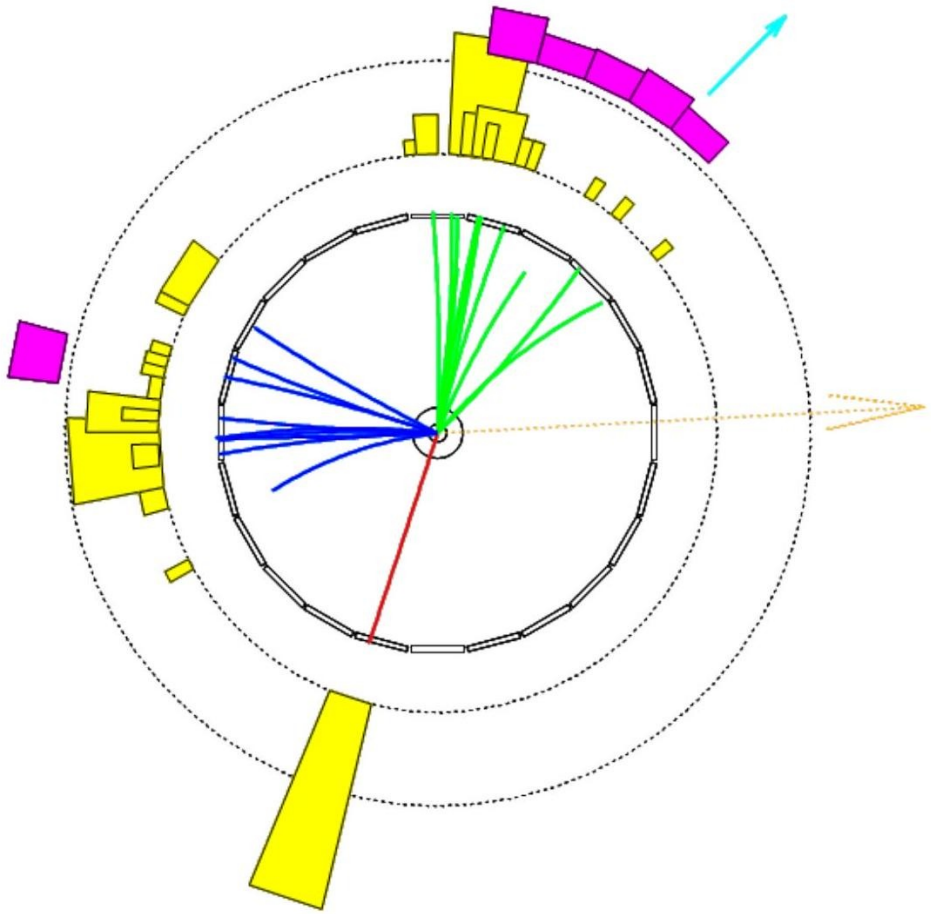
There is a great deal of experimental evidence for CP violation in the weak interactions of quarks

Some kind of *CP*-violation might explain matter-antimatter asymmetry in the Universe.

CP violation in the SM is not sufficient to explain the matter-antimatter asymmetry.

Either there is a different as-yet-undiscovered mechanism of *CP*-violation, **or** there was just more matter than antimatter to start with!

Prof. Alex Mitov, 2026



H13: Electroweak Unification and the W and Z Bosons

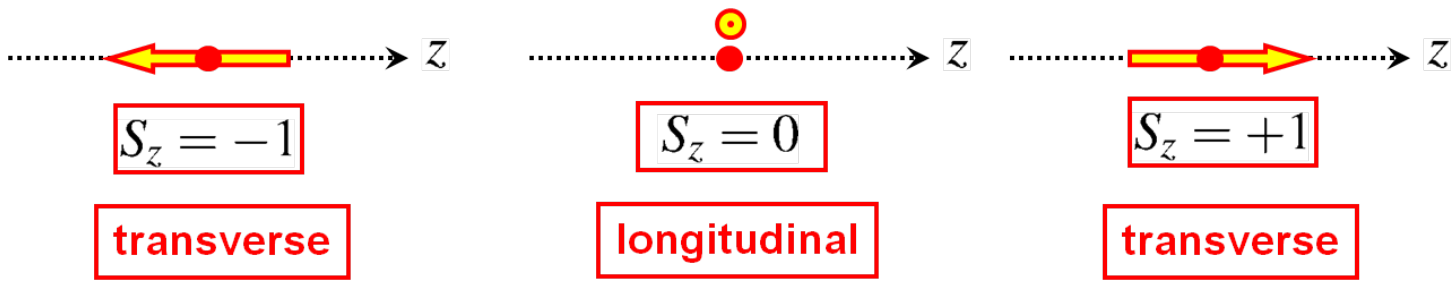
Boson Polarization Vectors

- Boson wave-functions are written in terms of **polarization four-vectors** ε^μ:

$$B^\mu = \varepsilon^\mu e^{-ip\cdot x} = \varepsilon^\mu e^{i(\vec{p}\cdot\vec{x}-Et)}.$$

- These **polarization four-vectors** ε^μ (in addition to p^μ) are the quantities used by Feynman rules to describe the state of spin-1 bosons entering or leaving a process.
- **Massless** spin-1 boson can exist in only two transverse polarization states (c.f. left and right circularly polarised light) – at least when they are not virtual.
- **Massive** spin-1 boson also can be longitudinally polarized.
- For a spin-1 boson travelling along the z-axis, the three **polarization four-vectors** are:

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = \frac{1}{m}(p_z,0,0,E); \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0) \qquad (155)$$

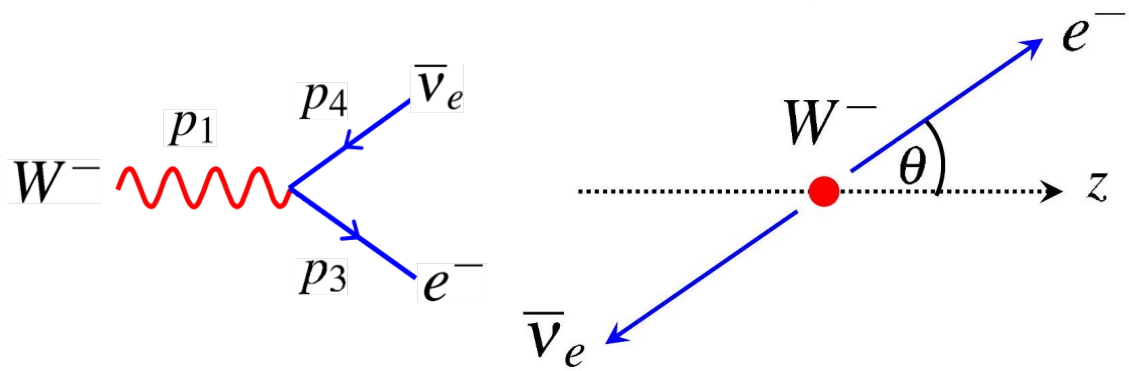


- All three polarisations satisfy $\varepsilon^\mu p_\mu = 0$ and $\varepsilon^\mu \varepsilon_\mu^* = -1$.

Results on this page are stated without proof. Nevertheless, some (non-examinable) partial justification can be found in Appendices XI, XII, XIII and XIV.

W-boson decay to fermions in centre of mass frame

- For each polarisation ε^μ we will calculate the matrix element for this process:



so parameterised by $\left\{\begin{array}{l} p_1 = (m_W, \quad 0, 0, 0) \\ p_3 = (E, +E\sin\theta, 0, +E\cos\theta) \\ p_4 = (E, -E\sin\theta, 0, -E\cos\theta) \end{array}\right\}$ with $E = \frac{1}{2}m_W$.

- Feynman rules give:

$$-iM_{fi} = \varepsilon_\mu(p_1) \cdot \bar{u}(p_3) \cdot -i\frac{g_W}{\sqrt{2}}\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right) \cdot v(p_4)$$

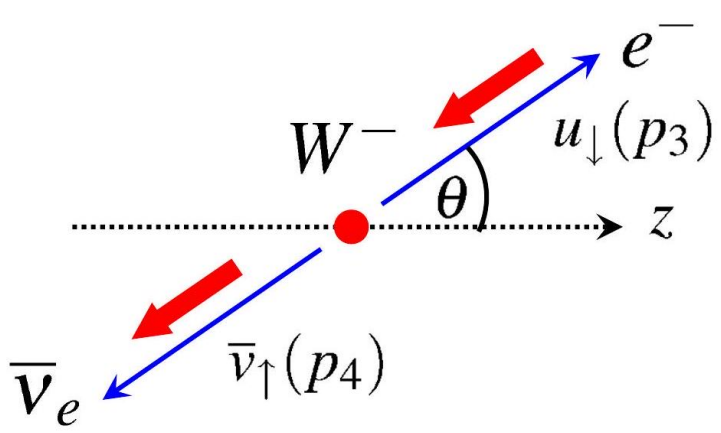
so $\boxed{M_{fi} = \frac{1}{\sqrt{2}} g_W \varepsilon_\mu(p_1) \cdot j^\mu}$ with $\boxed{j^\mu = \bar{u}(p_3)\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right) v(p_4)}$.

W-decay lepton current in ultra-relativistic limit

- Our leptons are in the ultra-relativistic limit ($m_w \gg m_e$).
- In this limit we know that the weak interaction only couples to left handed particles and right handed anti-particles. Hence

$$j^{\mu}_{\uparrow\uparrow} = j^{\mu}_{\downarrow\downarrow} = j^{\mu}_{\uparrow\downarrow} = 0$$

and it is only necessary to calculate $j^{\mu}_{\downarrow\uparrow}$ for the process shown to the right.



- By re-using the calculation of tha $(\mu^-_L \mu^+_R)$-current on page 110 which told us that:

$$j^{\mu}_{\uparrow\downarrow} = 2E(0, -\cos\theta, -i, \sin\theta)$$

we can substitute since $E = m_W/2$ to see that:

$$\boxed{j^{\mu}_{\uparrow\downarrow} = m_W(0, -\cos\theta, -i, \sin\theta)}. \tag{156}$$

- For a W-boson at rest, the three poloarisation vectors of (155) on page 402 become:

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = (0,0,0,1); \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0).$$

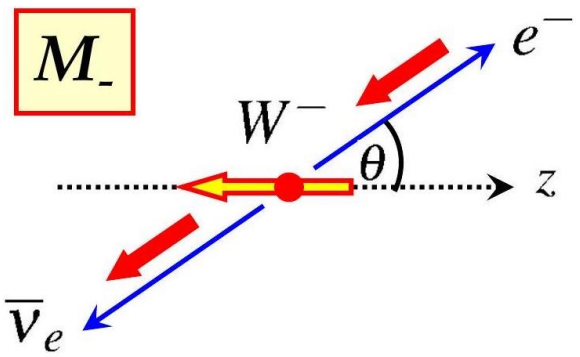
We can now calculate the three W-boson decay matrix elements $M_{fi} = \frac{g_W}{\sqrt{2}}\varepsilon_\mu(p_1)j^\mu$ by dotting the current (156) with each of the three polarization vectors shown above:

$$\begin{aligned}
M_- &= \frac{g_W}{\sqrt{2}} \times \tfrac{1}{\sqrt{2}}(0,1,-i,0)\cdot m_W(0,-\cos\theta,-i,\sin\theta) = \frac{1}{2}g_W m_W(1+\cos\theta)\\
M_L &= \frac{g_W}{\sqrt{2}} \times (0,0,0,1)\cdot m_W(0,-\cos\theta,-i,\sin\theta) = -\frac{1}{\sqrt{2}}g_W m_W \sin\theta\\
M_+ &= \frac{g_W}{\sqrt{2}} \times \tfrac{-1}{\sqrt{2}}(0,1,i,0)\cdot m_W(0,-\cos\theta,-i,\sin\theta) = \frac{1}{2}g_W m_W(1-\cos\theta).
\end{aligned}$$

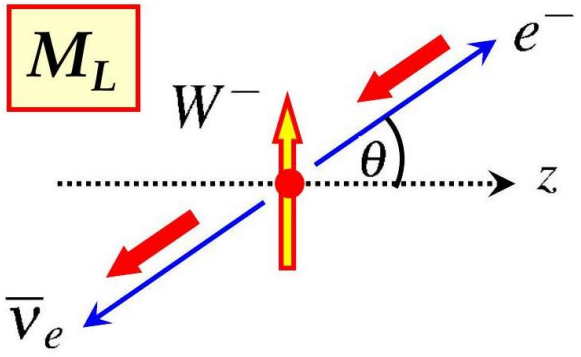
The W-boson decay matrix elements are therefore:

$$\left\{\begin{array}{l}
|M_-|^2 = g_W^2 m_W^2 \frac{1}{4}(1+\cos\theta)^2\\
|M_L|^2 = g_W^2 m_W^2 \quad \frac{1}{2}\sin^2\theta\\
|M_+|^2 = g_W^2 m_W^2 \frac{1}{4}(1-\cos\theta)^2
\end{array}\right\} \tag{157}$$

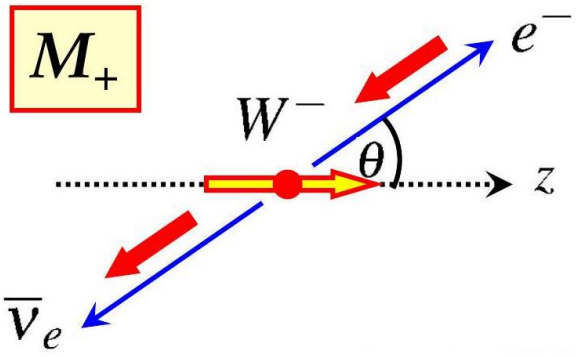
The angular distributions can be understood in terms of the spin of the particles:



$$\frac{1}{4}(1+\cos\theta)^2$$



$$\frac{1}{2}\sin^2\theta$$



$$\frac{1}{4}(1-\cos\theta)^2$$

Differential decay distributions for W-bosons of **definite polarization**

- The differential decay rate can be found using the expression given in (12):

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}|M|^2$$

  where $p^* = \frac{m_W}{2}$ is the C.O.M. momentum of the final state particles.
- For the three different polarisations we obtain:

$$\frac{\mathrm{d}\Gamma_-}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1+\cos\theta)^2; \quad \frac{\mathrm{d}\Gamma_L}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{2}\sin^2\theta; \quad \frac{\mathrm{d}\Gamma_+}{\mathrm{d}\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1-\cos\theta)^2.$$

- Integrating over all angles using

$$\int \frac{1}{4}(1\pm\cos\theta)^2 \,\mathrm{d}\phi\mathrm{d}\cos\theta = \int \frac{1}{2}\sin^2\theta\mathrm{d}\phi\mathrm{d}\cos\theta = \frac{4\pi}{3}$$

  gives

$$\boxed{\Gamma_- = \Gamma_L = \Gamma_+ = \frac{g_W^2 m_W}{48\pi}}. \tag{158}$$

$\Longrightarrow$ The decay rate for a W-boson does not depend on its direction of polarization.

This has to be the case! The decay rate should not depend on the arbitrary definition of the z-axis, even if the direction of the decay produces does.

Unpolarised W-boson decays

- For an unpolarized W-bosons, each of the three polarization states is equally likely.
- Thus, for the **average matrix element** sum over all possible matrix elements and average over the three initial polarization states:

$$\left\langle |M_{fi}|^2 \right\rangle = \frac{1}{3}\left(|M_-|^2 + |M_L|^2 + |M_+|^2\right)$$
$$= \frac{1}{3} g_W^2 m_W^2 \left[\frac{1}{4}(1+\cos\theta)^2 + \frac{1}{2}\sin^2\theta + \frac{1}{4}(1-\cos\theta)^2\right] = \boxed{\frac{1}{3} g_W^2 m_W^2}.$$

As one would expect, unpolarized W-bosons decay isotropically.

Such bosons have no special directions along which to favour certain decays!
For such decays, the differential decay rate is:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}\left\langle |M|^2 \right\rangle = \frac{\frac{1}{2}m_W}{32\pi^2 m_W^2}\frac{1}{3} g_W^2 m_W^2 = \frac{g_W^2 m_W}{192\pi^2}$$

so the total unploarised decay rate to this flavour of lepton is unchanged from (158):

$$\Gamma\left(W^- \to e^- \bar{\nu}\right) = 4\pi \cdot \frac{g_W^2 m_W}{192\pi^2} = \frac{g_W^2 m_W}{48\pi}. \tag{159}$$

W-boson width to all final states (i.e. total width or total decay rate)

- The calculation for the other decay modes (neglecting final state particle masses) is same. For quarks need to account for colour and CKM matrix. No decays to the top quark as its mass (173 GeV) is greater than the W-boson mass (80 GeV).
- Defining $X = \frac{g_W^2 m_W}{48\pi} = \Gamma(W^- \to e^- \bar{\nu}_e)$ from last page we therefore have:

$$\Gamma(W^- \to e^- \bar{\nu}_e) = X \qquad \Gamma(W^- \to d\bar{u}) = 3\,|V_{ud}|^2 X \qquad \Gamma(W^- \to d\bar{c}) = 3\,|V_{cd}|^2 X$$
$$\Gamma(W^- \to \mu^- \bar{\nu}_\mu) = X \qquad \Gamma(W^- \to s\bar{u}) = 3\,|V_{us}|^2 X \qquad \Gamma(W^- \to s\bar{c}) = 3\,|V_{cs}|^2 X$$
$$\Gamma(W^- \to \tau^- \bar{\nu}_\tau) = X \qquad \Gamma(W^- \to b\bar{u}) = 3\,|V_{ub}|^2 X \qquad \Gamma(W^- \to b\bar{c}) = 3\,|V_{cb}|^2 X$$

- Unitarity of CKM matrix gives, e.g. $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$, hence $\sum_q \Gamma\left(W^- \to qq'\right) = 3X + 3X = 6X$ and so $BR\left(W^- \to e\bar{\nu}_e\right) = \frac{X}{3X+6X} = \frac{1}{9}$ and $BR\left(W^- \to \text{hadrons}\right) = \frac{6X}{3X+6X} = \frac{2}{3}$, *etc.*
- Thus our leading-order prediction for the total W-boson decay rate is:

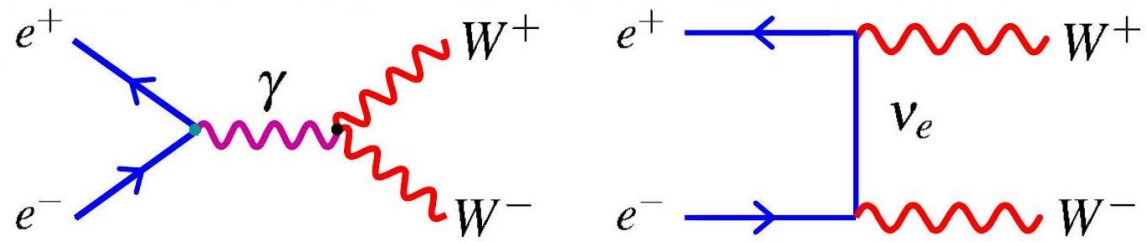
$$\Gamma_W \equiv \Gamma(W^- \to \text{anything}) = 6X + 3X = 9X = \frac{3g_W^2 m_W}{16\pi} \approx \boxed{2.07\ \text{GeV}}.$$

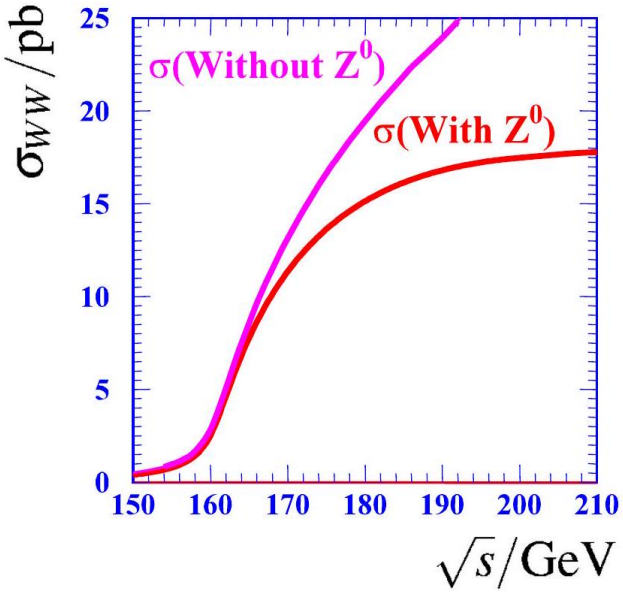
Experiment measures the W-boson width to be $\boxed{\Gamma_W = 2.085 \pm 0.042\ \text{GeV}}$.

Our leading-order calculation is remarkably good. Probably better than it deserves to be!

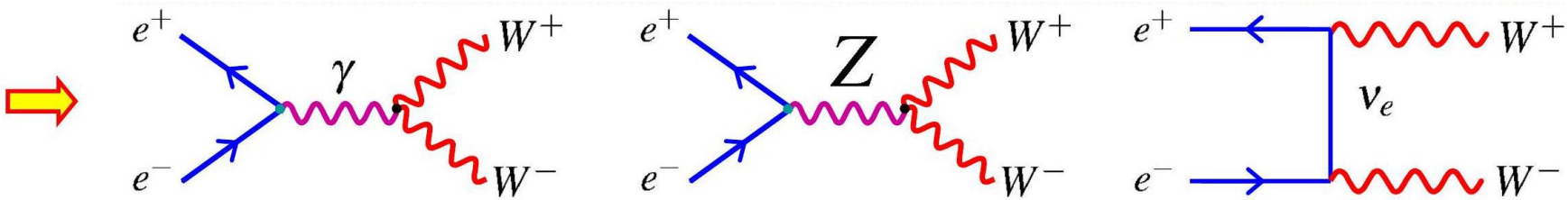
From W to Z

- W-bosons can be produced in e^+e^- annihilation:





- With just these two diagrams there is a problem: the cross section increases with C.o.M. energy and at some point **violates QM unitarity**
  [i.e. calculation would predict larger larger flux of W-bosons than incoming flux of electrons/positrons!]
- Problem could be 'fixed' by introducing a new boson, the Z. The new diagram could interferes negatively with the above two diagrams fixing the unitarity problem:



$$|M_{\gamma WW} + M_{ZWW} + M_{\nu WW}|^2 < |M_{\gamma WW} + M_{\nu WW}|^2$$

- Idea would only work if Z, γ and W couplings were appropriately related. Electroweak Unification will achieve this.

$SU(2)_L$: The Gauge Principle of the Weak Interaction

- The Weak Interaction arises from SU(2) local phase transformations:

$$\psi \rightarrow \psi' = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}\psi$$

where the $\vec{\sigma}$ are the three generators of the $SU(2)$, i.e. the three Pauli spin matrices.

Three generators imply three gauge bosons : $\boxed{W_1^\mu, W_2^\mu, W_3^\mu}$.

- The **left-chiral fermion** (and **anti-fermion**) components are placed in **'weak isospin'** DOUBLETS. The local gauge transformation therefore corresponds to:

$$\begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix} \longrightarrow \begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}' = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}} \begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}.$$

- The **right-chiral fermion** (and **anti-fermion**) components are placed in **'weak isospin'** SINGLETS. Singlets are always unmodified by gauge transformations!

Summary of fermion weak isospin assignments (anti-fermion assignments are opposite) :

$$\text{DOUBLETs} \quad I_W = \frac{1}{2} : \quad \begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L^- \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L^- \end{pmatrix}, \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \quad \begin{matrix} \leftarrow I_W^3 = +\frac{1}{2} \\ \leftarrow I_W^3 = -\frac{1}{2} \end{matrix}$$

$$\text{SINGLETs} \quad I_W = 0 : \quad (\nu_{eR}), (e_R^-), \ldots\ldots\ldots\ldots\ldots, (t_R), (b_R) \quad \leftarrow I_W^3 = 0.$$

This gauge symmetry reproduces the W-boson currents:

- $SU(2)$ has three generators (Pauli matrices) so has these three currents (one per generator) which we associate with the W_1, W_2, W_3 bosons respectively:

$$j_1^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2}\sigma_1 \chi_L; \quad j_2^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2}\sigma_2 \chi_L; \quad j_3^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2}\sigma_3 \chi_L.$$

- Define the $j_\pm^\mu$ currents as the following linear combinations of currents:

$$j_\pm^\mu \equiv \frac{1}{\sqrt{2}}\left(j_1^\mu \mp i\, j_2^\mu\right) \qquad \text{so} \qquad j_\pm^\mu = \frac{g_W}{\sqrt{2}} \bar{\chi}_L \gamma^\mu \frac{1}{2}\left(\sigma_1 \mp i\sigma_2\right)\chi_L.$$

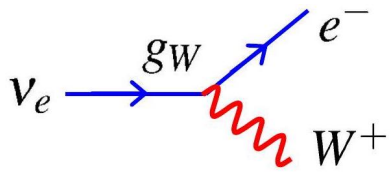
- Therefore, taking χ_L to be the weak isospin DOUBLET $\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$ we would find:

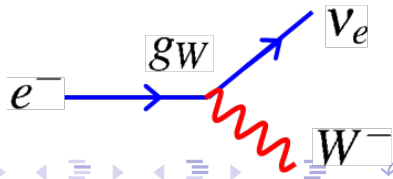
$$j_+^\mu = \frac{g_W}{\sqrt{2}} \begin{pmatrix} \bar{\nu}_{eL} & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} = \frac{g_W}{\sqrt{2}} \bar{e}_L \gamma^\mu \nu_{eL} = \frac{g_W}{\sqrt{2}} \bar{e} \gamma^\mu \frac{1}{2}(1-\gamma^5)\nu_e$$

and

$$j_-^\mu = \frac{g_W}{\sqrt{2}} \begin{pmatrix} \bar{\nu}_{eL} & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} = \frac{g_W}{\sqrt{2}} \bar{\nu}_{eL} \gamma^\mu e_L = \frac{g_W}{\sqrt{2}} \bar{\nu}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)e$$

and so

j_+^μ is the W^+-current: 

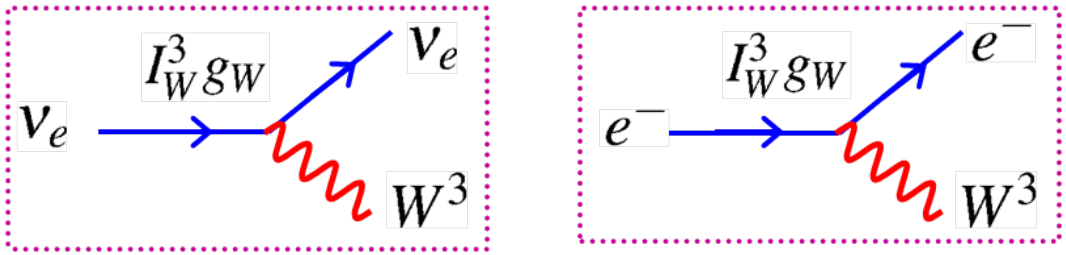
; j_-^μ is the W^--current: 

.

What about the remaining third current $j_3^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2} \sigma_3 \chi_L$?

$$
\begin{aligned}
j_3^\mu &= g_W \begin{pmatrix} \bar{\nu}_{eL} & \bar{e}_L \end{pmatrix} \gamma^\mu \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \\
&= g_W \left[\tfrac{1}{2} \bar{\nu}_{eL} \gamma^\mu \nu_{eL} - \tfrac{1}{2} \bar{e}_L \gamma^\mu e_L \right] \\
&= g_W \left[\quad (+\tfrac{1}{2}) \cdot \bar{\nu}_{eL} \gamma^\mu \nu_{eL} + \quad (-\tfrac{1}{2}) \cdot \bar{e}_L \gamma^\mu e_L + \quad (0) \cdot \bar{e}_R \gamma^\mu e_R + \quad (0) \cdot \bar{\nu}_{eR} \gamma^\mu \nu_{eR} \right] \\
&= g_W \left[I_W^3(\nu_{eL}) \cdot \bar{\nu}_{eL} \gamma^\mu \nu_{eL} + I_W^3(e_L) \cdot \bar{e}_L \gamma^\mu e_L + I_W^3(e_R) \cdot \bar{e}_R \gamma^\mu e_R + I_W^3(\nu_{eR}) \cdot \bar{\nu}_{eR} \gamma^\mu \nu_{eR} \right]
\end{aligned}
$$

- The above describes a **flavour-conserving NEUTRAL CURRENT INTERACTION**.
- The coupling strength is g_W but multiplied by the third component of weak isospin:



- Technically this interaction only affects members of the weak isospin DOUBLETs, (i.e. the left chiral fermions and anti-fermions).
- However, since members of weak isospin SINGLETs have $I_W^3 = 0$, **one may regard the interaction as applying to everything in proportion to I_W^3.** In effect, the SINGLETs are just 'neutral' so far as I_W^3 is concerned.

Electroweak Unification and $U(1)_Y$ Weak Hypercharge

- It would be tempting **(and wrong!)** to identify the W^3 as the Z.
- There are two physical neutral spin-1 gauge bosons: the photon (A) and the Z. By definition these are mass eigenstates. There is no reason that the **gauge bosons should correspond to mass eigenstates**. They can mix!

The Z and A (photon) are written in terms of the W^3 and a new neutral spin-1 'B-boson'.

In terms of an *a-priori* undetermined '**W**eak' (or '**W**einberg'?) mixing angle θ_W we define:

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W, \qquad \text{and} \tag{160}$$

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W \tag{161}$$

where the B is associated with a new $U(1)_Y$ gauge symmetry called **weak hypercharge**.

Hindsight is a wonderful thing ...

We realise that the $U(1)_{\text{em}}$ gauge symmetry we explored on page 222 was an emergent rather than a fundamental gauge symmetry of The Standard Model, which actually lives in:

$$U(1)_Y \times SU(2)_L \times SU(3)_C.$$

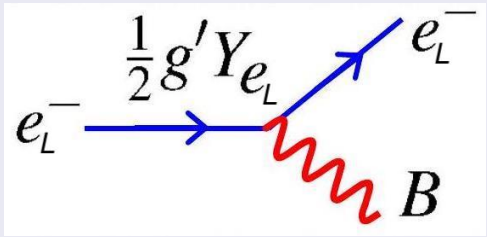
EM's (apparent) $U(1)_{\text{em}}$ symmetry, and the Z-boson, emerge from within $U(1)_Y \times SU(2)_L$.

Relationship of Weak Hypercharge Y to electric charge Q and I_W^3

- The coupling strength for Weak Hypercharge is denoted g'.
- By convention the coupling of any fermion to the B-boson is

$$\frac{1}{2}g'Y$$

where Y is the weak hypercharge of the fermion in question.



- Like I_W^3 (but unlike electric charge Q) **the weak hypercharge Y can be different for left and right chiral fermions**.
- We will shortly see that

$$\boxed{Y = 2Q - 2I_W^3}$$

where Q is the EM charge and I_W^3 the third component of weak isospin for the (left or right chiral) particle in question.

Example Weak Hypercharges:

$$e_L : Y = 2(-1) - 2\left(-\tfrac{1}{2}\right) = -1 \qquad \nu_{eL} : Y = 2(0) - 2\left(\tfrac{1}{2}\right) = -1$$
$$e_R : Y = 2(-1) - 2(0) = -2 \qquad \nu_{eR} : Y = 2(0) - 2(0) = 0$$

- The neutral-current for electrons (or indeed any fermion) take the form:

$$\begin{aligned} A\ (\gamma): \quad j_\mu^{em} &= eQ_e \cdot \bar{\psi}\gamma_\mu\psi = && eQ_e \cdot \overline{e}_L\gamma_\mu e_L + && eQ_e \cdot \overline{e}_R\gamma_\mu e_R \\ B: \quad j_\mu^{Y} &= \frac{g'}{2} Y_e \bar{\psi}\gamma_\mu\psi = && \frac{g'}{2} Y_{e_L} \cdot \overline{e}_L\gamma_\mu e_L + && \frac{g'}{2} Y_{e_R} \cdot \overline{e}_R\gamma_\mu e_R \\ W^3: \quad j_\mu^{W^3} &= && = g_W I_W^3(e_L) \cdot \overline{e}_L\gamma_\mu e_L + && g_W I_W^3(e_R) \cdot \overline{e}_R\gamma_\mu e_R \end{aligned}$$

- The relation $A_\mu = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W$ is equivalent to:

$$\boxed{j_\mu^{em} = j_\mu^{Y} \cos\theta_W + j_\mu^{W^3} \sin\theta_W}$$

and so consistency with earlier results will be achieved only if, for all particles:

$$\boxed{eQ_x = \cos\theta_W \frac{g'}{2} Y_{x_\bullet} + \sin\theta_W g_W I_W^3(x_\bullet)} \tag{162}$$

where $x \in \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, s, c, b, t\}$ labels the fermion, and '$\bullet$' $\in \{L, R\}$.

- Divide our parameters up according to whether they are old or new:

$$\begin{aligned} \text{pre-existing/known quantities are:} \quad & \{e, Q_x, g_W, I_W^3(x_\bullet)\}, \\ \text{new quantities are:} \quad & \{g', \theta_W, Y_{x_\bullet}\}. \end{aligned}$$

- We 'win' this 'unification game' if we can find values for our **new** parameters in terms of our **old** parameters that cause (162) to be satisfied.

Can we win the unification game? Yes!

- Given fixed values of our pre-existing/known quantities $\{e, Q_x, g_W, I_W^3(x_\bullet)\}$ we can easily ensure that the following relationship:

$$\boxed{eQ_x = \cos\theta_W \frac{g'}{2} Y_{x_\bullet} + \sin\theta_W g_W I_W^3(x_\bullet)}$$

is always satisfied **by requiring** the **three** new quantities $\{g', \theta_W, Y_{x_\bullet}\}$ to take the values which solve the following **three** simultaneous conditions:

$$\boxed{\begin{aligned} e &= g' \cos\theta_W \\ e &= g_W \sin\theta_W \\ Q_x &= \frac{1}{2} Y_{x_\bullet} + I_W^3(x_\bullet) \end{aligned}} \tag{163}$$

Aside

The relations in (163) can be solved by taking $\theta_W = \arcsin(e/g_W)$ and thence by $g' = \frac{e}{\cos\theta_W}$ together with

$$\boxed{Y_{x_\bullet} = 2Q_x - 2I_W^3(x_\bullet)} \tag{164}$$

which is the weak hypercharge assignment claimed earlier on page 415 !

In this model, we can now derive the couplings of the Z Boson.

- In (161) we learned that: $\boxed{Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W}$ which is equivalent to

$$j^Z_\mu = -j^Y_\mu \sin\theta_W + j^{W^3}_\mu \cos\theta_W$$

which, using our results from page 416, becomes

$$\begin{aligned} j^Z_\mu(x_\bullet) &= -\sin\theta_W \left(\frac{g'}{2} Y_{x_\bullet} \cdot \overline{x}_\bullet \gamma_\mu x_\bullet\right) + \cos\theta_W \left(g_W I^3_W(x_\bullet) \cdot \overline{x}_\bullet \gamma_\mu x_\bullet\right) \\ &= \left(\cos\theta_W g_W I^3_W(x_\bullet) - \sin\theta_W \frac{g'}{2} Y_{x_\bullet}\right) \cdot \overline{x}_\bullet \gamma_\mu x_\bullet \end{aligned} \tag{165}$$

if $x \in \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, s, c, b, t\}$ labels the fermion, and '$\bullet$' $\in \{L, R\}$.

- Using the conditions in (163) we can re-write (165) as

$$j^Z_\mu(x_\bullet) = \left[g' \frac{\left(I^3_W - Q \sin^2\theta_W\right)}{\sin\theta_W}\right] \cdot \overline{x}_\bullet \gamma_\mu x_\bullet$$

$$\text{or} \qquad j^Z_\mu(x_\bullet) = g_Z \left(I^3_W(x_\bullet) - Q_x \sin^2\theta_W\right) \cdot \overline{x}_\bullet \gamma_\mu x_\bullet \tag{166}$$

if we define g_Z by $\boxed{e = g_Z \cos\theta_W \sin\theta_W}$; *i.e.* $\boxed{g_Z = \dfrac{g_W}{\cos\theta_W}}$. [Check as exercise!]

- In contrast to the $W^{\pm}$, the Z-boson couples to both LH and RH chiral components, **but not equally ...**
- Illustrate the above by defining:

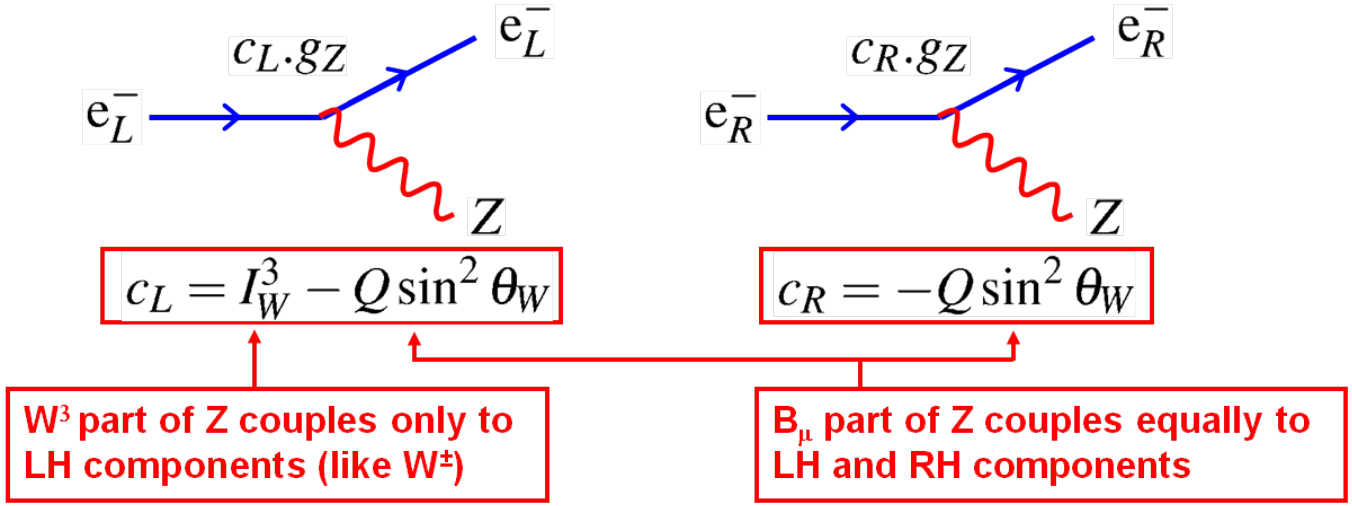
$$\boxed{c_{\bullet}(x) = \left(I_W^3(x_{\bullet}) - Q_x \sin^2\theta_W \right)} \tag{167}$$

- With L and R replacing $\bullet$ we see that:

$$c_L(x) = \left(I_W^3(x_L) - Q_x \sin^2\theta_W \right) \qquad \text{whereas} \tag{168}$$

$$c_R(x) = \left(\qquad - Q_x \sin^2\theta_W \right) \tag{169}$$

because right chiral fermions (and anti-fermions) have $I_W^3 = 0$.



Use projection operators to obtain vector and axial vector couplings.

- From last two pages:

$$j_\mu^Z(x_\bullet) = g_Z c_\bullet(x) \cdot \bar{x}_\bullet \gamma_\mu x_\bullet.$$

- Also:

$$\bar{x}_L \gamma_\mu x_L = \bar{x} \gamma_\mu \frac{1}{2}(1-\gamma_5) x \qquad \text{and} \qquad \bar{x}_R \gamma_\mu x_R = \bar{x} \gamma_\mu \frac{1}{2}(1+\gamma_5) x.$$

- Thus

$$\begin{aligned} j_\mu^Z(x) &= j_\mu^Z(x_L) + j_\mu^Z(x_R) \\ &= g_Z c_L \bar{x}_L \gamma_\mu x_L + g_Z c_R \bar{x}_R \gamma_\mu x_R \\ &= g_Z \bar{x} \gamma_\mu \left[c_L \frac{1}{2}(1-\gamma_5) + c_R \frac{1}{2}(1+\gamma_5) \right] x \\ &= \frac{g_Z}{2} \bar{x} \gamma_\mu \left[(c_L + c_R) + (c_R - c_L)\gamma_5 \right] x \\ &= \frac{g_Z}{2} \bar{x} \gamma_\mu \left[c_V - c_A \gamma_5 \right] x \end{aligned}$$

if one defines

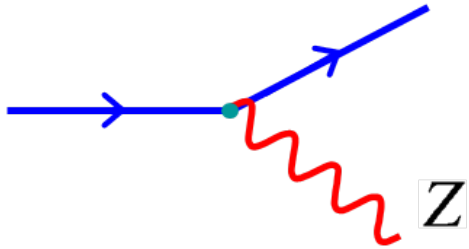
$$\boxed{c_V(x) = c_L(x) + c_R(x)} = I_W^3(x_L) - 2Q_x \sin^2\theta_W \qquad \text{and} \qquad (170)$$

$$\boxed{c_A(x) = c_L(x) - c_R(x)} = I_W^3(x_L). \qquad (171)$$

Summary of Z-boson couplings

- The vertex factor for the Z boson is:

$$-ig_Z \frac{1}{2}\gamma_\mu \left[c_V - c_A\gamma_5\right].$$



- Using the experimentally determined value of the weak mixing angle: $\sin^2\theta_W \approx 0.23$:

Fermion x	Q_x	$I_W^3(x_L)$	$I_W^3(x_R)$	$c_L(x)$	$c_R(x)$	$c_V(x)$	$c_A(x)$
ν_e, ν_μ, ν_τ	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
e, μ, τ	-1	$-\frac{1}{2}$	0	-0.27	0.23	-0.04	$-\frac{1}{2}$
u, c, t	$+\frac{2}{3}$	$+\frac{1}{2}$	0	0.35	-0.15	+0.19	$+\frac{1}{2}$
d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	0	-0.42	0.08	-0.35	$-\frac{1}{2}$

- Reminder:

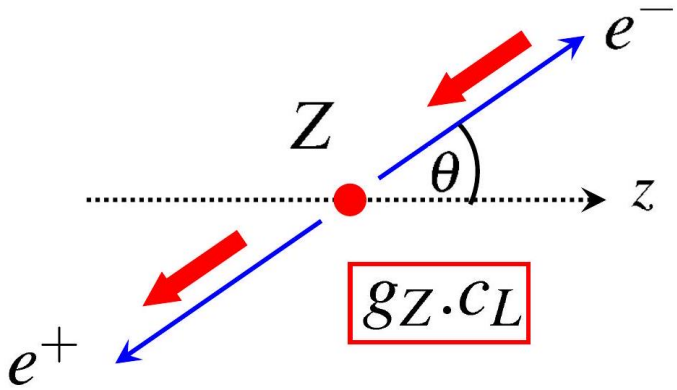
$$\boxed{c_\bullet(x) = I_W^3(x_\bullet) - Q_x \sin^2\theta_W}$$

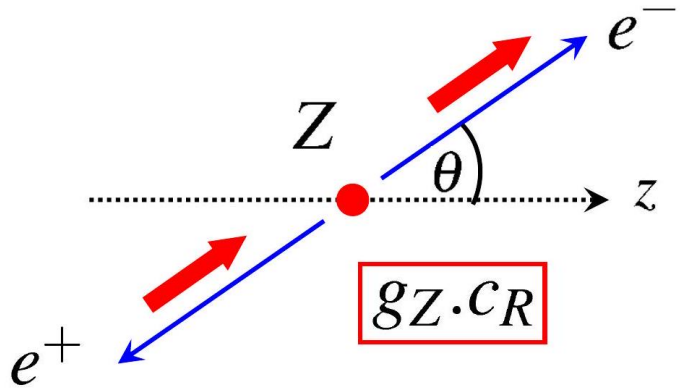
$$\boxed{c_V(x) = c_L(x) + c_R(x)} \quad = I_W^3(x_L) - 2Q_x \sin^2\theta_W \quad \text{and} \quad (172)$$

$$\boxed{c_A(x) = c_L(x) - c_R(x)} \quad = I_W^3(x_L). \quad (173)$$

Z Boson Decay: Γ_z

- In terms of left and right-handed combinations need to calculate:





- For unpolarized Z bosons: (Question 25):

$$\left\langle |M_{fi}|^2 \right\rangle = \frac{1}{3}\left[2c_L^2 g_Z^2 m_Z^2 + 2c_R^2 g_Z^2 m_Z^2\right] = \frac{2}{3} g_Z^2 m_Z^2 \left(c_L^2 + c_R^2\right)$$

(the 3 averages over the three polarization states of the Z)

- Using $c_V^2 + c_A^2 = 2\left(c_L^2 + c_R^2\right)$ and $\frac{d\Gamma}{d\Omega} = \frac{|p^*|}{32\pi^2 m_Z^2}|M|^2$ (from (12)) we get

$$\boxed{\Gamma\left(Z \to e^+ e^-\right) = \frac{g_Z^2 m_Z}{48\pi}\left(\left(c_V^e\right)^2 + \left(c_A^e\right)^2\right)}. \quad (174)$$

Z Branching Ratios

(Relating to Example Sheet Question 26):

- Neglecting fermion masses, we obtain the same expression for the other decays:

$$\boxed{\Gamma(Z \to f\bar{f}) = \frac{g_Z^2 m_Z}{48\pi}\left((c_V^f)^2 + (c_A^f)^2\right).} \tag{175}$$

- Using values for c_V and c_A from page 421 we obtain:

$$\begin{aligned}
&\mathrm{Br}\left(Z \to e^+e^-\right) = \mathrm{Br}\left(Z \to \mu^+\mu^-\right) = \mathrm{Br}\left(Z \to \tau^+\tau^-\right) &&\approx 3.5\% \\
&\mathrm{Br}\left(Z \to \nu_1\bar{\nu}_1\right) = \mathrm{Br}\left(Z \to \nu_2\bar{\nu}_2\right) = \mathrm{Br}\left(Z \to \nu_3\bar{\nu}_3\right) &&\approx 6.9\% \\
&\mathrm{Br}\left(Z \to d\bar{d}\right) = \mathrm{Br}\left(Z \to s\bar{s}\right) = \mathrm{Br}\left(Z \to b\bar{b}\right) &&\approx 15\% \\
&\mathrm{Br}\left(Z \to u\bar{u}\right) = \mathrm{Br}\left(Z \to c\bar{c}\right) &&\approx 12\%
\end{aligned}$$

- The Z-boson therefore predominantly decays to hadrons:

$$Br(Z \to \text{ hadrons }) \approx 69\% \quad \ldots \text{ mainly due to factor 3 from colour}$$

- Also predict total decay rate (total width)

$$\Gamma_Z = \sum_i \Gamma_i = 2.5 \text{ GeV}$$

Experiment: $\Gamma_Z = 2.4952 \pm 0.0023$ GeV

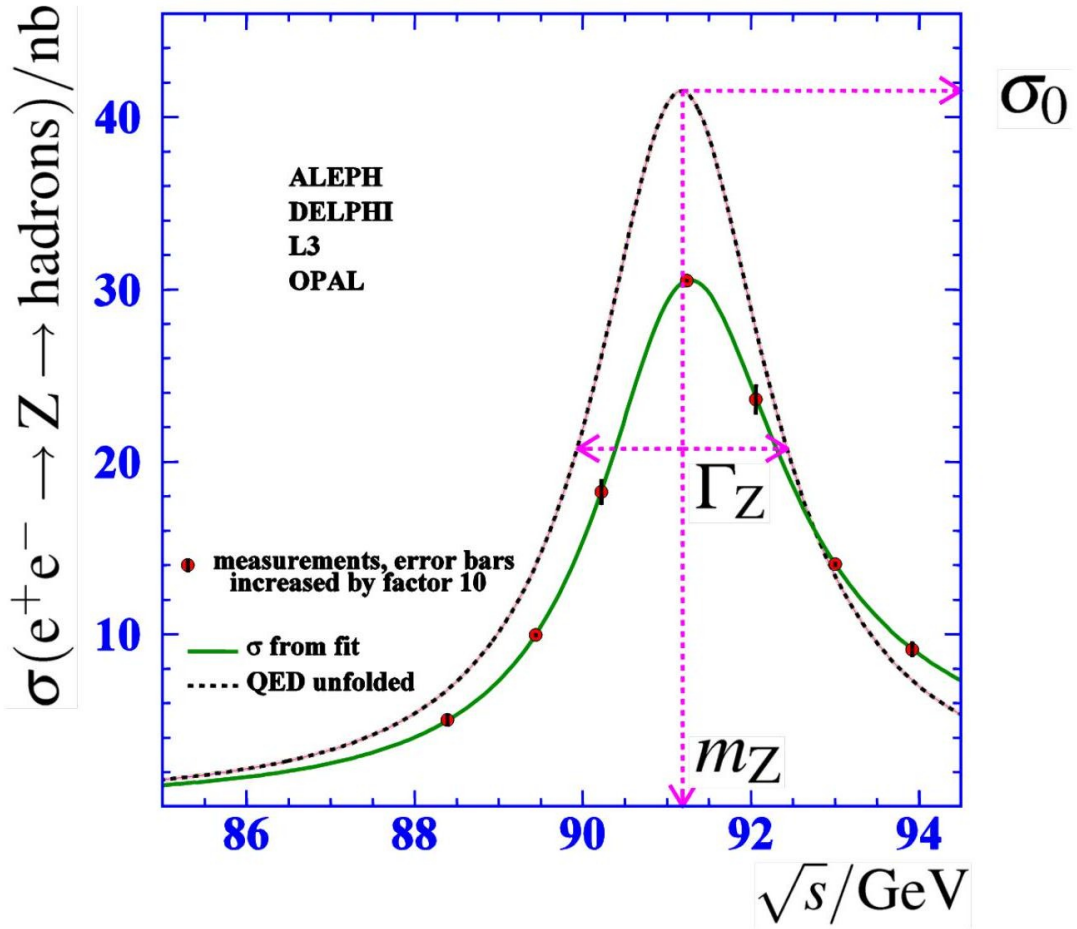
Summary

- The Standard Model interactions are mediated by spin-1 gauge bosons
- The form of the interactions are completely specified by the assuming an underlying local phase transformation $\Rightarrow$ GAUGE INVARIANCE, initially trying:

$$\begin{aligned} U(1)_{\text{em}} &\longrightarrow QED \\ SU(2)_L &\longrightarrow W^{\pm} \text{ and } W^3 \\ SU(3)_C &\longrightarrow QCD \end{aligned}$$

- In order to 'unify' the electromagnetic and weak interactions, introduced a new symmetry gauge symmetry : $U(1)_Y$ (weak hypercharge) with an associated B-boson.
- The physical Z-boson and the photon are mixtures of the neutral W-boson and B determined by the Weak Mixing angle θ_W which is such that $\sin^2\theta_W \approx 0.23$.
- Have we really unified the EM and Weak interactions ? Well not really:
  - We started with two independent theories with coupling constants g_W, e.
  - We ended up with coupling constants which are related but at the cost of introducing a new parameter in the Standard Model, θ_W.
  - Interactions were not unified from any higher theoretical principle ... but it works!
  - Didn't discuss why photon ends up massless and Z-boson massive. (Relates to Higgs, but beyond course.)

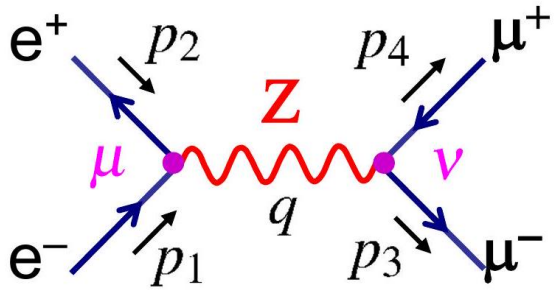
Prof. Alex Mitov, 2026



H14: Precision Tests of the Standard Model

The Z Resonance ($e^+e^- \to Z \to \mu^+\mu^-$)

- The Feynman rules from the last handout have told us that the matrix element for the process shown on the right is:



$$M_{fi} = \frac{-g_Z^2 g_{\mu\nu}}{q^2 - m_Z^2} \left[\bar{v}(p_2)\frac{\gamma^\mu}{2}\left(c_V(e) - c_A(e)\gamma^5\right)u(p_1)\right]\cdot\left[\bar{u}(p_3)\frac{\gamma^\nu}{2}\left(c_V(\mu) - c_A(\mu)\gamma^5\right)v(p_4)\right]$$

- As we have found previously, it is convenient to evaluate M_{fi} with external particles in helicity eigenstates in the ultra-relativistic limit so that helicity=chirality.
We will therefore prefer to use c_L and c_R in M_{fi} rather than c_V and c_A.
We can switch them over using (172) and (173):

$$\tfrac{1}{2}\left(c_V - c_A\gamma^5\right) = \underbrace{c_L\tfrac{1}{2}\left(1-\gamma^5\right) + c_R\tfrac{1}{2}\left(1+\gamma^5\right)}_{\text{LH and RH projections operators}}$$

- Thus:

$$M_{fi} = -\frac{g_Z^2 g_{\mu\nu}}{q^2 - m_Z^2}\left[c_L(e)\bar{v}(p_2)\gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right)u(p_1) + c_R(e)\bar{v}(p_2)\gamma^\mu \tfrac{1}{2}\left(1+\gamma^5\right)u(p_1)\right]$$
$$\times\left[c_L(\mu)\bar{u}(p_3)\gamma^\nu \tfrac{1}{2}\left(1-\gamma^5\right)v(p_4) + c_R(\mu)\bar{u}(p_3)\gamma^\nu \tfrac{1}{2}\left(1+\gamma^5\right)v(p_4)\right].$$

- Apply projection operators remembering that in the ultra-relativistic limit

$$\tfrac{1}{2}\left(1-\gamma^5\right)u = u_\downarrow; \quad \tfrac{1}{2}\left(1+\gamma^5\right)u = u_\uparrow, \quad \tfrac{1}{2}\left(1-\gamma^5\right)v = v_\uparrow, \quad \tfrac{1}{2}\left(1+\gamma^5\right)v = v_\downarrow$$

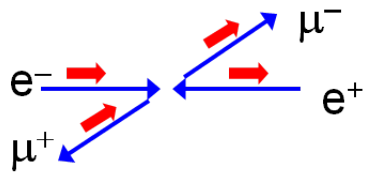
to get
$$M_{fi} = -\frac{g_Z}{q^2 - m_Z^2} g_{\mu\nu}\left[c_L(e)\bar{v}(p_2)\gamma^\mu u_\downarrow(p_1) + c_R(e)\bar{v}(p_2)\gamma^\mu u_\uparrow(p_1)\right]$$
$$\times\left[c_L(\mu)\bar{u}(p_3)\gamma^\nu v_\uparrow(p_4) + c_R(\mu)\bar{u}(p_3)\gamma^\nu v_\downarrow(p_4)\right]$$
$$= -\frac{g_Z^2}{q^2 - m_Z^2} g_{\mu\nu}\left[c_L(e)\bar{v}_\uparrow(p_2)\gamma^\mu u_\downarrow(p_1) + c_R(e)\bar{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1)\right]$$
$$\times\left[c_L(\mu)\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) + c_R(\mu)\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)\right]$$

since (in ultra-relativistic limit) $\bar{u}_\uparrow\gamma^\nu u_\uparrow = 0$ (see page 122).

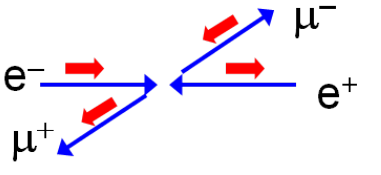
- M_{fi} is therefore a sum of the four terms M_{RR}, M_{RL}, M_{LR}, M_{LL} detailed overleaf.

Matrix elements for allowed Z-boson annihilation processes (ultra rel)

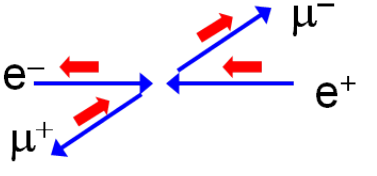
$$M_{RR} = -\frac{g_Z^2}{q^2 - m_Z^2} c_R(e) c_R(\mu) g_{\mu\nu} \left[\bar{v}_\downarrow(p_2) \gamma^\mu u_\uparrow(p_1)\right] \left[\bar{u}_\uparrow(p_3) \gamma^\nu v_\downarrow(p_4)\right]$$



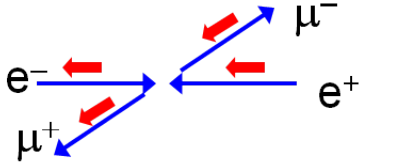
$$M_{RL} = -\frac{g_Z^2}{q^2 - m_Z^2} c_R(e) c_L(\mu) g_{\mu\nu} \left[\bar{v}_\downarrow(p_2) \gamma^\mu u_\uparrow(p_1)\right] \left[\bar{u}_\downarrow(p_3) \gamma^\nu v_\uparrow(p_4)\right]$$



$$M_{LR} = -\frac{g_Z^2}{q^2 - m_Z^2} c_L(e) c_R(\mu) g_{\mu\nu} \left[\bar{v}_\uparrow(p_2) \gamma^\mu u_\downarrow(p_1)\right] \left[\bar{u}_\uparrow(p_3) \gamma^\nu v_\downarrow(p_4)\right]$$



$$M_{LL} = -\frac{g_Z^2}{q^2 - m_Z^2} c_L(e) c_L(\mu) g_{\mu\nu} \left[\bar{v}_\uparrow(p_2) \gamma^\mu u_\downarrow(p_1)\right] \left[\bar{u}_\downarrow(p_3) \gamma^\nu v_\uparrow(p_4)\right]$$



(L/R refer are helicities of the initial/final state particles (not anti-particles).)

Fortunately we have calculated terms like these on pages 113 and 114 (e.g. (57)) when considering $e^+e^- \to \gamma \to \mu^+\mu^-$.

- Re-using QED results for the Z-exchange matrix elements on page 114 gives:

$$|M_{RR}|^2 = s^2 \left| \frac{g_Z^2}{s - m_Z^2} \right|^2 \cdot c_R^2(e) \cdot c_R^2(\mu) \cdot (1 + \cos\theta)^2$$

$$|M_{RL}|^2 = s^2 \left| \frac{g_Z^2}{s - m_Z^2} \right|^2 \cdot c_R^2(e) \cdot c_L^2(\mu) \cdot (1 - \cos\theta)^2$$

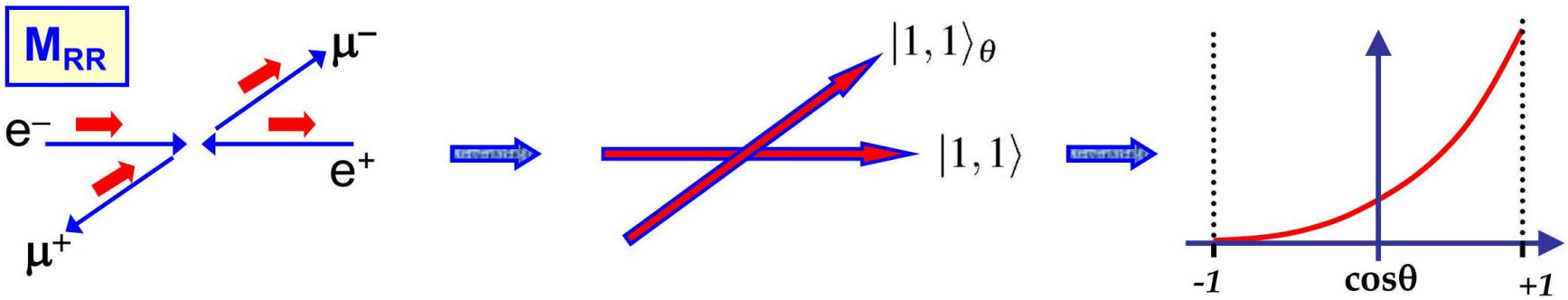
$$|M_{LR}|^2 = s^2 \left| \frac{g_Z^2}{s - m_Z^2} \right|^2 \cdot c_L^2(e) \cdot c_R^2(\mu) \cdot (1 - \cos\theta)^2$$

$$|M_{LL}|^2 = s^2 \left| \frac{g_Z^2}{s - m_Z^2} \right|^2 \cdot c_L^2(e) \cdot c_L^2(\mu) \cdot (1 + \cos\theta)^2$$

$$\frac{e^2}{q^2} \to \frac{g_Z^2}{q^2 - m_Z^2} c(e)c(\mu)$$

where $q^2 = s = 4E_e^2$

- As before, the angular dependence of the matrix elements can be understood in terms of the spins of the incoming and outgoing particles. For example:



The Breit-Wigner Resonance I

- Note that the propagator term $1/\left(s - m_Z^2\right)$ diverges when the C.o.M. energy is equal to the rest mass of the Z boson. Is this physical? No. It is an artefact of using only the simplest diagram. Other diagrams account for the Z-boson's instability.
- A stable particle at rest the time development of the wave-function is:

$$\psi \sim e^{-imt}.$$

- For an unstable particle the equivalent must amount to

$$\psi \sim e^{-imt} e^{-\Gamma t/2}$$

so that the particle probability decays away exponentially

$$\psi^* \psi \sim e^{-\Gamma t} = e^{-t/\tau} \quad \text{with} \quad \tau = \frac{1}{\Gamma_Z}.$$

- This change could be effected by making the replacement $m \to m - i\Gamma/2$ which motivates making the substitution:

$$m_Z \to m_Z - i\Gamma_Z/2$$

in the Z boson propagator to model some of the effects of higher order diagrams:

$$\left(s - m_Z^2\right) \longrightarrow \left[s - (m_Z - i\Gamma_Z/2)^2\right] = s - m_Z^2 + i m_Z \Gamma_Z + \frac{1}{4}\Gamma_Z^2 \approx s - m_Z^2 + i m_Z \Gamma_Z$$

in which the approximation has assumed that $\Gamma_Z \ll m_Z$.

The Breit-Wigner Resonance II

- The aforementioned substitution results in

$$\left|\frac{1}{s-m_Z^2}\right|^2 \to \left|\frac{1}{s-m_Z^2+im_Z\Gamma_Z}\right|^2 = \frac{1}{\left(s-m_Z^2\right)^2+m_Z^2\Gamma_Z^2}$$

and the Matrix elements change to things like this:

$$|M_{RR}|^2 = \frac{g_Z^4 s^2}{\left(s-m_Z^2\right)^2+m_Z^2\Gamma_Z^2}\cdot c_R^2(e)\cdot c_R^2(\mu)\cdot(1+\cos\theta)^2$$

which do not diverge as $s \to m_Z^2$ and are much more physical.

Physically reasonable c.o.m. differential cross sections for $e^+e^- \to Z \to \mu^+\mu^-$

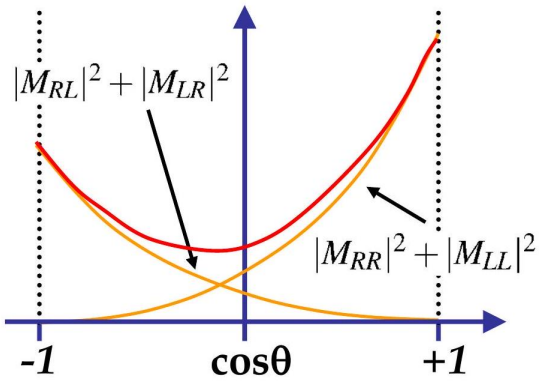
If we neglect m_e and m_μ, the result (10) on page 29 tells us that $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|M_{fi}|^2$ and so the differential cross sections for $e^+e^- \to Z \to \mu^+\mu^-$ broken down by particle helicty are:

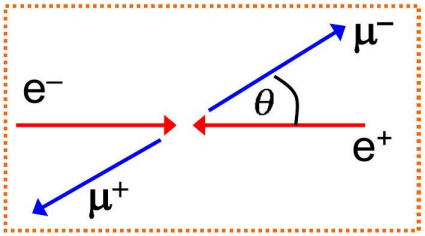
$$\frac{d\sigma_{RR}}{d\Omega} = \frac{1}{64\pi^2}\frac{g_Z^4 s}{(s-m_Z^2)^2 + m_Z^2\Gamma_Z^2} \cdot c_R^2(e) \cdot c_R^2(\mu) \cdot (1+\cos\theta)^2$$

$$\frac{d\sigma_{LL}}{d\Omega} = \frac{1}{64\pi^2}\frac{g_Z^4 s}{(s-m_Z^2)^2 + m_Z^2\Gamma_Z^2} \cdot c_L^2(e) \cdot c_L^2(\mu) \cdot (1+\cos\theta)^2$$

$$\frac{d\sigma_{LR}}{d\Omega} = \frac{1}{64\pi^2}\frac{g_Z^4 s}{(s-m_Z^2)^2 + m_Z^2\Gamma_Z^2} \cdot c_L^2(e) \cdot c_R^2(\mu) \cdot (1-\cos\theta)^2$$

$$\frac{d\sigma_{RL}}{d\Omega} = \frac{1}{64\pi^2}\frac{g_Z^4 s}{(s-m_Z^2)^2 + m_Z^2\Gamma_Z^2} \cdot c_R^2(e) \cdot c_L^2(\mu) \cdot (1-\cos\theta)^2$$





Aside regarding 'forward-backward asymmetry' (much more detail on this on slide 444)

Because $|M_{LL}|^2 + |M_{RR}|^2 \neq |M_{LR}|^2 + |M_{RL}|^2$, the differential cross section is asymmetric, i.e. parity violation (although not maximal as was the case for the W boson).

Total cross section for Z-boson production (unpolarized beams) I

- To calculate the total cross section need to sum over all matrix elements and average over the initial spin states. Here, assuming unpolarized beams (i.e. both $\mathrm{e}^+$and both e^-spin states equally likely) there a four combinations of initial electron/positron spins, so

$$\begin{aligned}\left\langle |M_{fi}|^2 \right\rangle =& \frac{1}{2}\cdot\frac{1}{2}\cdot\left(|M_{RR}|^2+|M_{LL}|^2+|M_{LR}|^2+|M_{RL}|^2\right)\\ =& \frac{1}{2}\cdot\frac{1}{2}\frac{g_Z^4 s^2}{\left(s-m_Z^2\right)^2+m_Z^2\Gamma_Z^2}\times\left\{\left[c_R^2(e)c_R^2(\mu)+c_L^2(e)c_L^2(\mu)\right](1+\cos\theta)^2\right.\\ &\left.+\left[c_L^2(e)c_R^2(\mu)+c_R^2(e)c_L^2(\mu)\right](1-\cos\theta)^2\right\}.\end{aligned}$$

- The $\{\ldots\}$ part of the expression can be rearranged:

$$\begin{aligned}\{\ldots\} =& \left[c_R^2(e)+c_L^2(e)\right]\left[c_R^2(\mu)+c_L^2(\mu)\right]\left(1+\cos^2\theta\right)+\\ & 2\left[c_R^2(e)-c_L^2(e)\right]\left[c_R^2(\mu)-c_L^2(\mu)\right]\cos\theta\end{aligned}$$

and using $c_V^2+c_A^2=2\left(c_L^2+c_R^2\right)$ and $c_V c_A = c_L^2-c_R^2$

$$\{\ldots\}=\tfrac{1}{4}\left[c_V^2(e)+c_A^2(e)\right]\left[c_V^2(\mu)+c_A^2(\mu)\right]\left(1+\cos^2\theta\right)+2c_V^{(e)}c_A^{(e)}c_V^{(\mu)}c_A^{(\mu)}\cos\theta.$$

Total cross section for Z-boson production (unpolarized beams) II

- Hence the complete expression for the unpolarized differential cross section is:

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\left\langle |M_{fi}|^2 \right\rangle \\ &= \frac{1}{64\pi^2}\cdot\frac{1}{4}\cdot\frac{g_Z^4 s}{\left(s-m_Z^2\right)^2+m_Z^2\Gamma_Z^2}\times \\ &\quad \left\{\tfrac{1}{4}\left[c_V^2(e)+c_A^2(e)\right]\left[c_V^2(\mu)+c_A^2(\mu)\right]\left(1+\cos^2\theta\right)+2c_V^{(e)}c_A^{(e)}c_V^{(\mu)}c_A^{(\mu)}\cos\theta\right\}.\end{aligned}$$

- Integrating over solid angle $\mathrm{d}\Omega = \mathrm{d}\phi\mathrm{d}(\cos\theta) = 2\pi\mathrm{d}(\cos\theta)$

$$\int_{-1}^{+1}\left(1+\cos^2\theta\right)\mathrm{d}(\cos\theta) = \int_{-1}^{+1}\left(1+x^2\right)dx = \frac{8}{3} \text{ and } \int_{-1}^{+1}\cos\theta\cdot\mathrm{d}(\cos\theta) = 0$$

and so the total cross section is:

$$\sigma_{e^+e^-\to Z\to\mu^+\mu^-} = \frac{1}{192\pi}\frac{g_Z^4 s}{\left(s-m_Z^2\right)^2+m_Z^2\Gamma_Z^2}\left[c_V^2(e)+c_A^2(e)\right]\left[c_V^2(\mu)+c_A^2(\mu)\right]. \quad (176)$$

Note: the total cross section is proportional to the sums of the squares of the vector- and axial-vector couplings of the initial and final state fermions $c_V^2(f)+c_A^2(f)$.

Relativistic Breit-Wigner Form of Z cross section

- The Z-boson decay rates (partial widths) from (175) on page 423 were:

$$\Gamma_{ee} = \Gamma\left(Z \to e^+e^-\right) = \frac{g_Z^2 m_Z}{48\pi}\left[c_V^2(e) + c_A^2(e)\right]$$

and

$$\Gamma_{\mu\mu} = \Gamma\left(Z \to \mu^+\mu^-\right) = \frac{g_Z^2 m_Z}{48\pi}\left[c_V^2(\mu) + c_A^2(\mu)\right] ..$$

- The above formulae contain **sums of vector and axial couplings** they can be used to eliminate those same sums from our last formula for the total cross section (176):

$$\sigma_{e^+e^-\to Z\to\mu^+\mu^-} = \frac{1}{192\pi}\frac{g_Z^4 s}{\left(s - m_Z^2\right)^2 + m_Z^2\Gamma_Z^2}\left[c_V^2(e) + c_A^2(e)\right]\left[c_V^2(\mu) + c_A^2(\mu)\right].$$

This leaves us with the much simpler formula for the total Z-boson cross section:

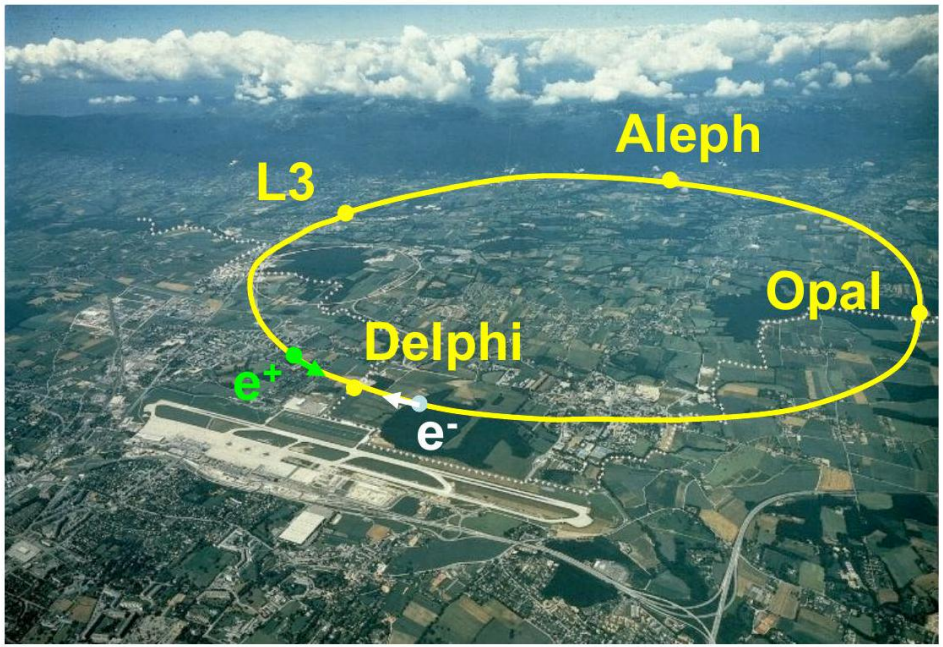
$$\sigma\left(e^+e^- \to Z \to f\bar{f}\right) = \frac{12\pi}{m_Z^2}\frac{s}{\left(s - m_Z^2\right)^2 + m_Z^2\Gamma_Z^2}\Gamma_{ee}\Gamma_{ff} \tag{177}$$

where $\boldsymbol{f}$ is the final state fermion flavour.

(A comparison of (177) with the non-relativistic Breit-Wigner formula seen in the Part II Course is given in Appendix XXV.)

Electroweak Measurements at LEP

The Large Electron Positron (LEP) Collider at CERN (1989-2000) was designed to make precise measurements of the properties of the Z and W bosons.

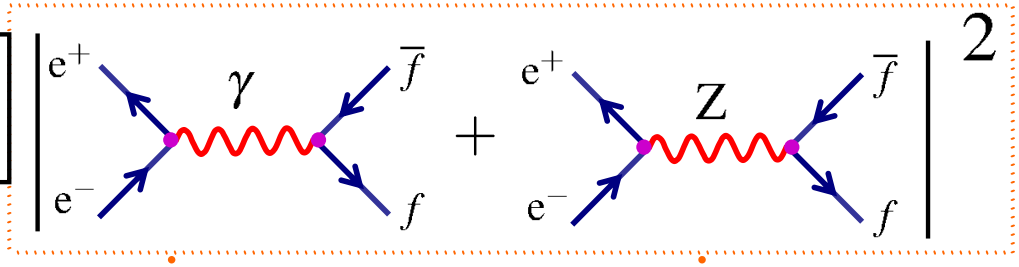


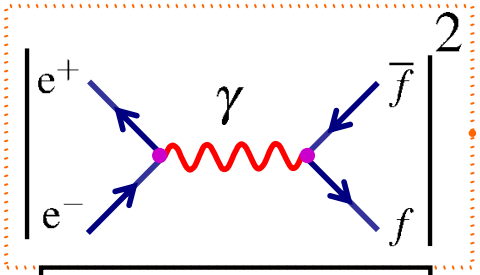
- 26 km circumference accelerator straddling French/Swiss border
- Electrons and positrons collided at 4 interaction points
- 4 large detector collaborations (each with 300-400 physicists):ALEPH, DELPHI, L3, OPAL

- 1989-1995: collisions at $\sqrt{s} = 91.2\ \mathrm{GeV}$
  - 17 Million Z bosons detected.
- 1996-2000 : collisions at $\sqrt{s} = 161 - 208\ \mathrm{GeV}$
  - 30,000 W^+W^- events detected.
- LEP was therefore sometimes referred to as a Z and W 'factory'.

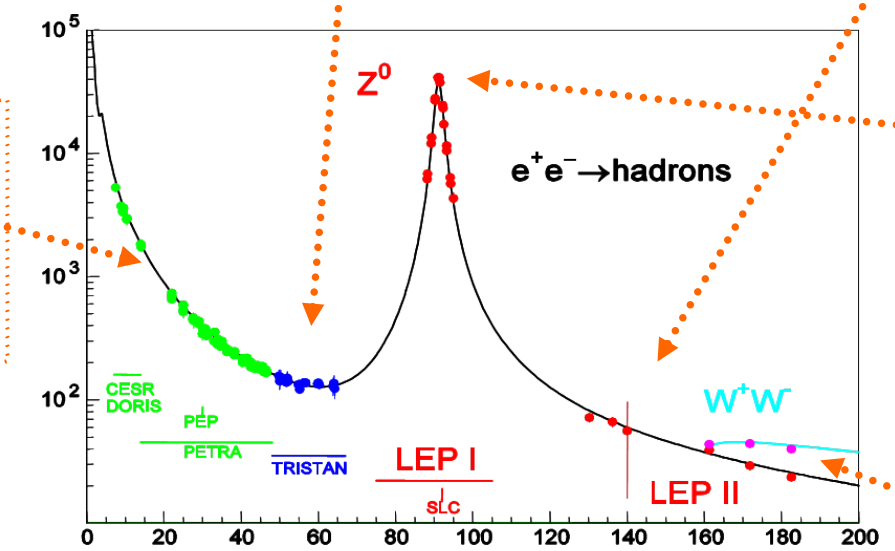
e^+e^- annihilation Feynman Diagrams – changes in importance with $\sqrt{s}$

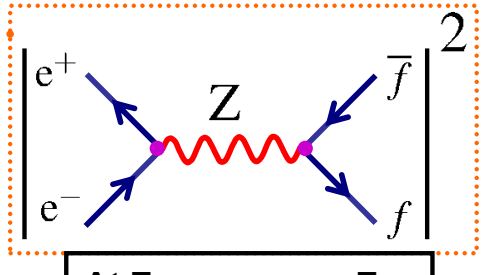
In general e⁺e⁻ annihilation involves both photon and Z exchange : + interference





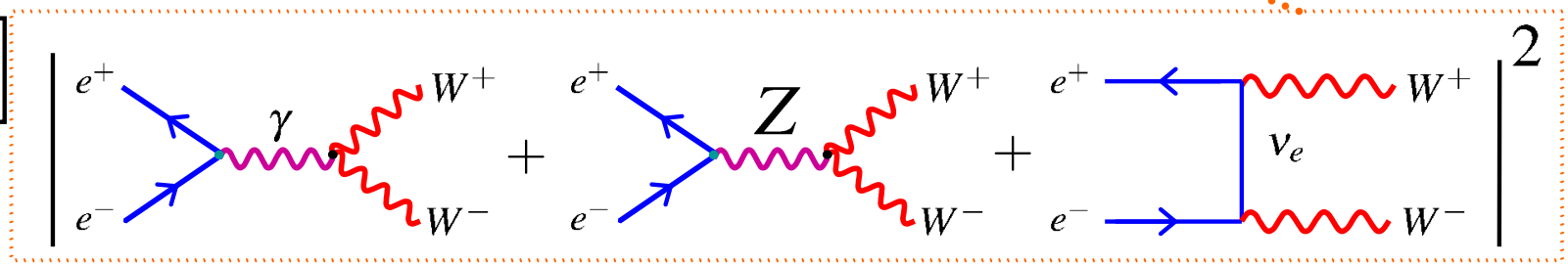
Well below Z: photon exchange dominant





At Z resonance: Z exchange dominant

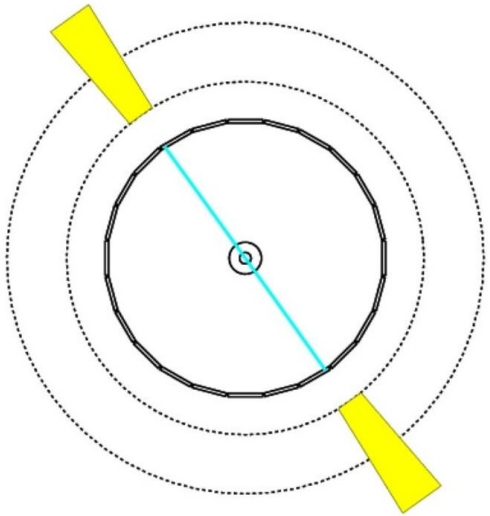
High energies: WW production



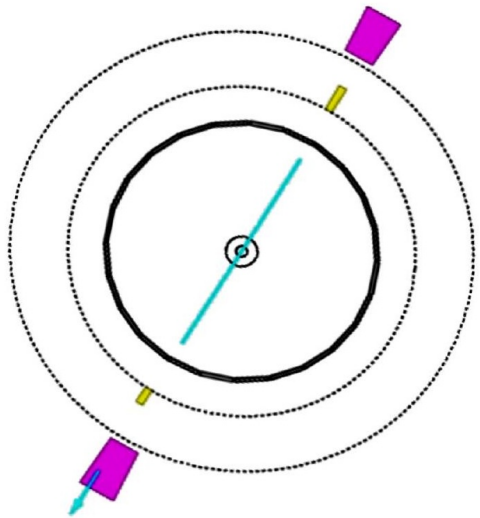
Z-boson cross section measurements and integrated luminosity

- On resonance there are a small number of event types including, e.g.:

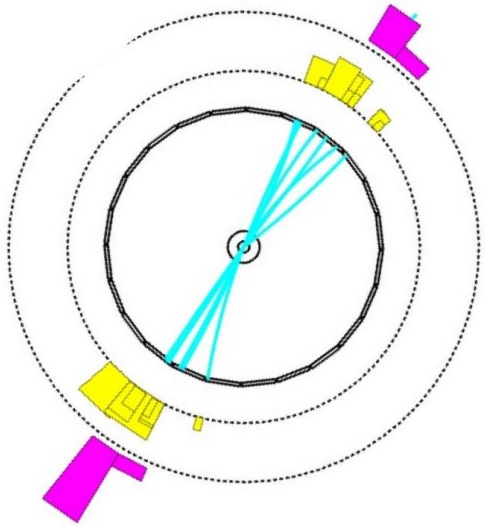
$e^+e^- \to Z \to e^+e^-$



$e^+e^- \to Z \to \mu^+\mu^-$



$e^+e^- \to Z \to \text{jet}^+\text{jet}^-$



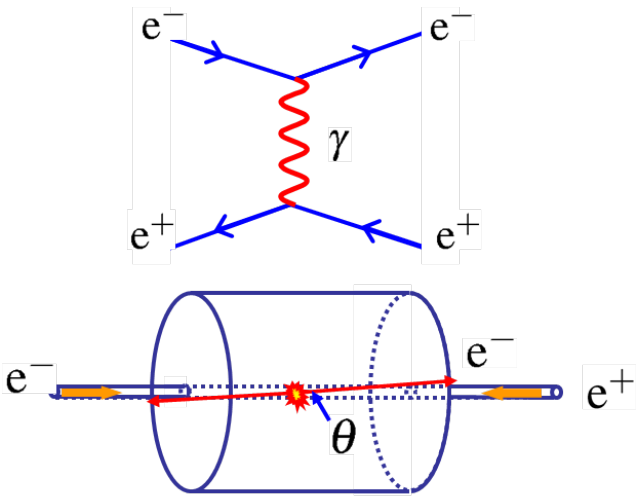
- To experimentally measure the cross section, σ, we can use $N_{\text{events}} = L\sigma$ where
  - N_{events} is the number of observed events of each type, and
  - L is the 'integrated luminosity'.
- Note that our original defintion of cross section in (6) (see page 25) did not use L. It defined σ instead in terms of incident flux and interactions per unit time.

No exciting new physics hides behind the introduction of the 'integrated luminosity', L.

View L just as the name for the correct constant of proportionality that links N_{events} to σ.

'Stealing' the integrated luminosity from a well understood process

- Calculating/predicting the collider's integrated luminosity from beam properties alone is very difficult. Uncertainty of $\sim 10\%$ as beam profiles are hard to measure.
- But since **'integrated luminosity' is a collider property** (not a property of any event) the value of L can be extracted from a (different) process that is well understood.



- Use the QED **Bhabha Scattering** process.
- σ_{Bhabha} can be calclated very precisely from QED.
- Reaction is very forward peaked — i.e. the electron tends not to get deflected much
  $\frac{d\sigma}{d\Omega} \propto \frac{1}{q^4} \propto \frac{1}{sin^4(\theta/2)} \rightarrow \frac{d\sigma}{d\theta} \propto \frac{1}{\theta^3}$
  (Photon propagator), (see (82) on page 134)
- V. large forward cross section (small statistical errors).
- Measure N_{Bhabha}, number of events where the electron is scattered in the very forward direction.
- $N_{\text{Bhabha}} = L\sigma_{\text{Bhabha}}$ is then used to find L.

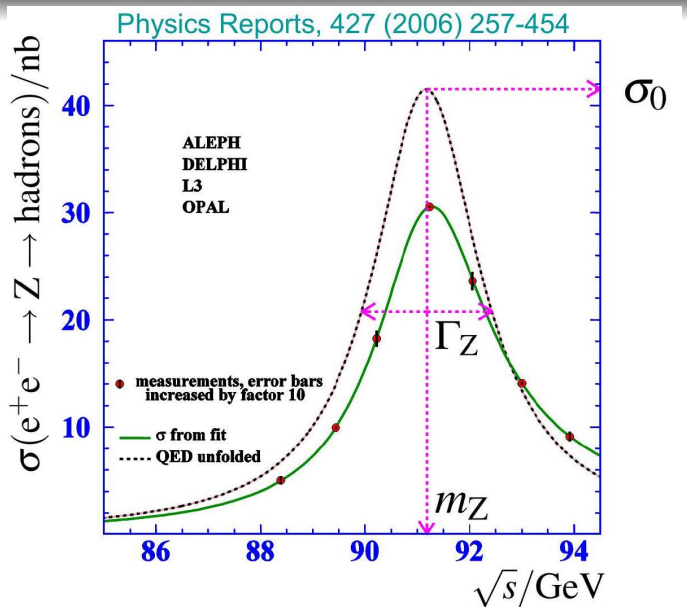
- Hence Z-boson (or other) cross sections can be expressed as multiple of Bhabha:

$$\boxed{\sigma_{Z\to ff} = \frac{N_{Z\to ff}}{N_{\text{Bhabha}}}\sigma_{\text{Bhabha}}}. \tag{178}$$

Nice! Z-boson cross section measuring is now 'mostly' counting!

Measurements of the Z line-shape

- The way the Z-boson's cross section varies as a function of $\sqrt{s}$ is referred to as its 'line-shape'.
- Measurements of the Z-boson line-shape determine:
  - m_Z : peak of the resonance,
  - Γ_Z : FWHM of resonance,
  - Γ_f : Partial decay widths,
  - N_V : Number of light neutrino generations.
- We have recently proved in (177) that



$$\sigma\left(e^+e^- \to Z \to f\bar{f}\right) = \frac{12\pi}{m_Z^2} \frac{s}{\left(s - m_Z^2\right)^2 + m_Z^2\Gamma_Z^2} \Gamma_{ee}\Gamma_{ff} \tag{179}$$

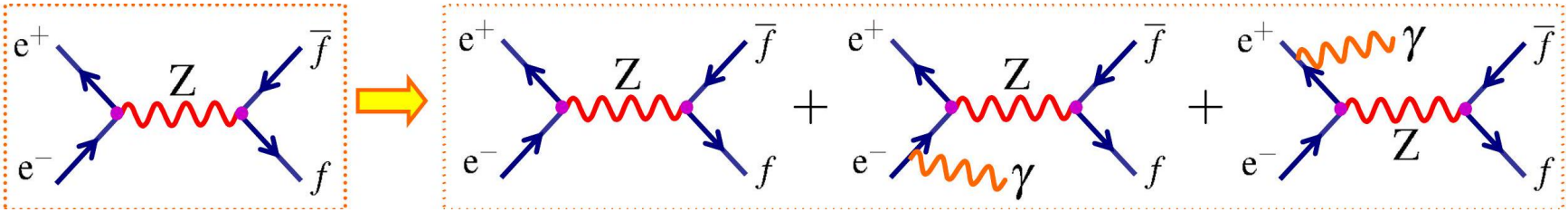
so maximum cross section occurs at $\sqrt{s} = m_Z$ and peak cross section is equal to

$$\boxed{\sigma^0_{f\bar{f}} = \frac{12\pi}{m_Z^2} \frac{\Gamma_{ee}\Gamma_{ff}}{\Gamma_Z^2}}. \tag{180}$$

That the cross section falls to half peak value at $\sqrt{s} \approx m_z \pm \frac{\Gamma_Z}{2}$ can be seen in (179).

Take home message: $\Gamma_Z = \frac{\hbar}{\tau_Z}$ can be 'seen' in the FWHM of the resonance lineshape.

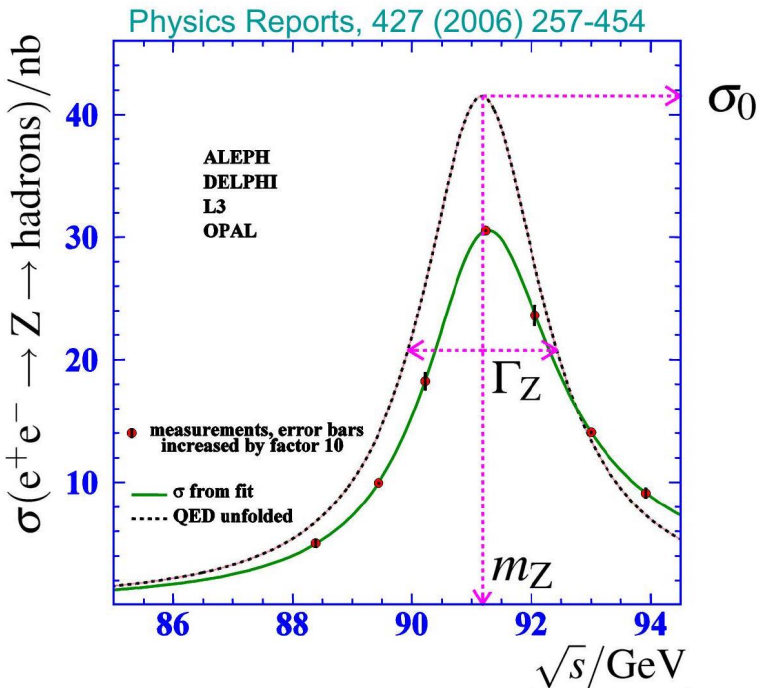
- In practice, it is not that simple, QED corrections distort the measured line-shape.
- One particularly important correction: initial state radiation (ISR):



- Initial state radiation reduces the centre-of-mass energy of the 'inner' e^+e^- collision.
- Measured cross section, σ_{meas}, can be written:

$$\sigma_{\text{meas}}\left(\sqrt{s}\right) = \int_0^{\sqrt{s}} \sigma(\sqrt{s'})p(\sqrt{s'} \mid \sqrt{s})\mathrm{d}\sqrt{s'}$$

where $p(\sqrt{s'} \mid \sqrt{s})$ is the probability density for the post-ISR $e^+e^- \to Z$ part of the collision to occur at $\sqrt{s'}$ given that the initial state energy available pre-ISR was $\sqrt{s}$, and where σ is the ISR-free cross section we calculated earlier (see (177) or (179)).



- Fortunately QED can be used to calculate $p(\cdot \mid \cdot)$.
- The ISR-distorted σ_{meas} measured lineshape can therefore be compared with calculations/predictions based on the convolution of the ISR-free σ and $p(\cdot \mid \cdot)$.

Precision measurements of Z-boson mass and width

World average at 2024 (largely dominated by LEP measurements) is:

$$m_Z = 91.1875 \pm 0.0021 \text{ GeV} \qquad \Gamma_Z = 2.4952 \pm 0.0023 \text{ GeV}$$

Not examinable:

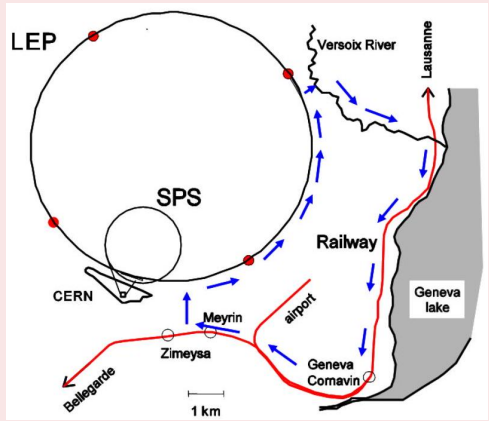
Achieving such high precision required other surprising factors to be taken into account beyond ISR. Examples include:

Position of the Moon:

- As the moon orbits the Earth it distorts the rock in the Geneva area very slightly and the nominal radius of the accelerator of 4.3 km varies by ± 0.15 mm.
- This changes beam energy by ~ 10 MeV, so need to correct for tidal effects!

Train tables:

- Leakage currents from the TGV railway line returned to Earth following the path of least resistance, a small part of which was via the Versoix river and using the LEP ring as a conductor.
- Each time a TGV train passed by, a small current circulated LEP slightly changing the magnetic field in the accelerator changing the LEP beam energy by ~ 10 MeV.



Count the number of generations via the Z's 'invisible width' ?

Observed Z-width, Γ_Z, is the sum of all partial widths including to invisible particles!

- 3-generation Standard Model predicts:

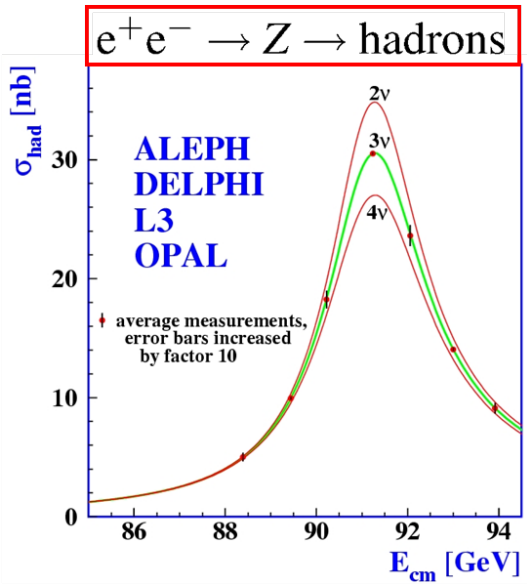
$$\Gamma_Z = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{\text{hadrons}} + \Gamma_{\nu_1\nu_1} + \Gamma_{\nu_2\nu_2} + \Gamma_{\nu_3\nu_3} = \boxed{\Gamma_{\text{visible}} + 3 \cdot \Gamma_{\nu_1\nu_1}}.$$

- If there were an additional 4th generation would expect $Z \to \nu_4 \bar{\nu}_4$ decays even if the charged leptons and fermions were too heavy (i.e. $> m_z/2$) so
- $\geq$ 4-generation Standard Model predicts:

$$\Gamma_Z \geq \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{\text{hadrons}} + \Gamma_{\nu_1\nu_1} + \Gamma_{\nu_2\nu_2} + \Gamma_{\nu_3\nu_3} + \Gamma_{\nu_4\nu_4} = \boxed{\Gamma_{\text{visible}} + 4 \cdot \Gamma_{\nu_1\nu_1}}.$$

Recall from (180) that

$$\sigma^0_{f\bar{f}} = \frac{12\pi}{m_Z^2} \frac{\Gamma_{ee}\Gamma_{ff}}{\Gamma_Z^2}.$$



Given:

- Γ_Z measured from Z lineshape,
- $\Gamma_{\nu_1\nu_1}$ calculated (e.g. as in qu. 26), and
- $\Gamma_{ee}, \Gamma_{\mu\mu}, \ldots$ measured from peak cross sections, $\sigma^0_{ee}, \sigma^0_{\mu\mu}, \ldots$,

a fit of N_ν to $\boxed{\Gamma_Z = \Gamma_{\text{visible}} + N_\nu \cdot \Gamma_{\nu_1\nu_1}}$ finds:

$$\boxed{N_\nu = 2.9840 \pm 0.0082}.$$

Forward-Backward Asymmetry for $e^+e^- \to Z \to \mu^+\mu^-$

On pages 433 and 434 we obtained the expression for the differential cross section:

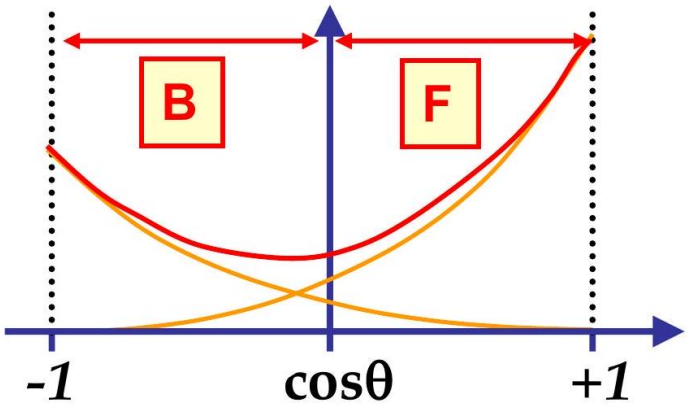
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \kappa\left[A\left(1+\cos^2\theta\right)+2B\cos\theta\right] \tag{181}$$

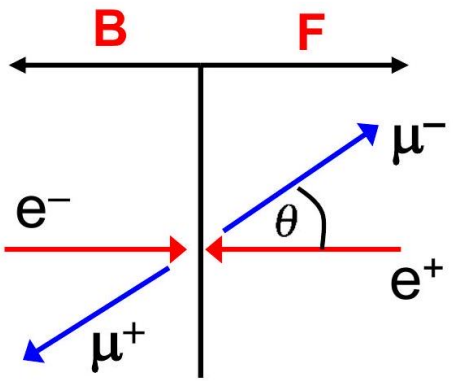
where κ was independent of the muon scattering angle, θ, and where A and B were

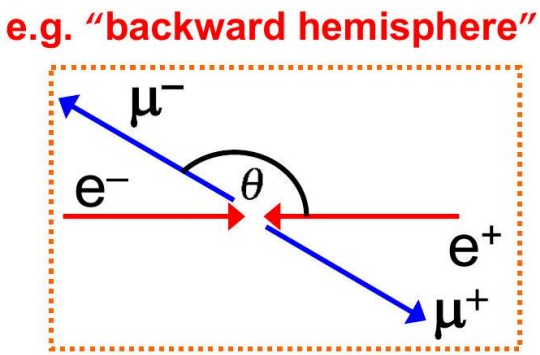
$$A = \left[c_L^2(e)+c_R^2(e)\right]\left[c_L^2(\mu)+c_R^2(\mu)\right] \quad \text{and} \quad B = \left[c_L^2(e)-c_R^2(e)\right]\left[c_L^2(\mu)-c_R^2(\mu)\right].$$

We define the FORWARD and BACKWARD cross sections as follows:

$$\sigma_F \equiv \sigma_{\cos\theta>0} = \int_0^1 \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\mathrm{d}\cos\theta \quad \text{and} \quad \sigma_B \equiv \sigma_{\cos\theta<0} = \int_{-1}^0 \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\mathrm{d}\cos\theta.$$

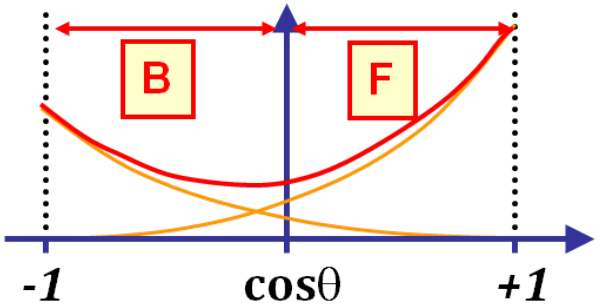






- The Forward-Backward Asymmetry, A_{FB}, is **defined** by:

$$\boxed{A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}}.$$



- Integrating equation (181) (while replacing $\cos\theta$ with x) gives:

$$\sigma_F \propto \int_0^1 \left[A\left(1+x^2\right)+2Bx\right]\mathrm{d}x = \frac{4}{3}A + B \propto \boxed{4A+3B}$$

$$\sigma_B \propto \int_{-1}^0 \left[A\left(1+x^2\right)+2Bx\right]\mathrm{d}x = \frac{4}{3}A - B \propto \boxed{4A-3B}, \qquad \text{so}$$

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{6B}{8A} = \frac{3}{4}\left[\frac{c_L^2(e)-c_R^2(e)}{c_L^2(e)+c_R^2(e)}\right]\cdot\left[\frac{c_L^2(\mu)-c_R^2(\mu)}{c_L^2(\mu)+c_R^2(\mu)}\right] = \frac{3}{4}A_eA_\mu$$

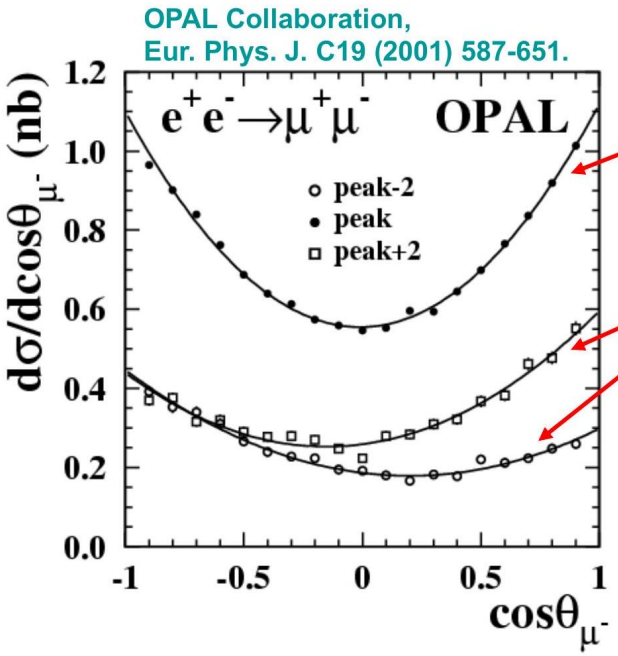
where

$$\boxed{A_x \equiv \frac{c_L^2(x)-c_R^2(x)}{c_L^2(x)+c_R^2(x)} = \frac{2c_V(x)c_A(x)}{c_V^2(x)+c_A^2(x)} = \frac{2\cdot\left(c_V(x)/c_A(x)\right)}{1+\left(c_V(x)/c_A(x)\right)^2} = f\left(\frac{c_V(x)}{c_A(x)}\right)}. \quad (182)$$

- We observe a non-zero asymmetry A_{FB} because the couplings of the Z to L and R particles differ so $A_e \neq 0 \neq A_\mu$. Contrast with QED where the L and R couplings are the same (parity is conserved) and the interaction is FB symmetric.

Measured Forward-Backward Asymmetries

These are the observed muon $\cos\theta$ distributions from the LEP experiment 'OPAL':



Because $\sin^2\theta_w \approx 0.25$, the value of A_{FB} for leptons is almost zero

For data above and below the peak of the Z resonance interference with $e^+e^- \to \gamma \to \mu^+\mu^-$ leads to a larger asymmetry

★LEP data combined:

$$A_{FB}^{0,e} = 0.0145 \pm 0.0025$$
$$A_{FB}^{0,\mu} = 0.0169 \pm 0.0013$$
$$A_{FB}^{0,\tau} = 0.0188 \pm 0.0017$$

- A_e, A_μ and A_τ values can be extracted from the above A_{FB} values via $A_{FB}^{\mu} = \frac{3}{4} A_e A_\mu$.

- A mixture of the above measurements and other related measurments (e.g. A_{LR} of Appendix XXVI) finds that

$$\begin{aligned} A_e &= 0.1514 \pm 0.0019, \\ A_\mu &= 0.1456 \pm 0.0091, \\ A_\tau &= 0.1449 \pm 0.0040. \end{aligned} \tag{183}$$

Measuring the Weak Mixing Angle, θ_W, from F-B asymmetries

- We saw already in (182) that A_x is a function of just $c_V(x)/c_A(x)$.

- Proof below shows that $c_V(x)/c_A(x)$ is only a function of the weak mixing angle θ_W.

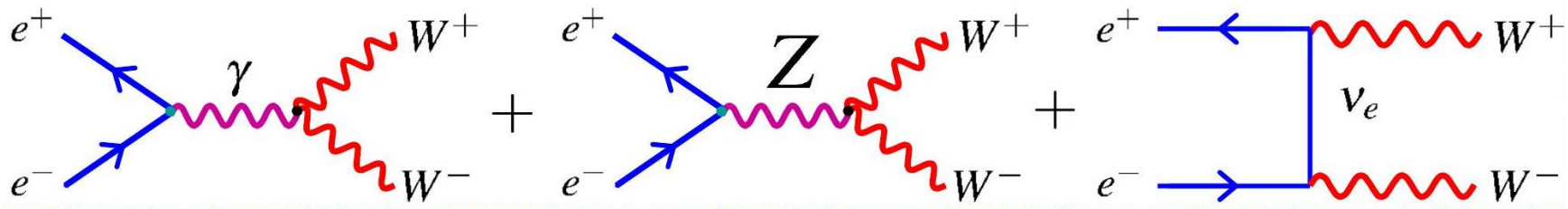
$$\begin{aligned}
\frac{c_V(x)}{c_A(x)} &= \frac{I_W^3(x_L) - 2Q_x \sin^2\theta_W}{I_W^3(x_L)} && \text{(by (170) and (171))} \\
&= 1 - \frac{2Q_x}{I_3(x_L)} \sin^2\theta_W \\
&= 1 - 4|Q_x| \sin^2\theta_W && \text{(true for every row of the table on page 421)} \\
&= 1 - 4\sin^2\theta_W && \text{(only true for } x \in \{e, \mu, \tau\}\text{).}
\end{aligned}$$

Therefore a clean measurement of the weak mixing angle θ_W can be obtained from the measurements of A_e, A_μ and A_τ listed in (183). It is found that:

$$\sin^2\theta_W = 0.23154 \pm 0.00016.$$

W^+W^- Production

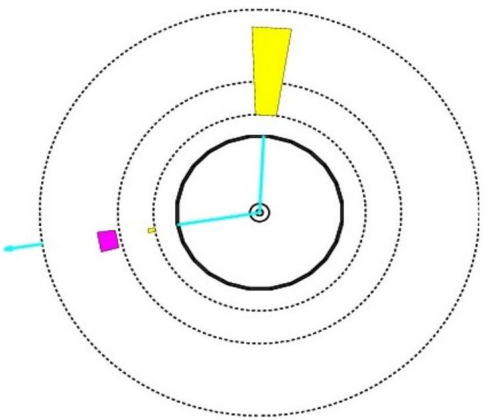
- From 1995-2000 LEP operated above the threshold for W-pair production.
- The three tree level diagrams for W-pair production are shown below.



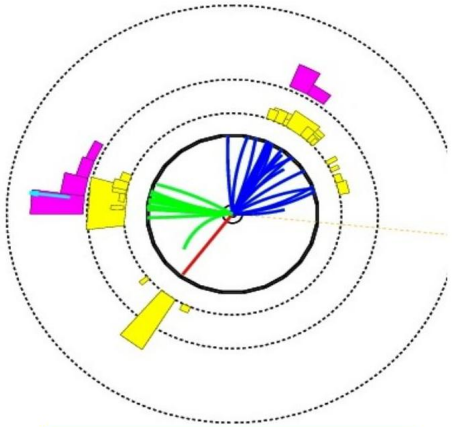
- [Page 409]: W-bosons decay to either to leptons or hadrons with branching fractions:

$$\text{Br}\left(W^- \to \text{ hadrons }\right) \approx 0.67 \quad \text{Br}\left(W^- \to e^-\bar{\nu}_e\right) \approx 0.11$$
$$\text{Br}\left(W^- \to \mu^-\bar{\nu}_\mu\right) \approx 0.11 \quad \text{Br}\left(W^- \to \tau^-\bar{\nu}_\tau\right) \approx 0.11$$

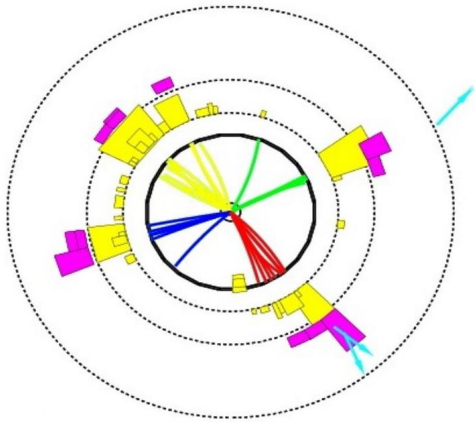
giving rise to three distinct signatures:



$W^+W^- \to \ell^+\nu\ell^-\bar{\nu}$



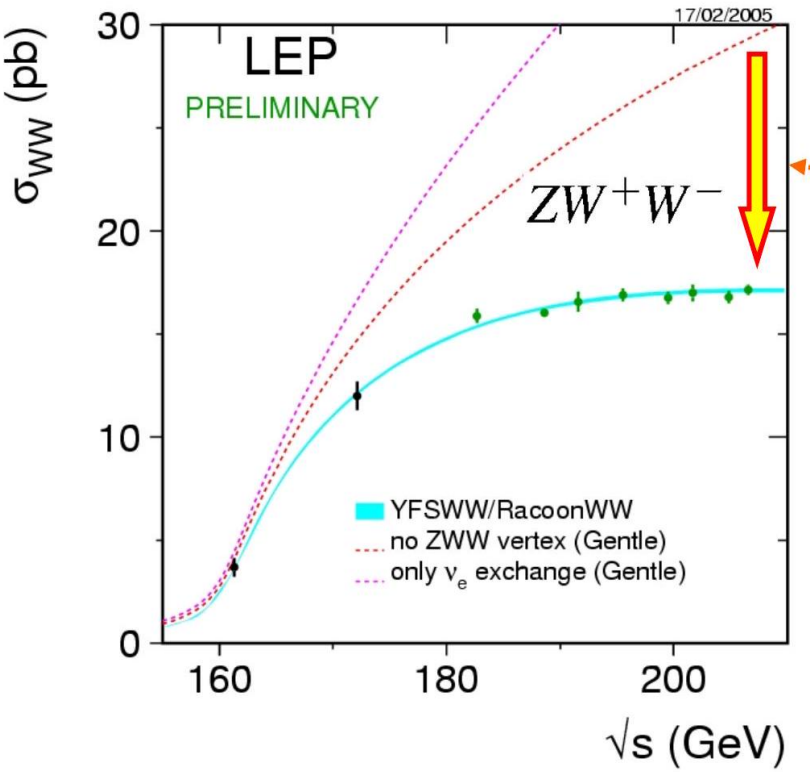
$W^+W^- \to q\bar{q}\ell\nu$



$W^+W^- \to q\bar{q}q\bar{q}$

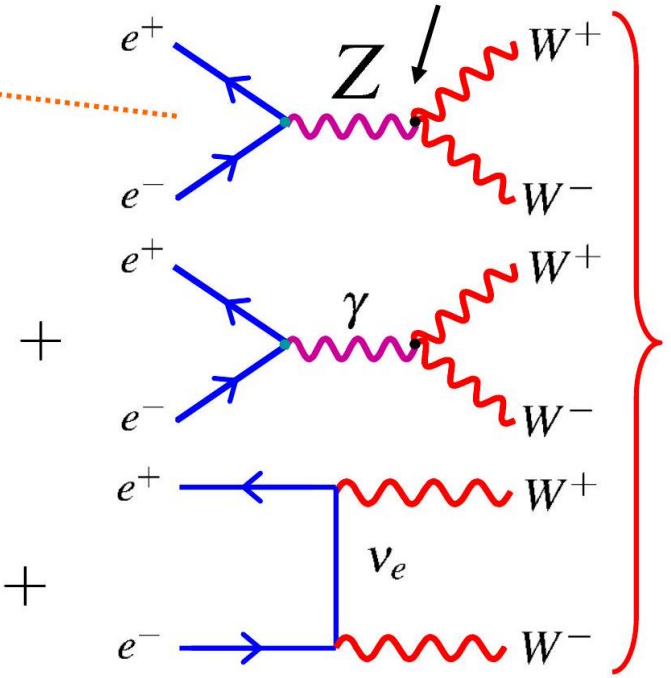
$e^+e^- \to W^+W^-$ line shape provides confirmation of ZWW vertex

Cross section is again normalised against Bhabha scattering (see (178) on page 439).



★ **Data consistent with SM expectation**

★ **Provides a direct test of ZW^+W^- vertex**



- Without the Z diagram the WW cross section would violate unitarity.
- The presence of the Z fixes this problem.

W-mass and W-width

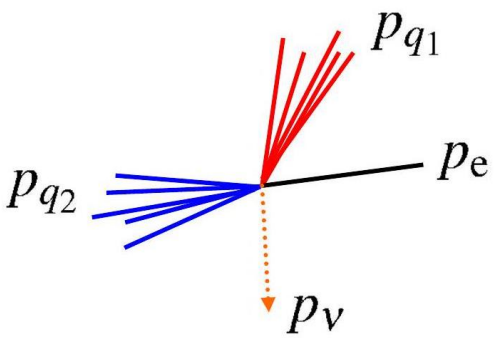
- The W line-shape is not used to find the W mass as production is not resonant. Instead, W masses are directly reconstructed in each event (often twice).

- E.g. in $W^+W^- \to q\bar{q}e^-\bar{\nu}_e$ the two mass estimates would be defined by

$$m_{W1}^2 = (p_e + p_{\nu_e})^\mu (p_e + p_{\nu_e})_\mu$$
$$m_{W2}^2 = (p_{q_1} + p_{q_2})^\mu (p_{q_1} + p_{q_2})_\mu$$

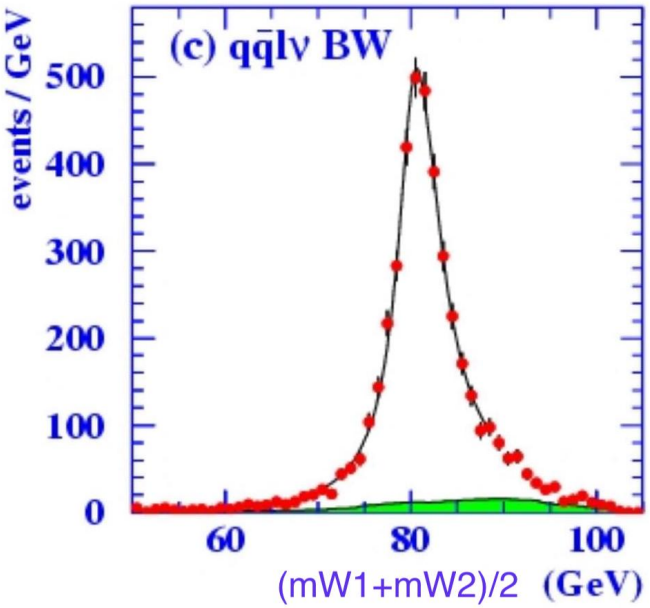
  in which the momentum $p_{\nu_e}^\mu$ of the unobserved neutrino would be deduced by solving

$$(p_e + p_{\nu_e} + p_{q_1} + p_{q_2})^\mu = (\sqrt{s}, 0)^\mu.$$



- Shape of reconstructed mass distribution gives:

$$m_W = 80.376 \pm 0.033 \text{ GeV} \quad \text{and}$$
$$\Gamma_W = 2.196 \pm 0.083 \text{ GeV}.$$



Hadron Colliders after LEP (Tevatron, LHC) have improved upon these results.

The Higgs Mechanism

- Higgs Fields can be used to give masses to both fermions and gauge bosons - but mechanism is different in the two cases.

- Explaining how the Higgs mechanism gives the W and Z gauge bosons masses, while leaving the photon massless, is (unfortunately) beyond this course. [See Gauge Field Theory minor option).]

- We do, howwever, attempt to describe the way interactions with the Higgs field gives masses to fermions.

Higgs Learning Objectives Overview

Higgs field is more important than Higgs Boson

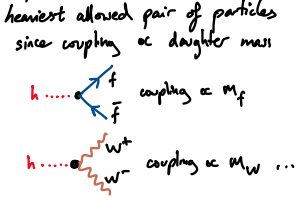
Higgs field gives mass to bosons and fermions in different ways.

Mass given to $W^\pm$ & Z bosons by:
- Spontaneous Symmetry Breaking
- Goldstone Bosons
- Gauge Field Theory Course!

Higgs Boson

Higgs Boson must be a scalar (spin-0) because Yukawa interaction (see later) has no spare indices to contract with a spinor or vector field.

prefers to decay to heaviest allowed pair of particles since coupling $\propto$ daughter mass



Mass for Fermions is supplied by Yukawa interactions:

[Complex Scalar Higgs Field $SU(2)_L$ Doublet]

$$\mathcal{L} = \left(\ldots + y_e \overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L} \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} e_R + \ldots\right) + \text{complex conjugate}$$

not Dirac mass term:

$$\mathcal{L} = \ldots + m_e \bar{e} e + \ldots$$

because:
- Yukawa mass $\mathcal{L}$ is $SU(2)_L$ Gauge Invariant
- Dirac mass $\mathcal{L}$ is not $SU(2)_L$ Gauge Invariant

} Why?

and because
- Yukawa interactions "act like" Dirac mass terms if $\langle\phi^0\rangle \neq 0$.

} Why?

$\langle\phi^0\rangle$ is called the "Higgs V.E.V." (vacuum expectation value)

[Measured value: $\langle\phi^0\rangle = 246\,\text{GeV}$]

ASIDE (Not Examinable!)

Strictly one must use something like:

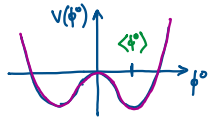
$$a\left[\begin{pmatrix}\phi^{0\dagger} & \phi^{+\dagger}\end{pmatrix}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}\right]^2 - b\left(\begin{pmatrix}\phi^{0\dagger} & \phi^{+\dagger}\end{pmatrix}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}\right)^2$$

to maintain $SU(2)_L$ Gauge Invariance

Give Lagrangian a Higgs Potential $V(\phi^0)$ which is minimised at $\phi^0 = \langle\phi^0\rangle_{VEV} \neq 0$.

e.g.:

$$V(\phi^0) = a(\phi^0)^4 - b(\phi^0)^2$$



But why should $\langle\phi^0\rangle$ bother to take a non-zero value?

Under $SU(2)_L$ transformation:

$$\left\{\begin{array}{l}\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L \to e^{i\theta\cdot\sigma}\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L\\ \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \to e^{i\theta\cdot\sigma}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}\\ e_R \to e_R\end{array}\right\}$$

hence:

$$y_e \overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L} \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} e_R \to y_e \overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L} e^{-i\theta\cdot\sigma} e^{i\theta\cdot\sigma} \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} e_R = y_e \overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L} \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} e_R$$ ✓ invariant

but:

$$\begin{aligned} m_e \bar{e} e &= m_e \bar{e}(P_L + P_R) e && (P_L + P_R = 1)\\ &= m_e \bar{e}(P_L^2 + P_R^2) e && (P_L^2 = P_L \;\&\; P_R^2 = P_R)\\ &= m_e \bar{e} P_L^2 e + m_e \bar{e} P_R^2 e\\ &= m_e \bar{e}_R e_L + m_e \bar{e}_L e_R && (\bar{\psi} P_{R/L} = \bar{\psi}_{L/R} \;\&\; P_{R/L}\psi = \psi_{R/L}) \end{aligned}$$

$$\to m_e \bar{e}_R (c\,\nu_{eL} + d\,e_L) + m_e (\bar{c}\,\bar{\nu}_{eL} + \bar{d}\,\bar{e}_L) e_R .$$

Not $m_e \bar{e} e$!
× Not invariant!

Yukawa Mass Term $= y_e \overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L} \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} e_R + \text{c.c.}$ ("complex conjugate")

$$\begin{aligned} &= \left(y_e \bar{\nu}_L \phi^+ e_R + y_e \bar{e}_L \phi^0 e_R\right) + \text{c.c.}\\ &= \left(y_e \bar{e}_L \phi^0 e_R + y_e \bar{e}_R \phi^0 e_L\right) + \left(y_e \bar{\nu}_{eL} \phi^+ e_R + \text{c.c.}\right)\\ &= (y_e \phi^0)\left(\bar{e}_L e_R + \bar{e}_R e_L\right) + \left(y_e \bar{\nu}_{eL} \phi^+ e_R + \text{c.c.}\right)\\ &= \underbrace{(y_e \phi^0)}\,\bar{e} e + \underbrace{\left(y_e \bar{\nu}_{eL} \phi^+ e_R + y_e \bar{e}_R \phi^- \nu_{eL}\right)} \end{aligned}$$

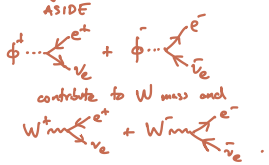
Behaves like $m_e \bar{e} e$ (Dirac mass term)

provided ϕ^0 does not wobble too much

and provided $\langle y_e \phi^0\rangle = m_e$

i.e. provided $\langle\phi^0\rangle_{\text{HIGGS VEV}} = \frac{m_e}{y_e} \neq 0$.

ASIDE



Higgs field is more important than Higgs Boson

Higgs field gives mass to bosons and fermions in different ways.

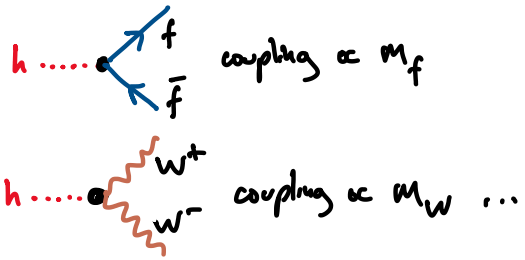
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prefers to decay to heaviest allowed pair of particles since coupling $\propto$ daughter mass



Mass for Fermions is supplied by

Yukawa interactions:

[Complex Scalar Higgs Field $SU(2)_L$ Doublet]

$$\mathcal{L} = \left(\ldots + y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R + \ldots \right) + \text{complex conjugate}$$

not **Dirac mass term:**

$$\mathcal{L} = \ldots + m_e \bar{e} e + \ldots$$

because:

- Yukawa mass $\mathcal{L}$ is $SU(2)_L$ Gauge Invariant
- Dirac mass $\mathcal{L}$ is not $SU(2)_L$ Gauge Invariant

} Why?

and because

- Yukawa interactions "act like" Dirac mass terms if $\langle \phi^0 \rangle \neq 0$.

} Why?

• Yukawa interactions act

mass terms if $\langle \phi^0 \rangle \neq 0$.

$\langle \phi^0 \rangle$ is called the "Higgs V.E.V." (vacuum expectation value)

[Measured value: $\langle \phi^0 \rangle = 246\,\text{GeV}$]

Under $SU(2)_L$ transformation:

$$\left\{ \begin{array}{l} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \to e^{i\underline{\theta}\cdot\underline{\sigma}} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \to e^{i\underline{\theta}\cdot\underline{\sigma}} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \\ e_R \to e_R \end{array} \right\}$$

hence:

$$y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R \to y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} e^{-i\underline{\theta}\cdot\underline{\sigma}} e^{i\underline{\theta}\cdot\underline{\sigma}} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R = y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R$$ ✓ invariant

but:

$$\begin{aligned} m_e \bar{e} e &= m_e \bar{e} (P_L + P_R) e && (P_L + P_R = 1) \\ &= m_e \bar{e} (P_L^2 + P_R^2) e && (P_L^2 = P_L \ \& \ P_R^2 = P_R) \\ &= m_e \bar{e} P_L^2 e + m_e \bar{e} P_R^2 e \\ &= m_e \bar{e}_R e_L + m_e \bar{e}_L e_R && (\bar{\psi} P_{R/L} = \bar{\psi}_{L/R} \ \& \ P_{R/L} \psi = \psi_{R/L}) \end{aligned}$$

$$\to m_e \bar{e}_R (c\, \nu_{e_L} + d\, e_L) + m_e (\bar{c}\, \bar{\nu}_{e_L} + \bar{d}\, \bar{e}_L) e_R .$$

Not $m_e \bar{e} e$!
× Not invariant!

Yukawa Mass Term $= y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R + \text{c.c.}$ ("complex conjugate")

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$= \left(y_e \bar{e}_L \phi^0 e_R + y_e \bar{e}_R \phi^0 e_L \right) + \left(y_e \bar{\nu}_{e_L} \phi^+ e_R + \text{c.c.} \right)$

$= (y_e \phi^0)(\bar{e}_L e_R + \bar{e}_R e_L) + \left(y_e \bar{\nu}_{e_L} \phi^+ e_R + \text{c.c.} \right)$

$= \underbrace{(y_e \phi^0) \bar{e} e}_{\downarrow} + \underbrace{\left(y_e \bar{\nu}_{e_L} \phi^+ e_R + y_e \bar{e}_R \phi^- \nu_{e_L} \right)}_{\downarrow}$

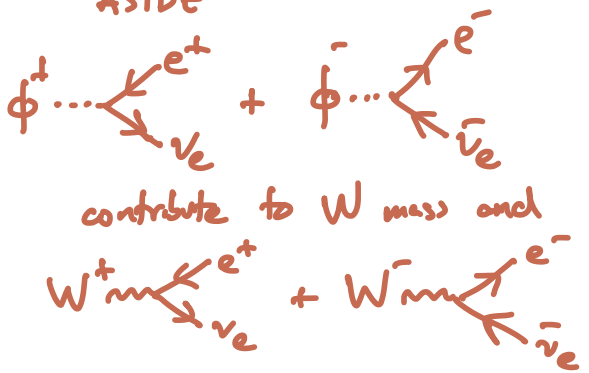
Behaves like $m_e \bar{e} e$ (DIRAC MASS TERM)

provided ϕ^0 does not wobble too much

and provided $\langle y_e \phi^0 \rangle = m_e$

i.e. provided $\underbrace{\langle \phi^0 \rangle}_{\text{HIGGS VEV}} = \frac{m_e}{y_e} \neq 0.$

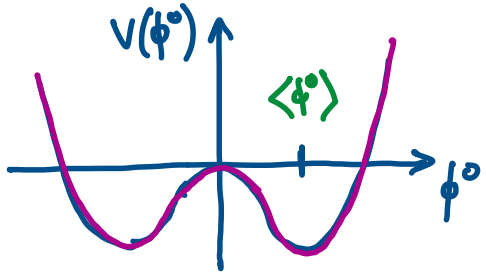
ASIDE



Give Lagrangian a Higgs Potential $V(\phi^0)$ which is minimised at $\phi^0 = \langle \phi^0 \rangle_{\text{VEV}} \neq 0$.

e.g. :

$$V(\phi^0) = a(\phi^0)^4 - b(\phi^0)^2$$



But why should $\langle \phi^0 \rangle$ bother to take a non-zero value?

Behaves like $m_e \bar{e} e$ (DIRAC MASS TERM)

provided ϕ^0 does not wobble too much

and provided $\langle y_e \phi^0 \rangle = m_e$

i.e. provided $\underset{\text{HIGGS VEV}}{\langle \phi^0 \rangle} = \frac{m_e}{y_e} \neq 0$.

Not examinable

ASIDE (Not Examinable!)

Strictly one must use something like:

$$V = a\left[\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}^\dagger\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}\right]^2 - b\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}^\dagger\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$$

to maintain $SU(2)_L$ Gauge Invariance

Not examinable

Higgs theory summary for fermions:

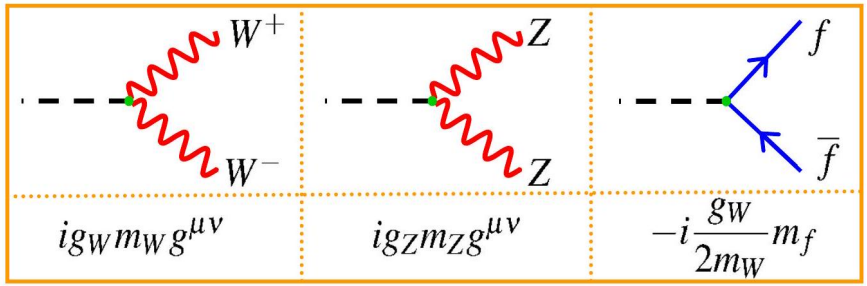
- Fermions are intrinsically massless, and need to be so to be Gauge Invariant.
- Nevertheless, interactions with the Higgs field make fermions look like they have mass at "low temperature" (i.e. when the Higgs field is near its ground state)
- Apparent fermion masses are controlled by free parameters called Yukawa Couplings (the strength of the coupling to the Higgs field).
- A Higgs Boson is an excitation of the Higgs Field.
- The Higgs Boson must be a scalar particle to make everything work.
- The Higgs Boson has a mass, but the mass is not predicted by the theory - we have to find it experimentally.
- The Higgs Boson has couplings to all the particles it gives mass to (and indeed to gauge bosons too!) and so has many ways it could decay, all fully calculable and determined by the theory as a function of its (as yet unknown) mass.
- For proper discussion of the Higgs mechanism see the Gauge Field Theory minor option.

Higgs mechanism for gauge bosons:

Not examinable

- The Higgs mechanism results in absolute predictions for masses of gauge bosons.
- In the SM, fermion masses are also ascribed to interactions with the Higgs field - however, here no prediction of the masses - just put in by hand.

- The Higgs is electrically neutral but carries weak hypercharge of $\frac{1}{2}$.
- The photon does not couple to the Higgs field and remains massless.
- The W and Z bosons couple to weak hypercharge and become massive.



Feynman Vertex factors

Within the SM, electroweak unification and the Higgs mechanism implies the following relations between SM parameters:

$$m_W = \left(\frac{\pi\alpha_{em}}{\sqrt{2}G_F}\right)^{\frac{1}{2}} \frac{1}{\sin\theta_W} \quad \text{and} \quad m_Z = \frac{m_W}{\cos\theta_W}.$$

Hence, if you know any three of : $\alpha_{em}, G_F, m_W, m_Z, \sin\theta_W$ predict the other two.

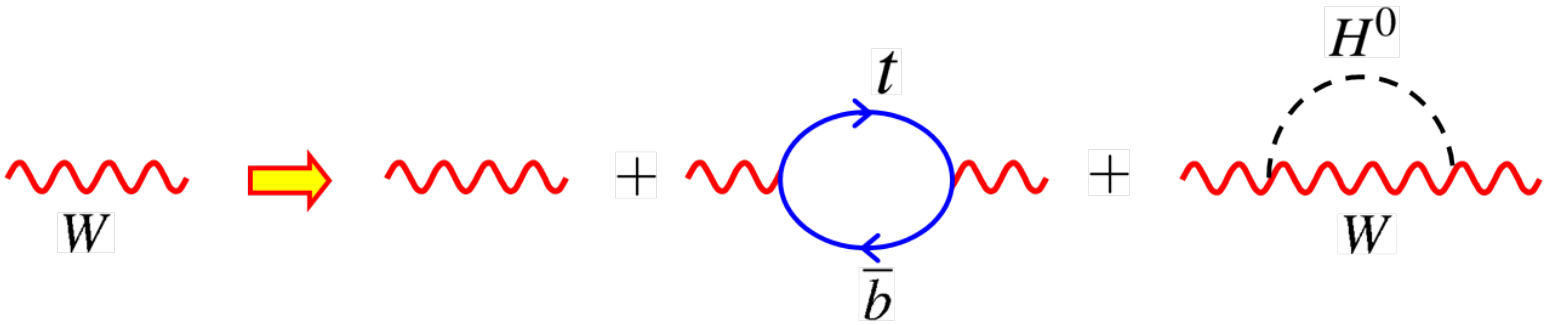
Not examinable

So called 'Precision Electroweak' tests at LEP

- At leading order, our theory predicts $m_W = m_Z \cos\theta_W$.
- Since we measure:

$$\left\{ \begin{array}{ll} m_Z & = 91.1875 \pm 0.0021 \text{ GeV} \\ \sin^2\theta_W & = 0.23154 \pm 0.00016 \end{array} \right\}$$

  we expect: $m_W = 79.946 \pm 0.008$ GeV but measure $m_W = 80.376 \pm 0.033$ GeV.
- Close, but not quite right! We have only considered lowest order diagrams!
- Higher order calculation shows that preduction for W boson mass also includes terms from virtual loops:



$$m_W = m_W^0 + a \cdot m_t^2 + b \cdot \ln\left(\frac{m_H}{m_W}\right)$$

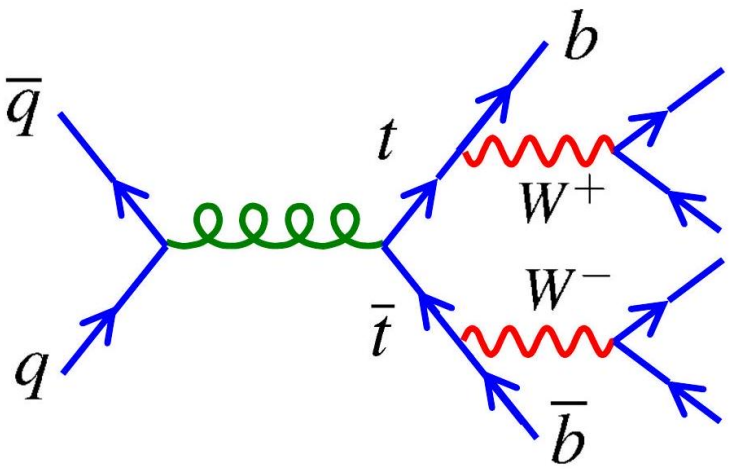
- So very high precision measurements provide sensitivity to the masses of particles inside the virtual loops!

The Top Quark

- Virtual loop corrections and precise LEP data predicted the top quark mass would be:

$$\boxed{m_t^{\text{loop}} = 173 \pm 11 \text{ GeV}}.$$

- This was subsequently confirmed by the discovery in 1994 of the top quark at Fermilab's Tevatron (proton anti-proton collider) at the predicted mass!



- The top quark almost exclusively decays to a bottom quark since

$$|V_{tb}|^2 \gg |V_{td}|^2 + |V_{ts}|^2 .$$

leading to complicated final states like:

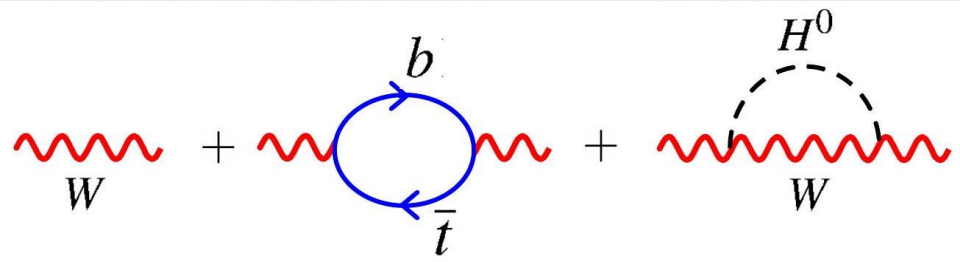
$$t\bar{t} \to b\bar{b}q\bar{q}q\bar{q} \to \text{6jets}$$
$$t\bar{t} \to b\bar{b}q\bar{q}\ell\nu \to \text{4jets} + \ell + \nu$$
$$t\bar{t} \to b\bar{b}\ell\nu\ell\nu \to \text{2jets} + 2\ell + 2\nu.$$

- The top-quark mass is usually determined by direct reconstruction (methods similar to those used for W-boson mass, page 450). Including later LHC results, the 2024 world estimate is:

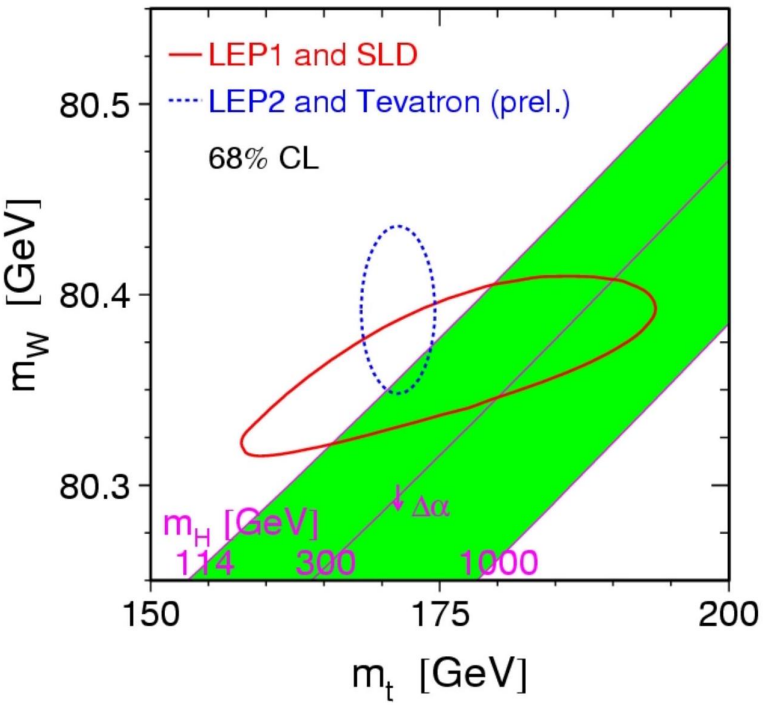
$$\boxed{m_t^{\text{meas}} = 172.76 \pm 0.3 \text{ GeV}}.$$

Logarithmic corrections to W-mass placed LEP limits on Higgs Mass

The W mass depends on the Higgs mass (albeit only logarithmically):

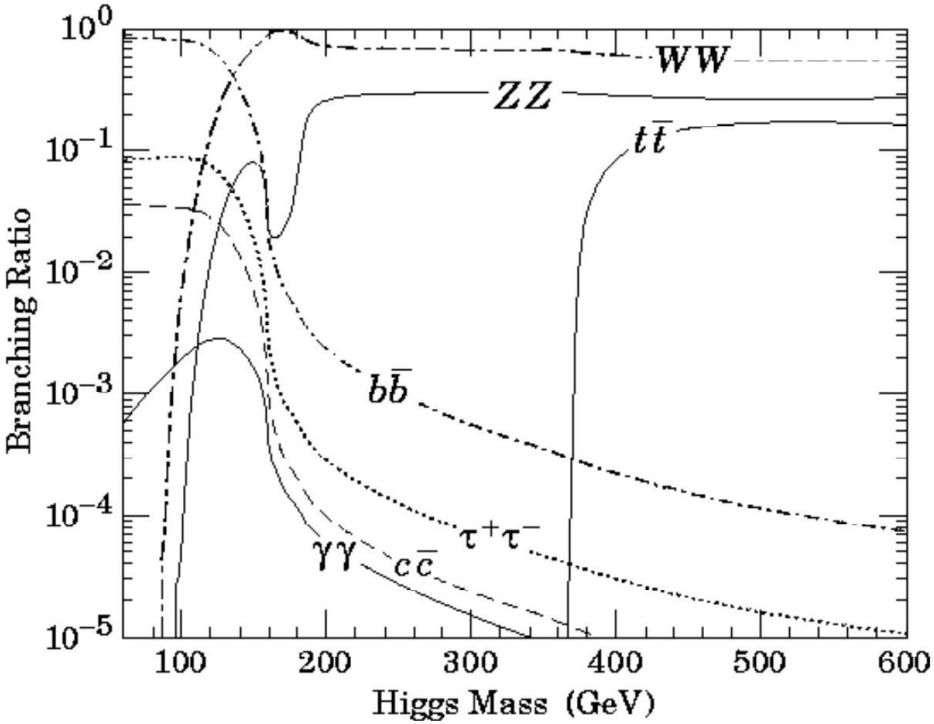


$$m_W = m_W^0 + a \cdot m_t^2 + b \cdot \ln\left(\frac{m_H}{m_W}\right)$$

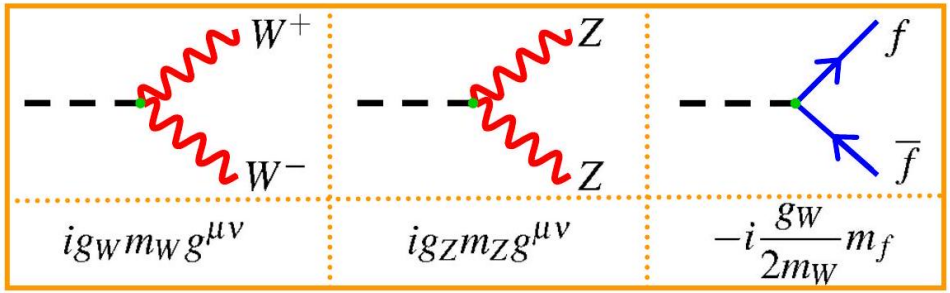


- LEP measurements were sufficiently precise to have some sensitivity to the Higgs mass.
- Direct and indirect values of the top-quark and W mass could be compared to prediction for different Higgs masses.
  - Direct: W and top masses from direct reconstruction
  - Indirect: from SM interpretation of Z mass, θ_W etc.
- Data favoured a light Higgs: $\boxed{m_H < 200\,GeV}$.

Possible Higgs Boson Decays at LEP



- The LEP searches needed to pay attention to how the Higgs decays.



- Since Higgs boson couplings are proportional to mass, most Higgs decays are to the heaviest kinematically allowed particle:

Summary of typical Higgs decay modes at LEP

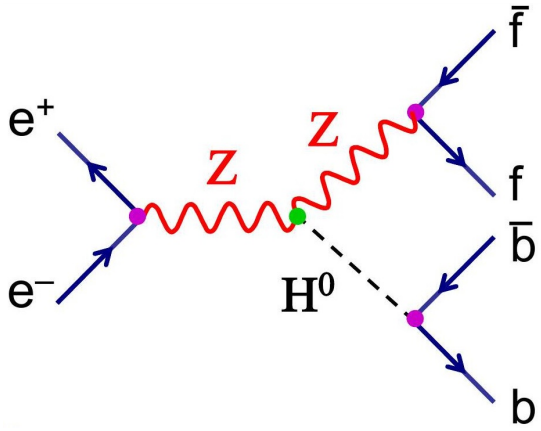
$m_H < 2m_W$: mainly $H^0 \to b\bar{b}$ but also around 10% of $H^0 \to \tau^+\tau^-$.

$2m_W < m_H < 2m_t$: almost entirely $H^0 \to W^+W^-$ and $H^0 \to ZZ$.

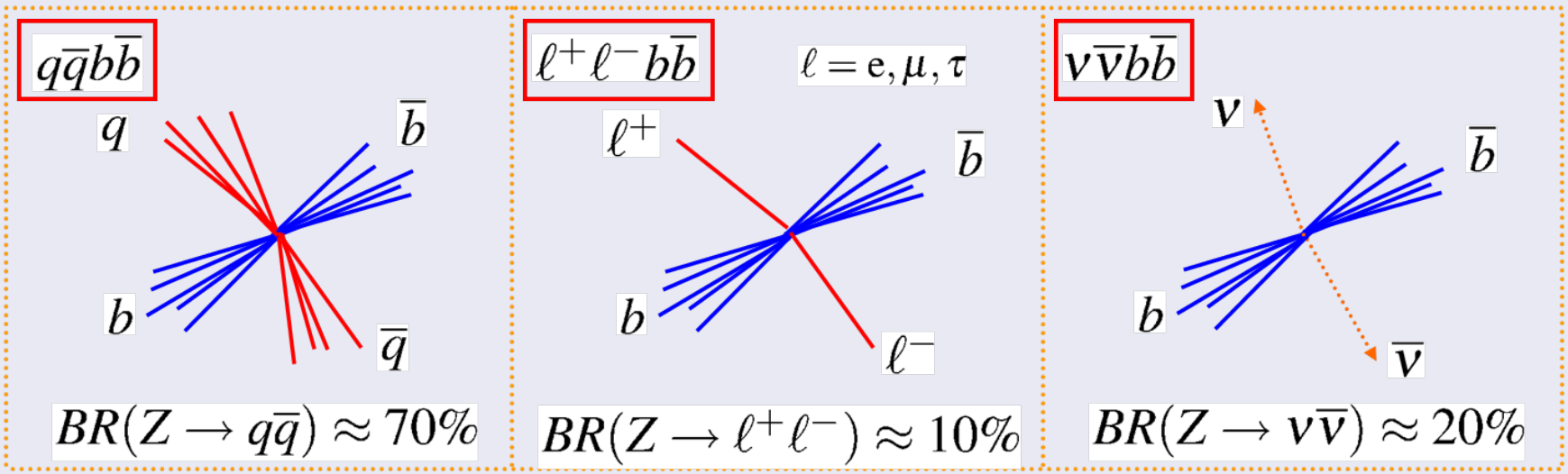
$m_H > 2m_t$: either $H^0 \to W^+W^-$, $H^0 \to ZZ$, or $H^0 \to t\bar{t}$.

Could LEP observe Higgs Bosons ?

- LEP operated with a C.o.M. energies up to 207 GeV.
- At these energies the main production mechanism would be the 'Higgsstrahlung' process (shown right).
- The Z-boson in the final state means that process is viable only for Higgs bosons with masses below ~ 207 GeV $- m_Z \sim 116$ GeV.

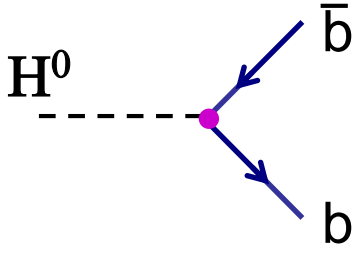


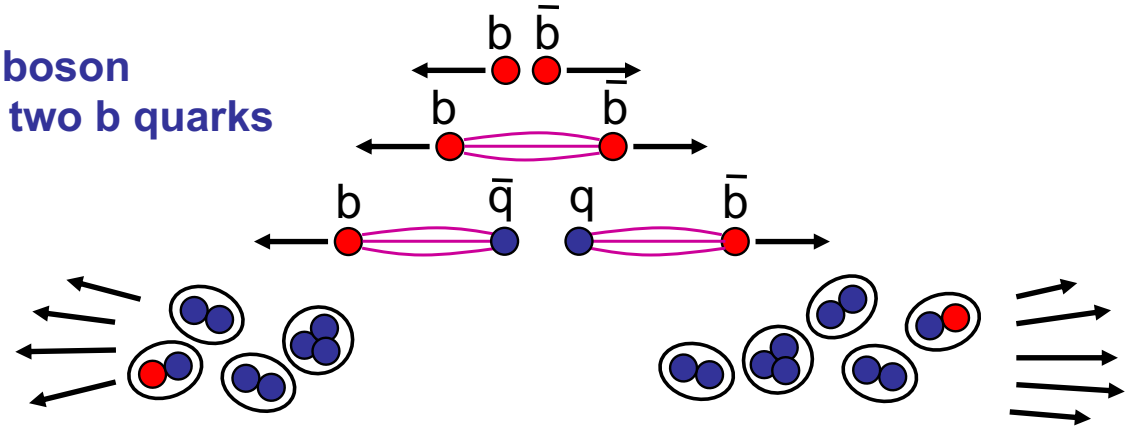
The Higgs predominantly decays to the heaviest particle possible. For Higgs boson masses below 116 GeV the main Higgs decay is therefore to a pair of b-quarks, so LEP Higgs signatures would be:



Tagging *b*-jets in the Higgs Boson Decays

★ One signature for a Higgs boson decay is the production of two b quarks



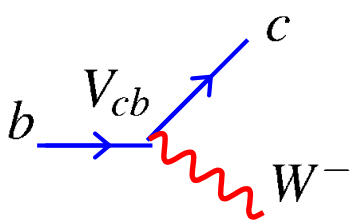


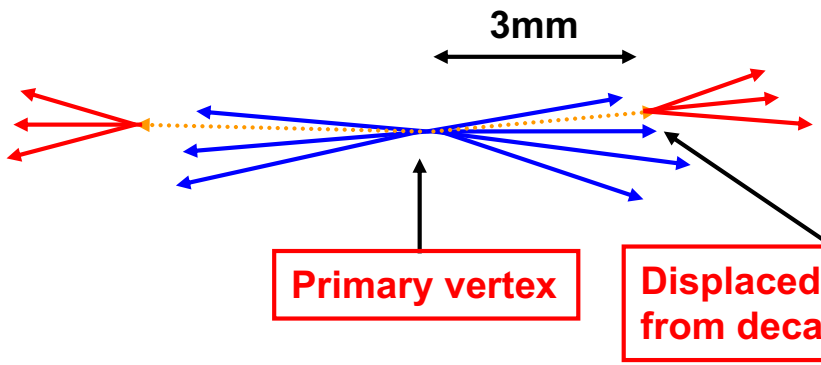
★ Each jet will contain one b-hadron which will decay weakly

★ Because V_{cb} is small $(V_{cb} \approx 0.04)$ hadrons containing b-quarks are relatively long-lived

★ Typical lifetimes of $\tau \sim 1 \times 10^{-12}\,\mathrm{s}$

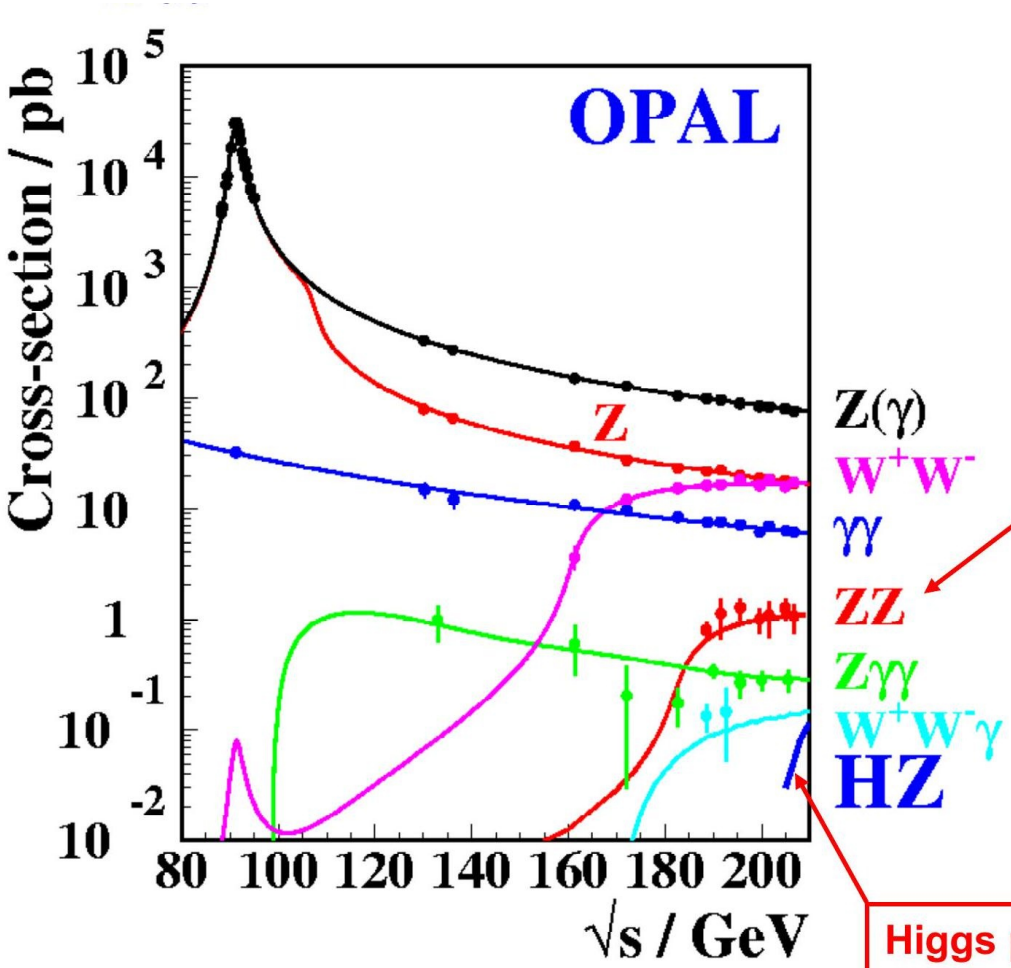
★ At LEP b-hadrons travel approximately **3mm** before decaying



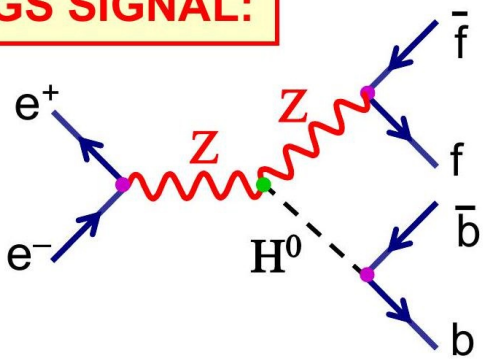


★ Can efficiently identify jets containing b quarks

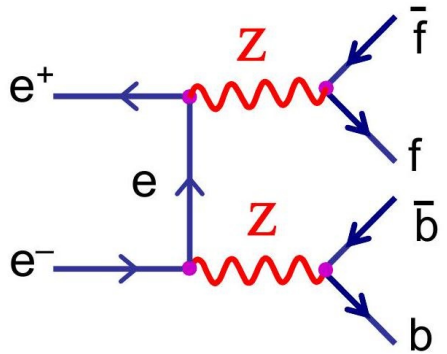
- Clear experimental signature, but small cross section, e.g. for $m_H \approx 115$ GeV would only produce a few tens of $e^+e^- \to H^0$ events at LEP.
- In addition, there are large backgrounds.



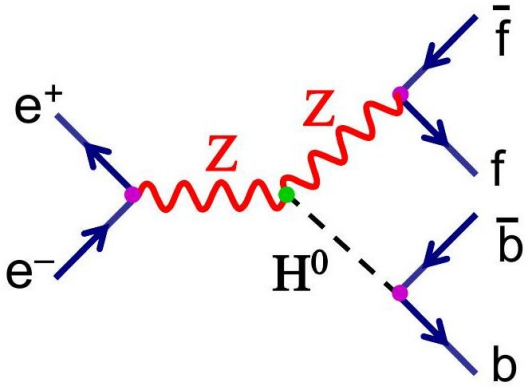
HIGGS SIGNAL:

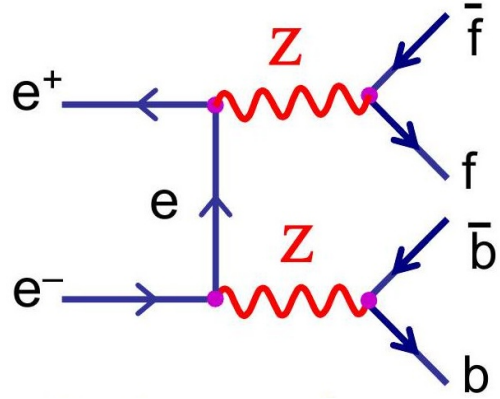


MAIN BACKGROUND:



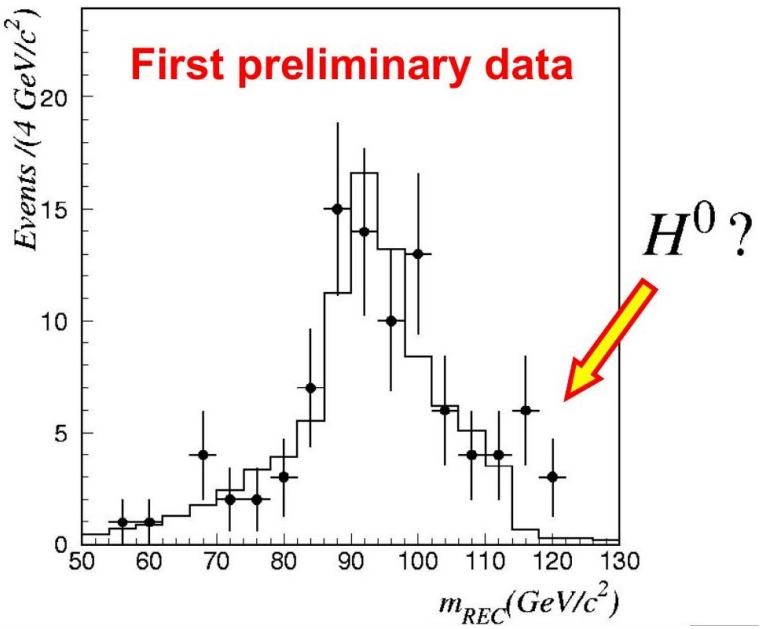
Higgs production cross section (m_H=115 GeV)

- The only way to distinguish 

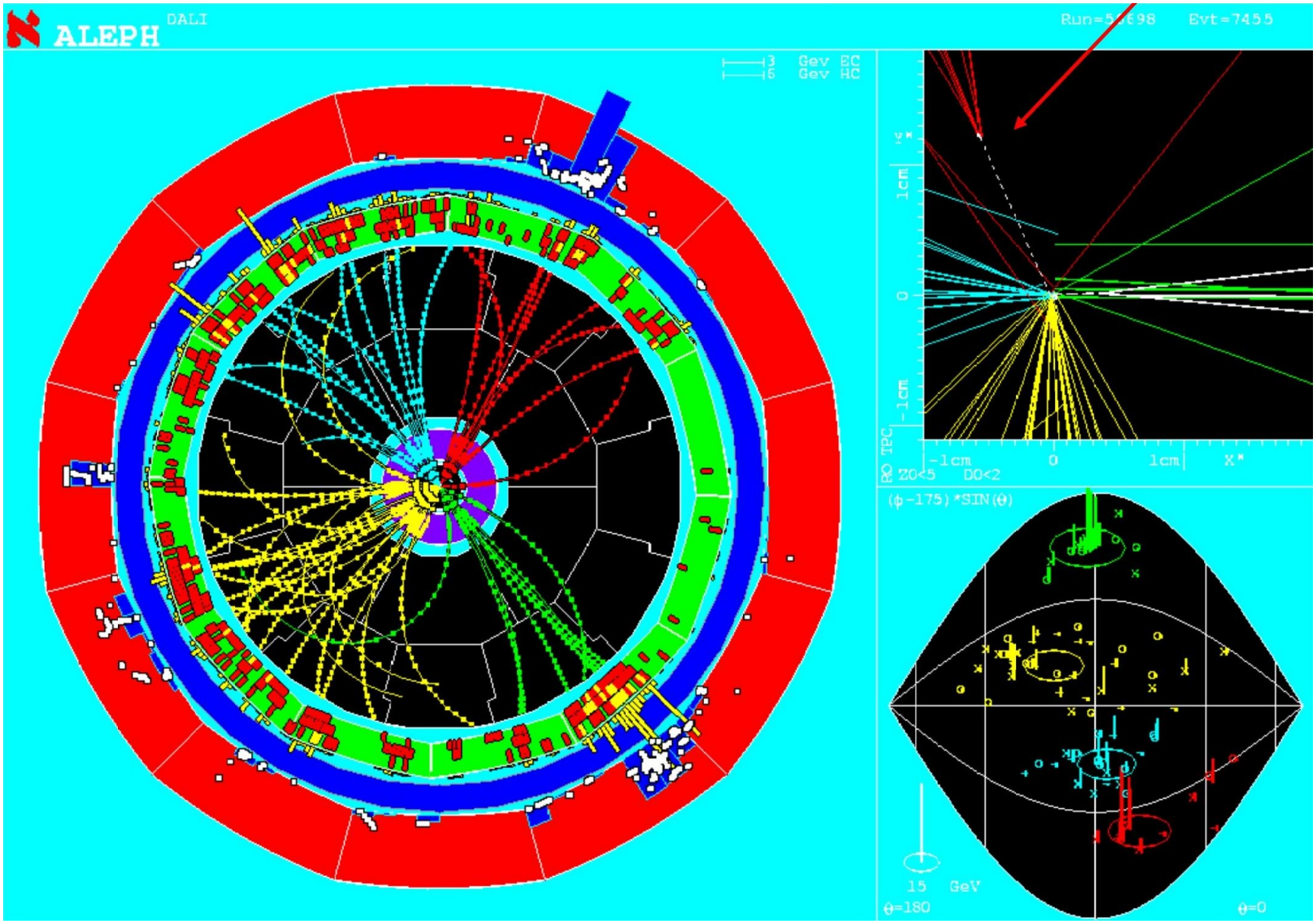
from 

is via the invariant mass of the pair of jets from the boson decays.

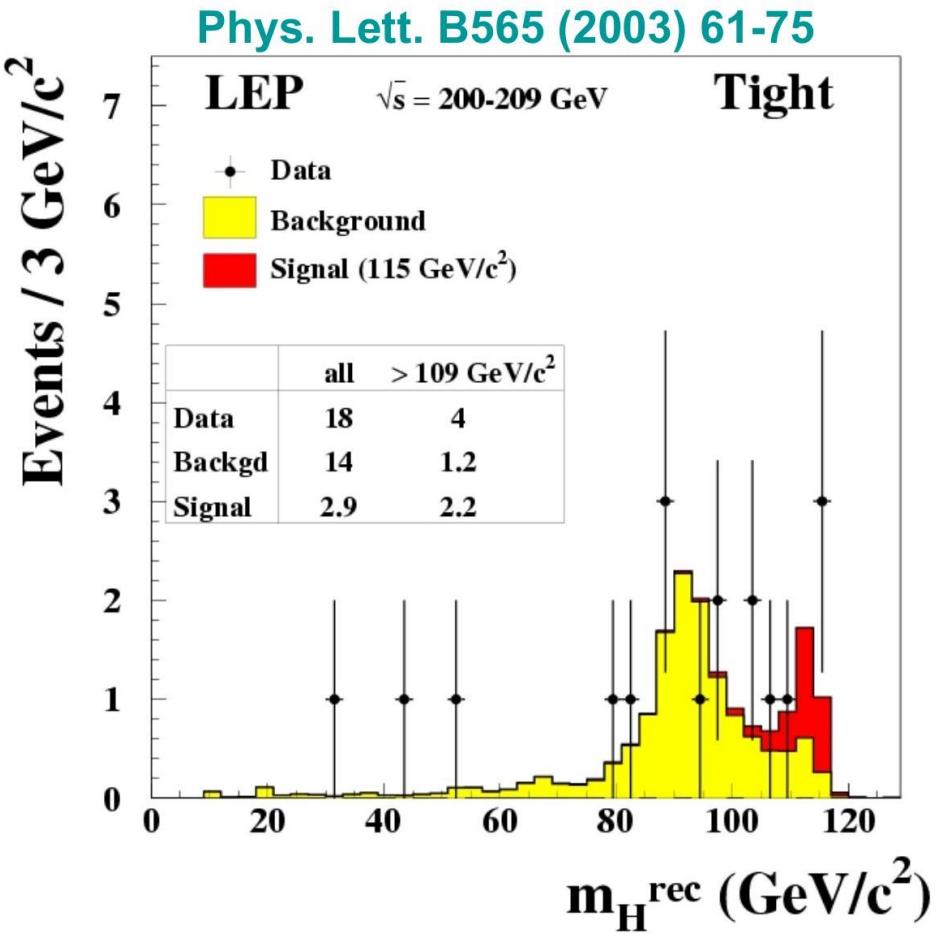
- In 2000 (the last year of LEP running) the ALEPH experiment reported an excess of events consistent with being a Higgs boson with mass 115 GeV.
- ALEPH found 3 events which were high relative probability of being signal.
- L3 found 1 event with high relative probability of being signal.
- OPAL and DELPHI found none.



Example event: Displaced vertex from b-decay



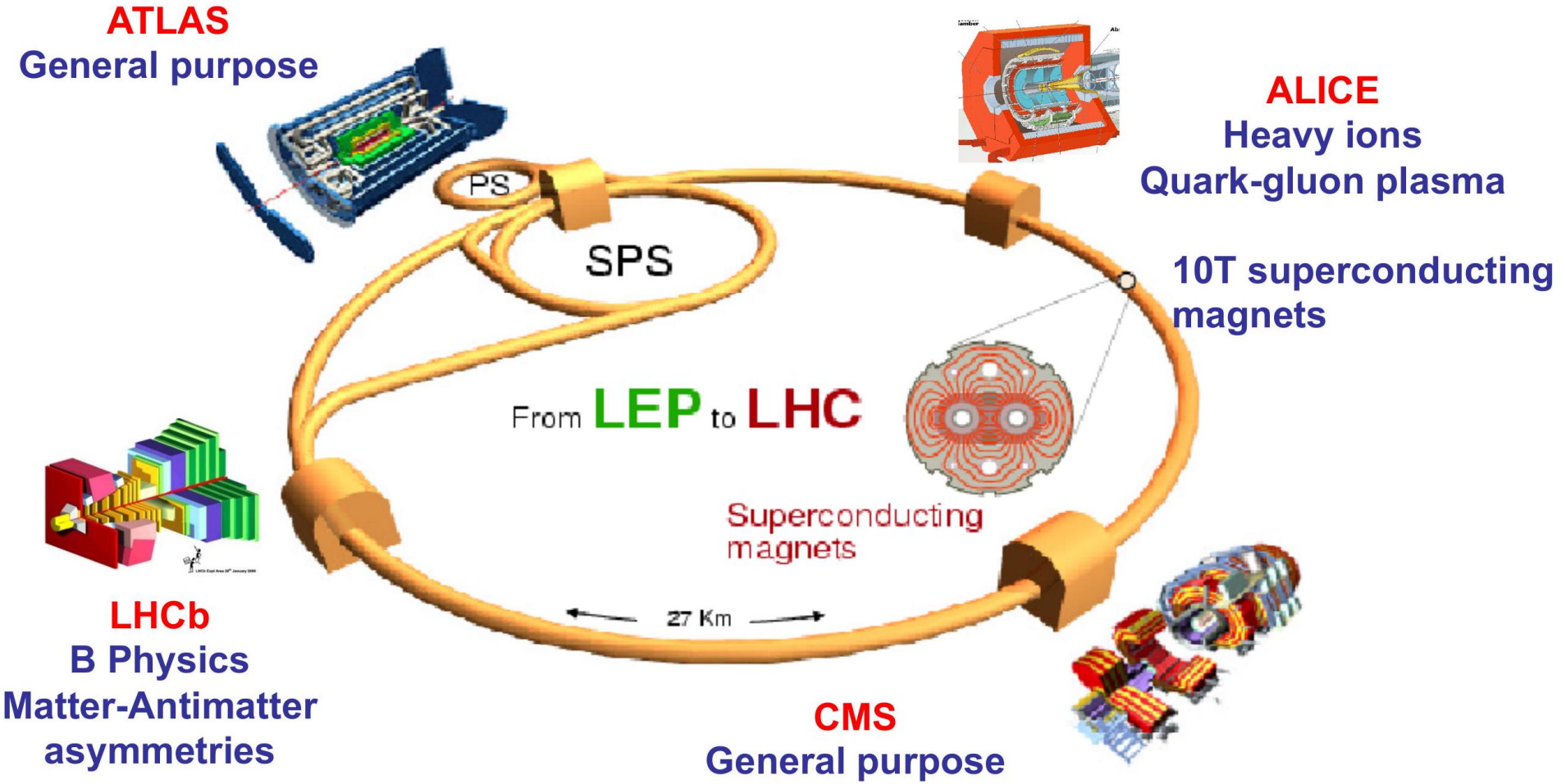
Combined LEP Results



- Final combined LEP results showed $m_H > 114$ GeV.

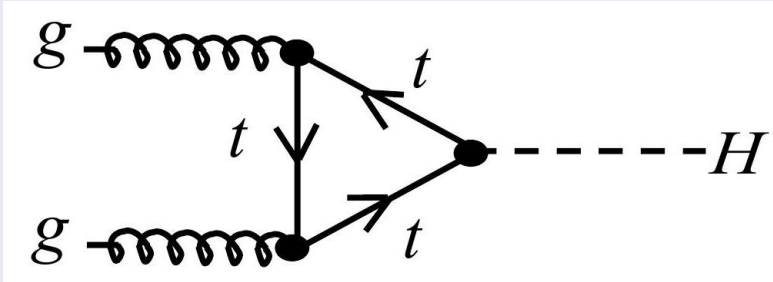
The Large Hadron Collider

The LHC is a proton-proton collider now running in the old LEP tunnel at CERN.

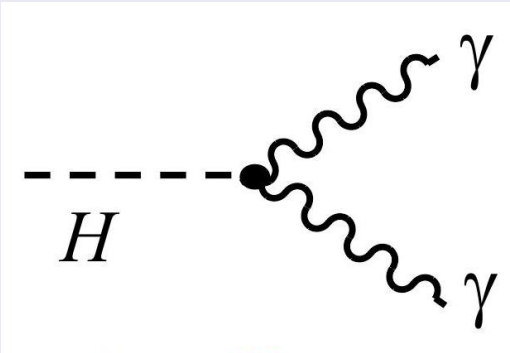


Higgs Production and Decay at the Large Hadron Collider

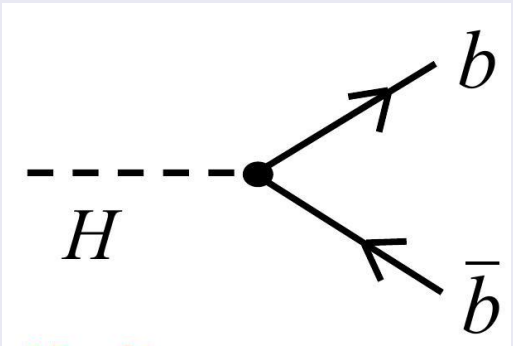
The dominant Higgs production mechanism at the LHC is 'gluon fusion':



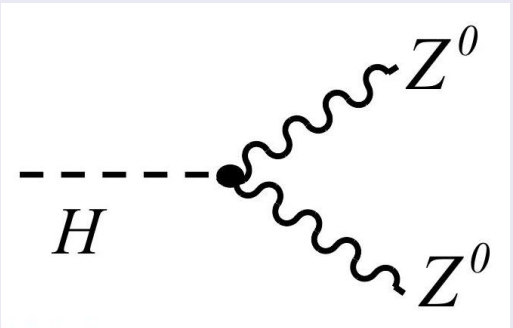
Depending on the mass of the Higgs boson, m_H, it could decay in different ways:



Low Mass



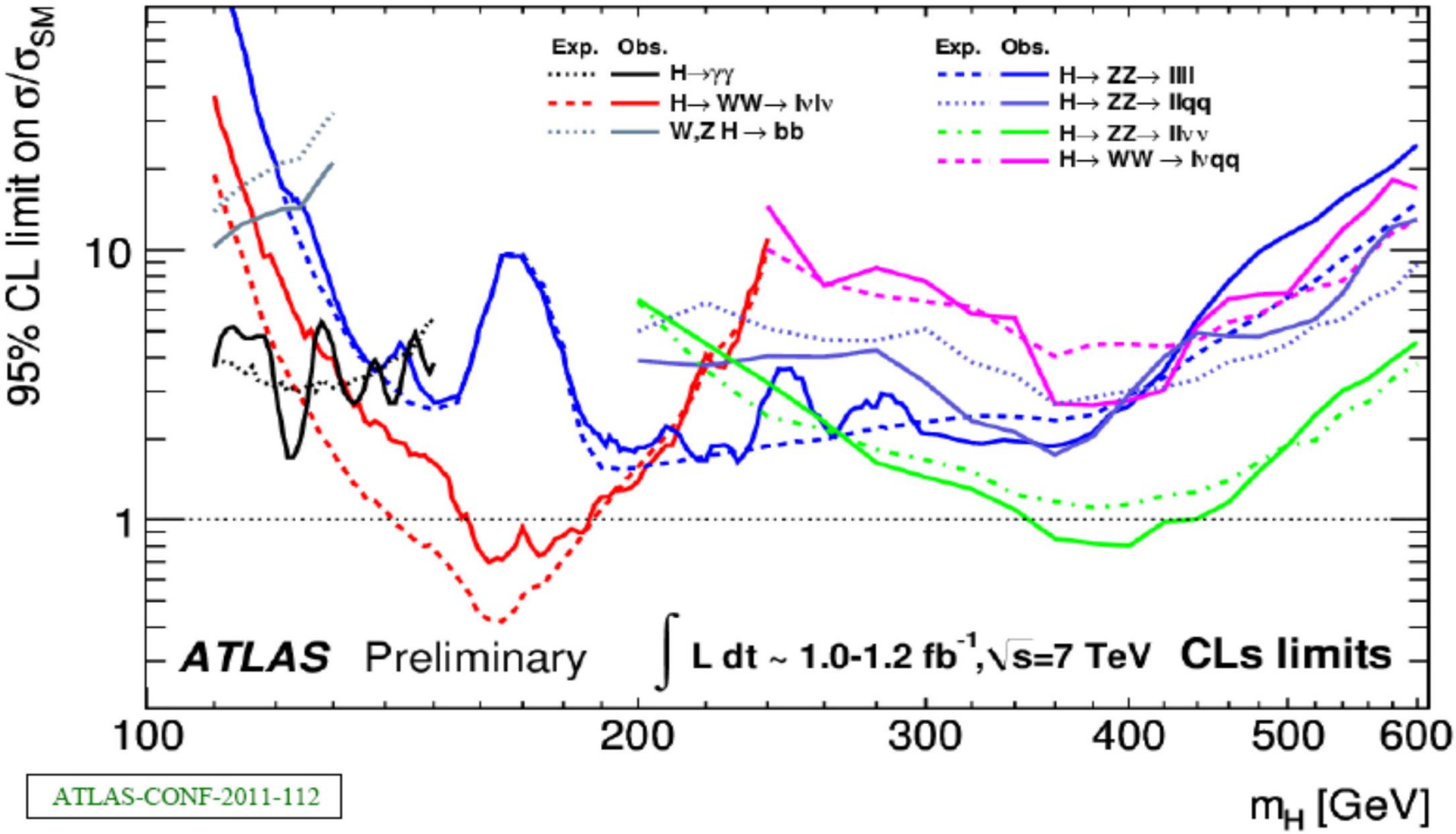
Medium mass



High mass

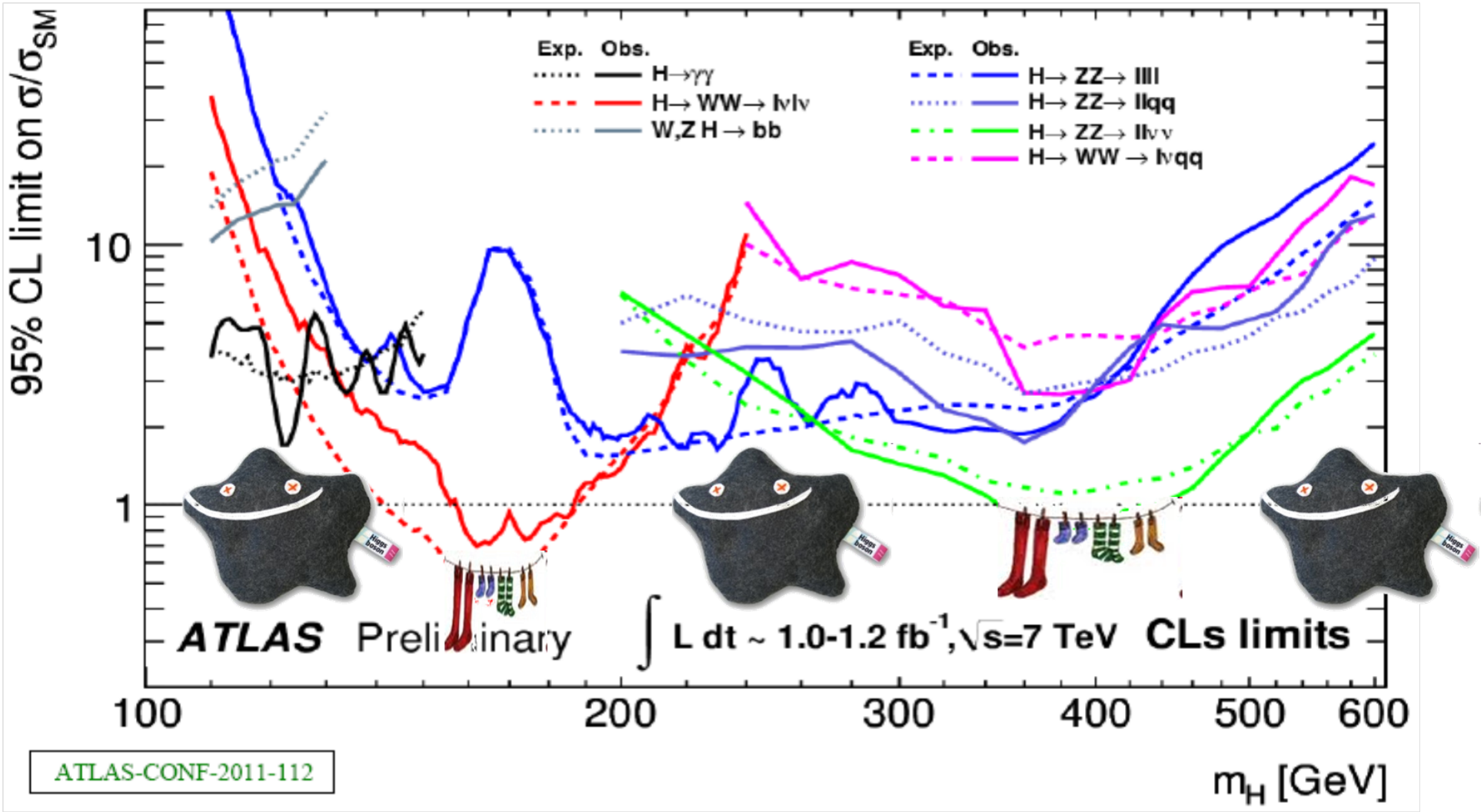
Higgs Boson mass (and number of Higgs Bosons!) was not known before LHC start.

So LHC experiments needed to design **separate search strategies** for each m_H.



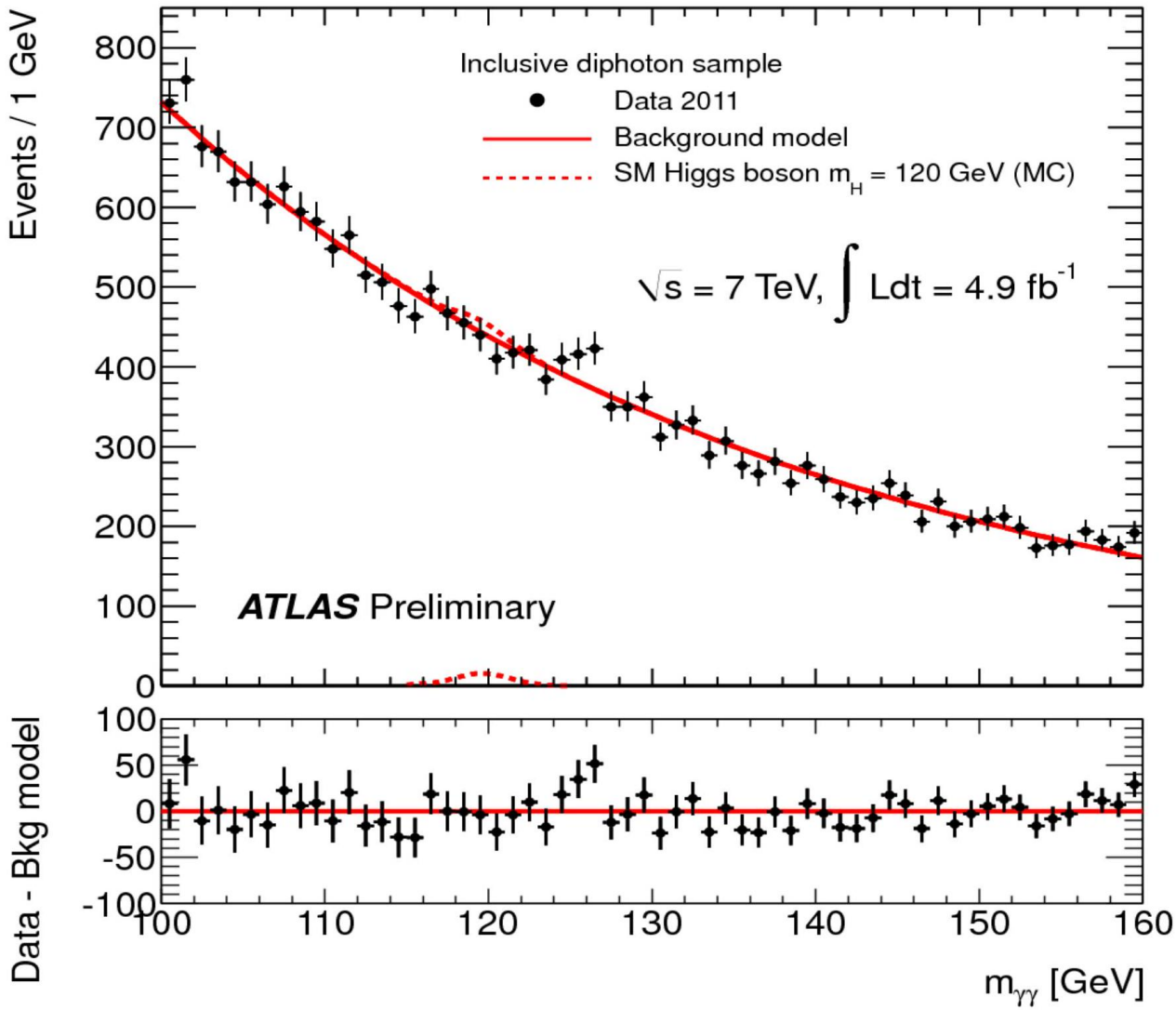
At each possible Higgs mass, m_H, each individual LHC search places a 95% confidence upper bound on the cross section for Higgs-Boson-Like events divided by the expected Higgs cross section at that mass. I.e. a curve passing through '10' on the y-axis at $m_H = 125\ \mathrm{GeV}$ means that a Higgs boson at this mass could have been produced at up to 10 times the rate expected in the Standard Model and yet could still (just) have gone un-noticed, at 95% confidence, by that search.

As data arrives it should lower the curves, unless support from a Higgs boson can prevent curve from passing through dotted line at " 1 "



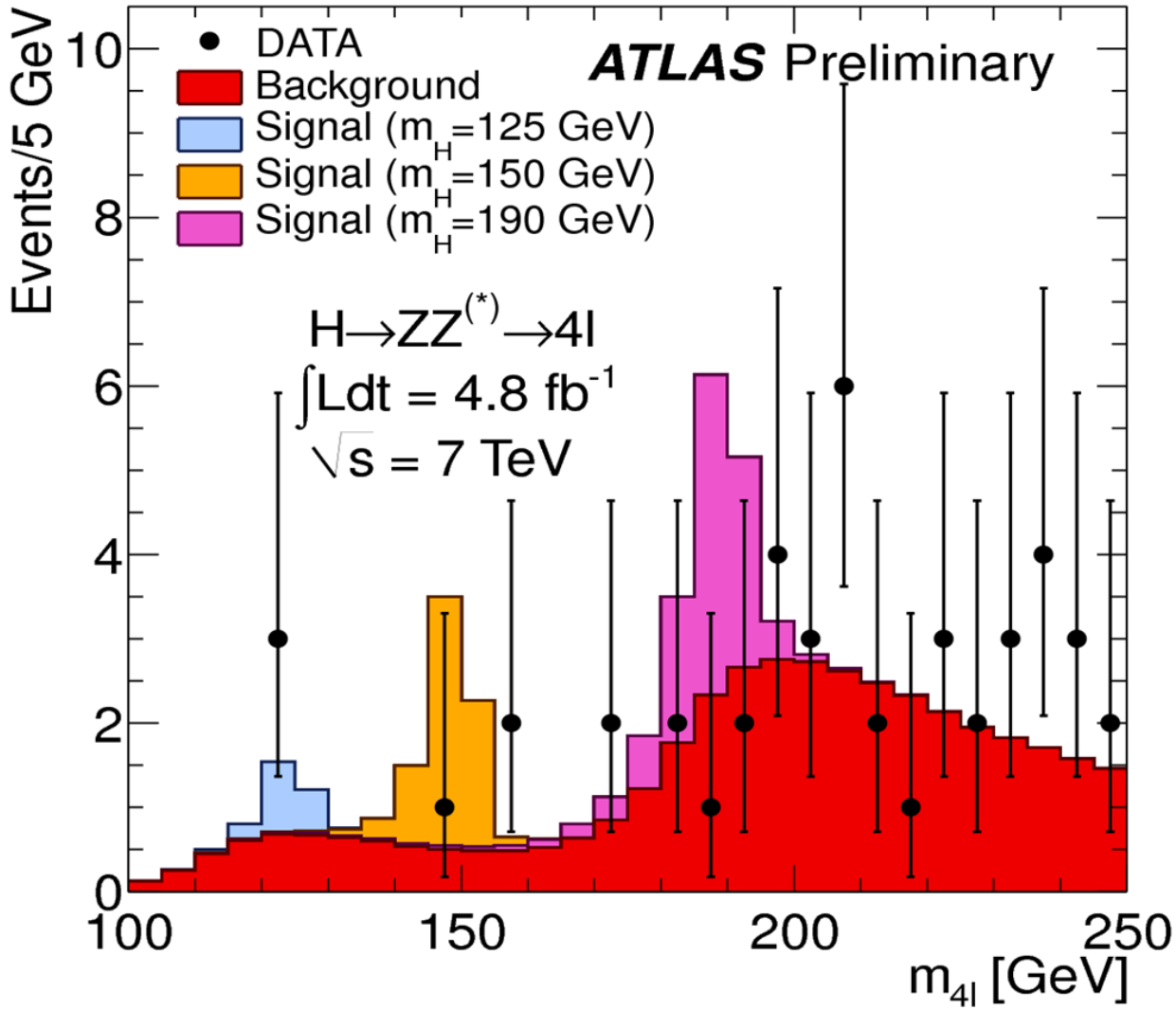
Here is the (unconvincing) data that was made public in February 2012

- The black blobs are data. The smooth curve is the expected background shape.
- The small dotted "bump" indicate how a Higgs signal might change the shape of the distribution if the Higgs boson mass was 120 GeV.
- The variable on the x axis is the invariant mass two photons.

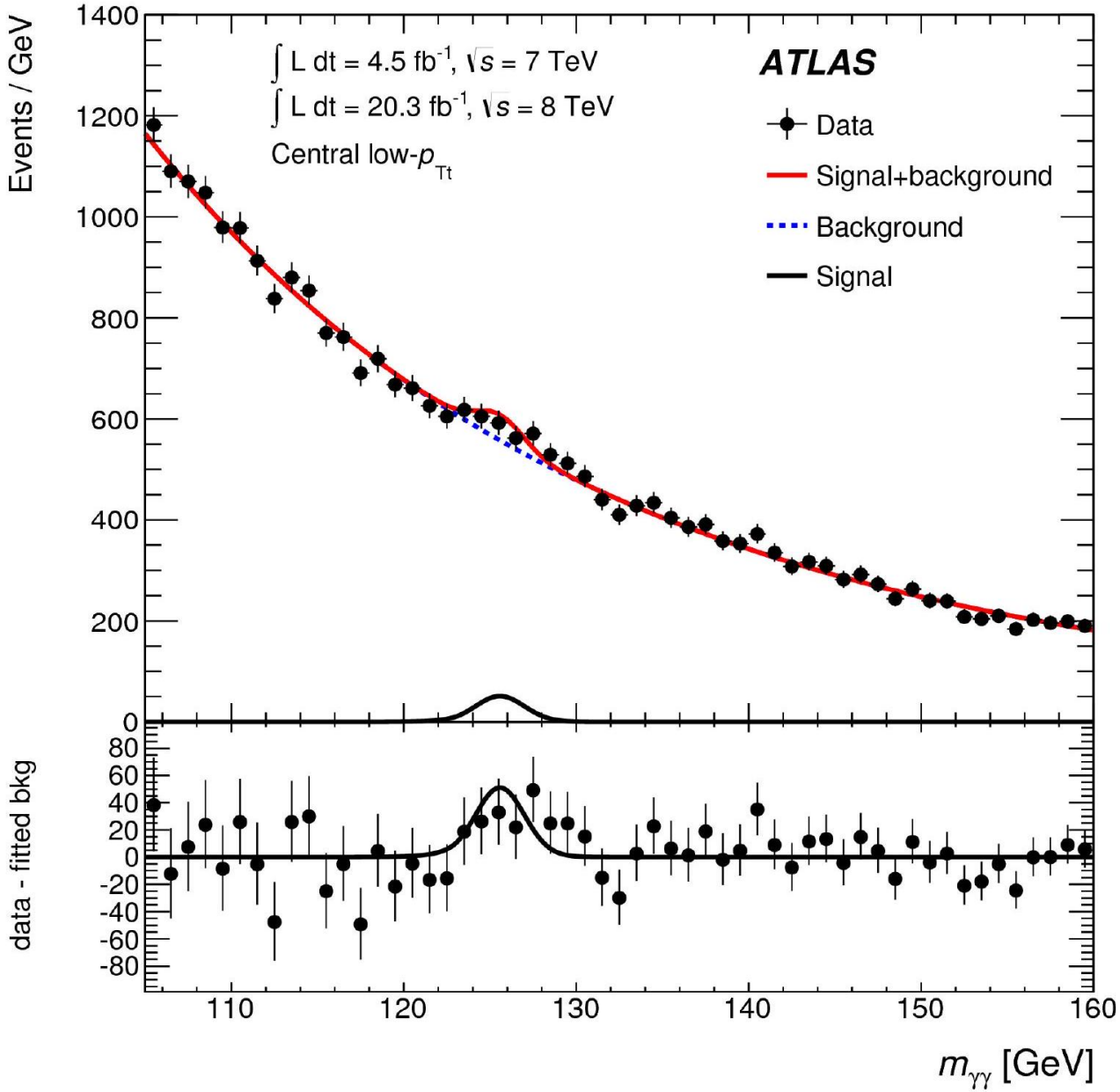


The astonishingly (un?)convincing evidence in the analysis looking for Higgs decays to pairs of Z bosons

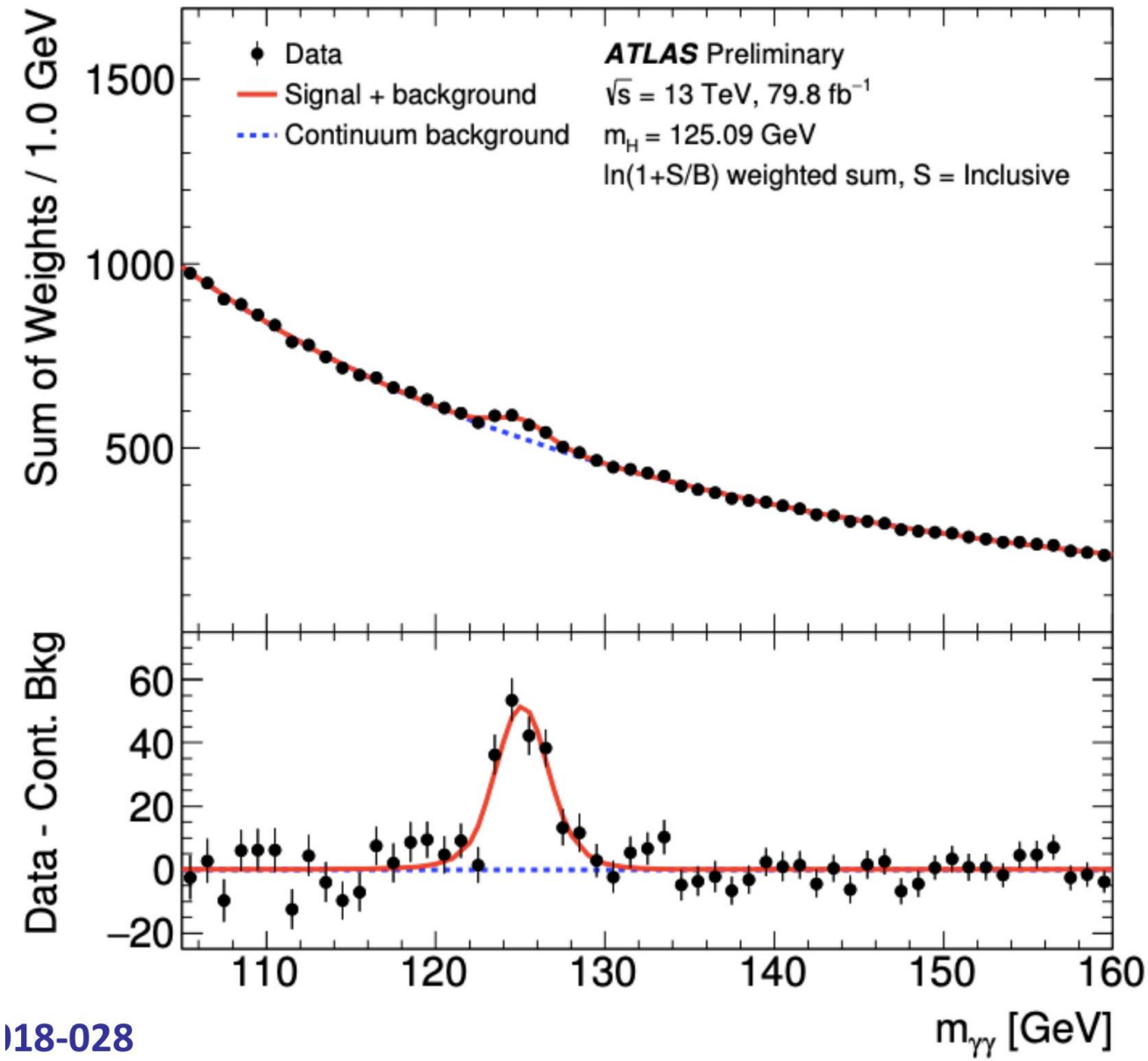
- The black blobs are data.
- The three triangular lumps indicate what a Higgs signal might look at (for three different Higgs boson masses).
- The variable on the x axis is the invariant mass of four leptons which seem to have come from two Z bosons.



The 2015 public ATLAS data for Higgs turning into two photons

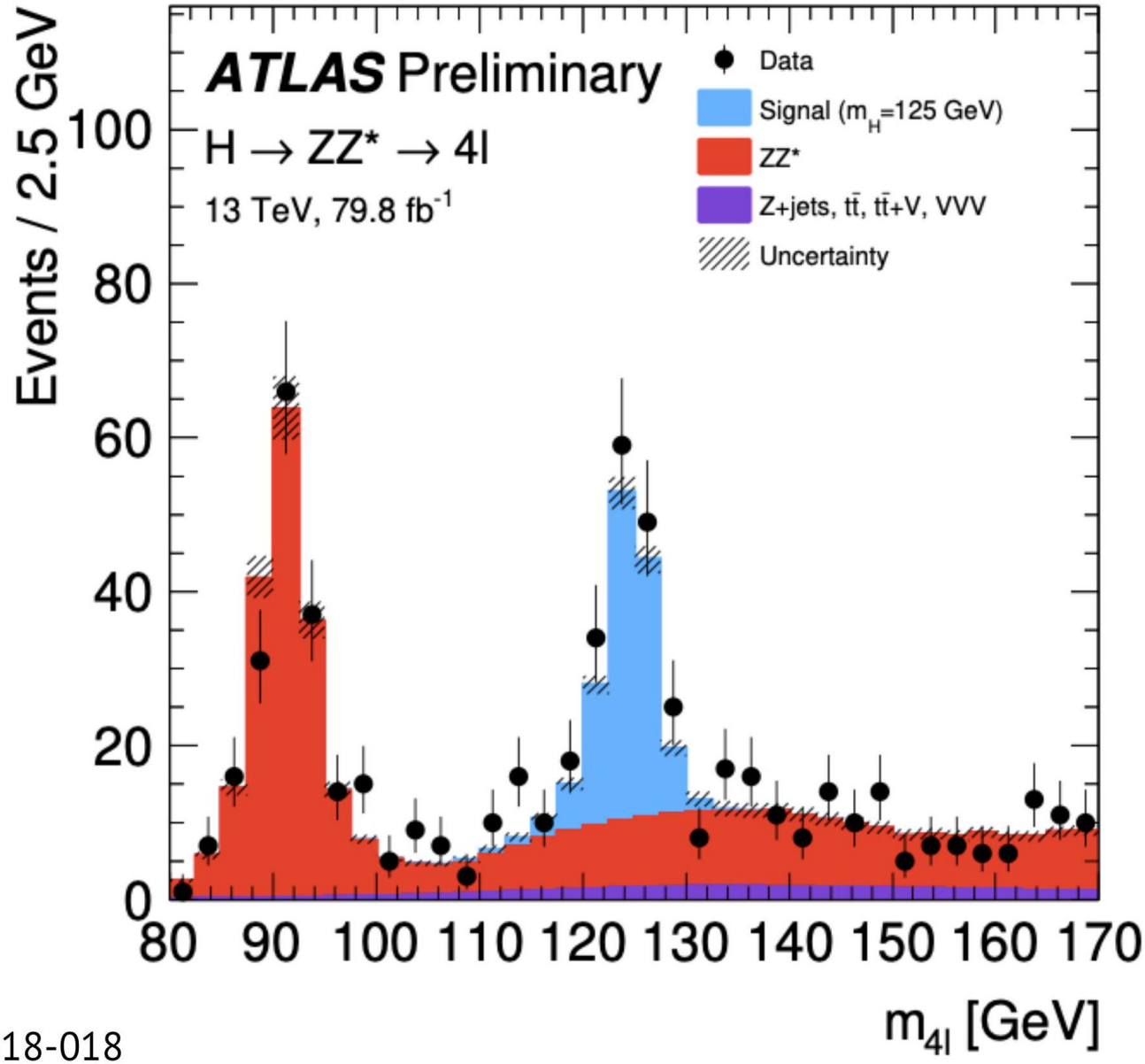


The 2018 public ATLAS data for Higgs turning into two photons



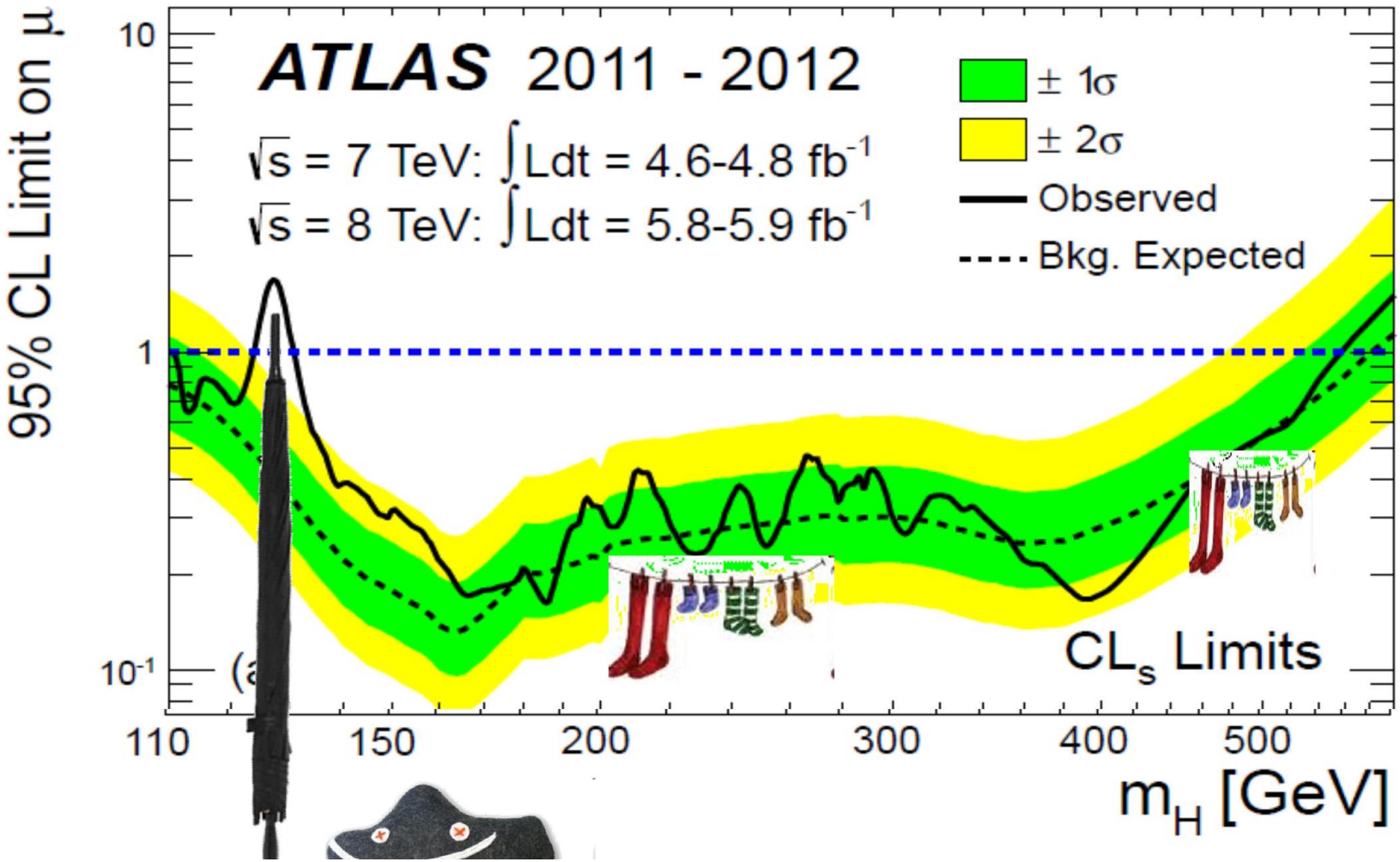
)18-028

... or turning into four leptons



)18-018

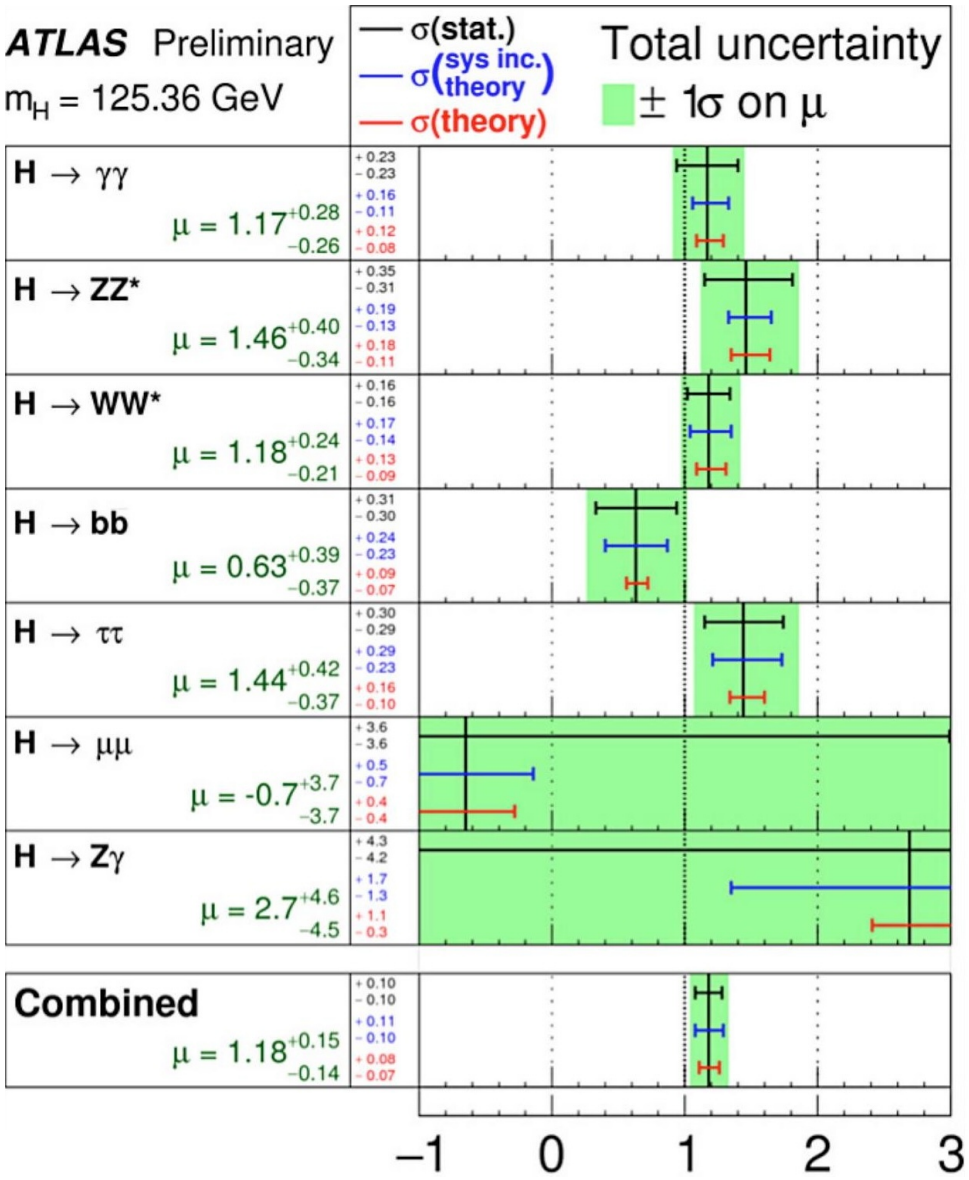
The discovery plot. Supposedly 2×10^{-9} probability of fluctuation!



Spring 2012 data ... this is the data that took us past the 5-sigma 'discovery' threshold.

But is it 'really' the (a?) Higgs boson?

- It is considered to be 'discovered'. Nobel Prize 2013.
- What has been discovered is a bump in the sort of place you'd expect to find a Higgs Boson. In other words, a particle consistent with the Higgs Boson.
- To be really sure its **The** Higgs Boson, we are acquiring more information on its spin and couplings (e.g. data shown to the right). So far everything checks out. The Higgs looks 'as expected'. Nonetheless, other (non-standard) Higgs Bosons could yet be found.



Concluding Remarks (1 of 2) — concerns/questions

The Standard Model with massless neutrinos is controlled by 19 free parameters, e.g.:

- three lepton masses, $(m_e,\ m_\mu,\ m_\tau)$
- six quark masses, $(m_d,\ m_u,\ m_s,\ m_c,\ m_b,\ m_t)$
- four CKM mixing matrix parameters (θ_{12}, θ_{13}, θ_{23} and δ_{CP}),
- three gauge couplings $(g',\ g_w,\ g_s)$,
- two boson-sector parameters (e.g. Higgs mass plus one of Higgs vev, m_W or m_Z), and
- the QCD theta parameter (not discussed in this course).

Massive neturinos would add at least another seven (three masses and four mixing angles).

Some open questions:

- Why three generations ?
- Why $SU(3)_c \times SU(2)_L \times U(1)_Y$?
- No quantum gravity?
- Origin of CP violation in early universe?
- What is Dark Matter made of?
- Why are neutrinos so light?

Concluding Remarks (2 of 2) — the future

The main aims of this course have been:

- to show the main experimental methods and results which have lead to **The Standard Model** being considered (at present) to be our best description of fundamental physics,
- to emphasise that **the evidence is sometimes messy or indirect**, and
- to make the point that there is considerable scope for people of the future (you) to make discoveries!

All you have to do is find something better than The Standard Model which is able to retrodict results of all past experiments.

- Be sceptical of The Standard Model, but mindful of the evidence!
- Look for the gaps in that evidence!
- Don't waste your time on untestable meta theories.

The End

The End

References I

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The CPLEAR-experiment at CERN.
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Proceedings of the Fourth KEK Topical Confernce on Flavor Physics.

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Precision measurement of the B_s^0-$\bar{B}_s^0$ oscillation frequency with the decay $B_s^0 \to D_s^- \pi^+$.
New J. Phys., 15:053021, 2013.

List of appendices (non-examinable)

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II: Detailed derivation of the cross section formula (page 509)
III: Logitudinal invariance of 'Lorentz Invariant Flux' (page 515)
IV: General $2 \to 2$ Body Scattering in lab frame (page 516)

Handout 2
V: Dimensions of the Dirac Matrices (page 519)
VI: Magnetic Moment (page 541)
VII: Generators of Lorentz Transformations (page 544)

Handout 4
VIII: Spin-1 Rotation Matrices (page 560)

Handout 5
IX: Crossing Symmetry (page 564)

Handout 7
X: the $SU(2)$ anti-quark representation (page 565)

Handout 8
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XII: Gauge Invariance (page 569)
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XVII: Alternative evaluation of colour factors (page 580)
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Handout 12
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XXIII: CP Violation via Mixing (page 610)
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Handout 14
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Appendix I: Decay rate derivation from Fermi's Golden Rule

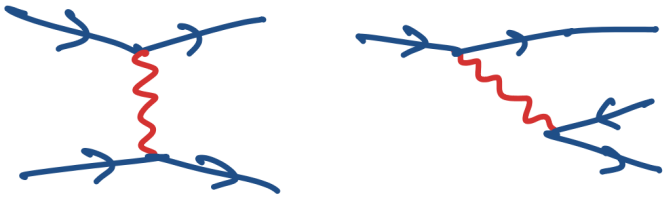
Not examinable

This appendix contains the full, hand-written derivation of the decay rate, eq. (2), from Fermi's Golden Rule.

Not examinable

Not examinable

- We want to precisely calculate decay rates and scattering probabilities



- To do so, we use **FERMI'S GOLDEN RULE**

$$\Gamma_{fi} = 2\pi \left| T_{fi} \right|^2 \rho(E_f) \qquad \text{(A)}$$

which calculates:

Γ_{fi} = Rate (number of transitions per unit time) at which initial state $|i\rangle$ transitions to final state $|f\rangle$

under various assumptions (see next page).

Not examinable

Not examinable

The assumptions of FERMI'S GOLDEN RULE.

In the form in which it was given ($\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$) :

- All basis States are normalised such that

$$\langle i | j \rangle = \int_V \psi_i^* \psi_j \, dV = \delta_{ij}$$



(where V is a box of volume V inside which the states are confined)

- States $|i\rangle$ are eigenstates of $\hat{H}_{free}$
- T_{fi} is the "Transition Matrix Element"

(A2) $$T_{fi} = \langle f | \hat{H}_{int} | i \rangle + \sum_{j \neq i} \frac{\langle f | \hat{H}_{int} | j \rangle \langle j | \hat{H}_{int} | i \rangle}{E_i - E_j} + \dots$$

- $\hat{H}_{int}$ is the "interaction" part of the full Hamiltonian:

$$\hat{H}_{full} = \hat{H}_{free} + \hat{H}_{int}, \quad \text{and}$$

- $\rho(E_f)$ is the density of final states, defined by

$$\rho(E_f) = \left| \frac{dN}{dE} \right|_{E=E_f}$$

where dN is the number of final states with energy between E_f and $E_f + dE$

(see revision in next slide)

(E_f = Final state energy = Initial State Energy)

Not examinable

Not examinable

ASIDE

A proof of Fermi's Golden Rule in the above form is given in sec 2.3.6 of Mark Thomson's book. It is assumed that you have seen similar proofs in earlier courses, so the origin of F.G.R. is not discussed here. For an alternative more in-depth proof see section 46 of P.A.M. Dirac's book "The Principles of Quantum Mechanics".

Not examinable

Not examinable

Learning/take home points for Fermi's Golden Rule:

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f) \quad \text{(A)}$$

Reaction rates are controlled by and

- Matrix Element ← Quantum Physics/ Particle Content/ SM/SUSY/Leptoquarks
- Phase Space ← Relativity, Mom Cons, Energy Cons.

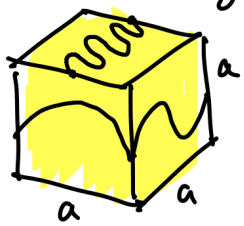
Not examinable

Revision of Phase Space (non-relativistic, single particle)

Not examinable

When a state is confined to a box, allowed momenta are only able to take values on a lattice.

Suppose V is a cube of side "a": $(a^3 = V)$



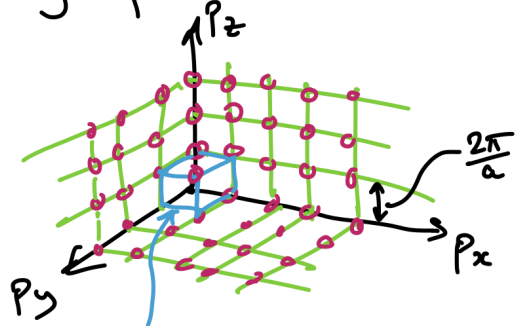
then allowed states $\psi \propto e^{\pm i(\underline{p}\cdot\underline{x} - \omega t)}$ must have $\underline{p} = (p_x, p_y, p_z) = \frac{2\pi}{a}(n_x, n_y, n_z)$

n_x, n_y, n_z: integers

so that periodic B.C.s are satisfied

$$\psi(x+a, y, z) = \psi(x, y, z)$$
$$\psi(x, y+a, z) = \psi(x, y, z)$$
$$\psi(x, y, z+a) = \psi(x, y, z),$$

Hence single particle states live on this lattice



(Volume of SINGLE STATE in mom space) =

$$= \left(\frac{2\pi}{a}\right)\left(\frac{2\pi}{a}\right)\left(\frac{2\pi}{a}\right) = \frac{(2\pi)^3}{V}$$

$\therefore$ number of states dn in element $d^3\underline{p}$ is

$$dn = \frac{d^3\underline{p}}{\left(\frac{(2\pi)^3}{V}\right)} = V\frac{d^3\underline{p}}{(2\pi)^3} \quad (B)$$

Not examinable

Not examinable

ASIDE ON YUCKY PHASE-SPACE (NOT MANIFESTLY LORENTZ COVARIANT)

We want $\rho(E_g) \equiv \left|\frac{dn}{dE}\right|_{E=E_g}$.

We COULD get $\rho(E_g)$ by using

(1) $dn = V\frac{d^3p}{(2\pi)^3} = V\frac{4\pi p^2 dp}{(2\pi)^3} \Rightarrow \boxed{\frac{dn}{dp} = V\frac{4\pi p^2}{(2\pi)^3}}$

(2) $E^2 = m^2 + p^2 \Rightarrow 2E\,dE = 2p\,dp \Rightarrow \boxed{\frac{dp}{dE} = \frac{E}{p} = \frac{1}{\beta}}$

to deduce that

$$\rho(E_g) \equiv \left|\frac{dn}{dE}\right|_{E=E_g} = \left|\frac{dn}{dp}\frac{dp}{dE}\right|_{E=E_g}$$

$$\therefore \boxed{\rho(E_g) = V\frac{4\pi p^2}{(2\pi)^3}\frac{1}{\beta}}$$

but this is SUPER UGLY!

See nice trick on next slide that avoids β usage!

Not examinable

Not examinable

26 Writing Golden Rule in nicer form.

Ideally we'd like to write the Golden Rule in a form that avoids it being polluted by β's and other things that are frame dependant (except where absolutely necessary). Want to aim for explicit "Lorentz Covariance".

Step 1: write density of states more nicely! ☺.

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE}\,\delta(E - E_i)\,dE \qquad \textcircled{c}$$

Why is this better?

- NOT because $E_f \to E_i$ ... that's just Energy Conservation.
- Better because integral scans "ALL ENERGIES" rather than "just one".
  - "ALL ENERGIES" $\longrightarrow$ "ALL ENERGIES" in a boost.
  - "ALL ENERGIES" is a "nice set" under Lorentz Trans.

Not examinable

Not examinable

What does F.G.R. look like now?

$$\Gamma_{i\to f} = 2\pi \, |T_{fi}|^2 \, \rho(E_f) \quad \text{(A)}$$

$$\Rightarrow \Gamma_{i\to f} = 2\pi \int |T_{fi}|^2 \frac{dn}{dE} \delta(E - E_i)\, dE \quad \text{using (C)}$$

$$\Rightarrow \Gamma_{i\to f} = 2\pi \int |T_{fi}|^2 \delta(E - E_i)\, dn \quad \text{(D)}$$

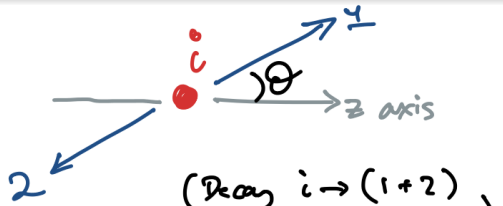
NICER

Integral is now over all allowed final states of any energy. ☺

Not examinable

Not examinable

Let's consider a special case:



(Decay $i \to (1+2)$ in REST FRAME (i))

Momentum conservation says that final state $(1+2)$ can be parameterised by the momentum of just one of the two final state particles (w.l.o.g particle 1) since other one has momentum equal and opposite*.

Thus it suffices to replace $dn \to V \frac{d^3 p_1}{(2\pi)^3}$ (uses (B)) and since also $E \equiv E_f \equiv E_1 + E_2$ we find that:

* in QFT course you will see this is not an ad-hoc additional principle. Comes from way interactions are constrained to be local.

$$\Gamma_{i\to f} = 2\pi \int |T_{fi}|^2 \delta(E - E_i)\, dn \qquad \text{(D)}$$

GENERAL

$$\Rightarrow \Gamma_{i\to f} = 2\pi \int |T_{fi}|^2 \delta(E_1 + E_2 - E_i)\, V \frac{d^3 p_1}{(2\pi)^3} \qquad \text{(uses (B))}$$

$i \to 1+2$

"Because we can!"

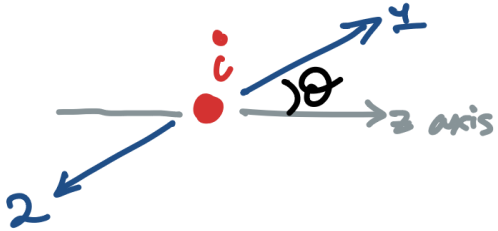
$$\Rightarrow \Gamma_{i\to f} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \delta(E_1 + E_2 - E_i)\, \delta^3(p_1 + p_2 - p_i) \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \qquad \text{(E)}$$

$i \to 1+2$

MATHS | PHYSICS ENERGY CONS | MOM CONS | DENSITY OF STATES

Not examinable

Not examinable



(Decay $i \to (1+2)$ in REST FRAME (i))

$i \to 1+2$

$$\Gamma_{i\to f} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \, \delta(E_1+E_2-E_i) \, \delta^3(\underline{p}_1+\underline{p}_2-\underline{p}_i) \frac{d^3\underline{p}_1}{(2\pi)^3} \frac{d^3\underline{p}_2}{(2\pi)^3} \quad (E)$$

MATHS | PHYSICS | ENERGY CONS | MOM CONS | DENSITY OF STATES



Nicer Still!

Integral is now over all final state momenta, not just some special ones! ☺

Becoming "more" Lorentz Covariant

Meaning is starting to "factorise".

Not examinable

Not examinable

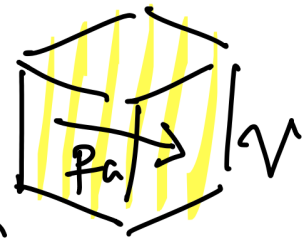
27 Let's try to move to a "better" wave func (or state) normalisation.

("better" = more manifestly Lorentz Covariant)

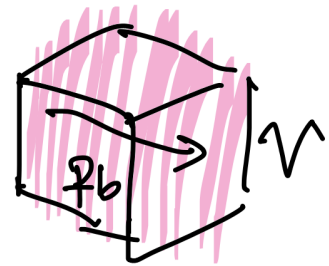
Up to now, we have been using single particle states satisfying

$$\int_V \psi^* \psi \, dV = 1 \quad \text{with} \quad \langle i|j\rangle = \delta_{ij}$$

ie $|\underline{a}\rangle$ = "The box of volume V contains a single particle with energy E_a and mom $\underline{p}_a$"



$|\underline{b}\rangle$ = "The box of volume V contains a single particle with energy E_b and mom $\underline{p}_b$"



Not examinable

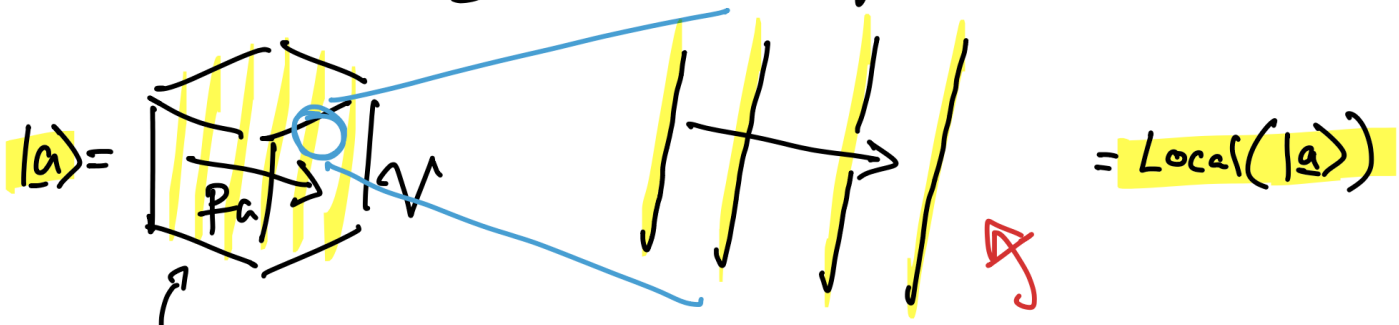
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EQUIVALENTLY:

Within a "small bit" of the box, $|a\rangle$ & $|b\rangle$ each look locally like a bit of a

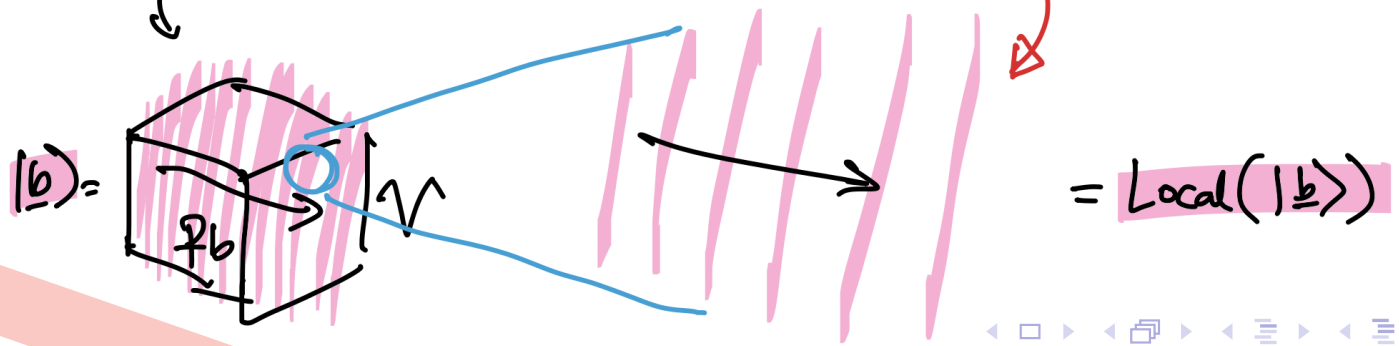
"$\frac{1}{V}$-particles-per-unit-volume soup"

of particles with energy E_a or E_b respectively.



1 particle in box in both cases

$\frac{1}{V}$ particles-per-unit-volume in both cases!



Not examinable

Not examinable

LENGTH CONTRACTION caused by RUNNING PAST

Local regions of $|a\rangle$ changes wavelength and thus also momentum, as expected by Lorentz Tfm:



Alas 😟 !! If we run past $|a\rangle$ at the right rate so as to exactly transform

$$E_a \to E'_a = E_b$$

and

$$p_a \to p'_a = p_b$$

we do **NOT** find that $|a\rangle \to |a\rangle' = |b\rangle$

Instead we find that $|a\rangle' = \sqrt{\gamma}\,|b\rangle$.

↑ γ = gamma-factor associated with the boost belonging to our "run"

Why?

Because the LENGTH CONTRACTION also increases the local particle density by a factor of γ! 😟

Not examinable

Not examinable

$$\left[\int_V \psi^* \psi \, dV = 1 \text{ and } \langle i | j \rangle = \delta_{ij}, \text{ so } \frac{1}{V} \text{ particles per unit volume as 1 particle in box of volume } V. \right]$$

Summary:

- One-particle states which are correctly normalised for one observer (according to our current convention) will not appear correctly normalised to observers moving at other velocities.

Not examinable

Not examinable

Choose, therefore, to adopt a new normalization convention which (unlike the last one) is consistent for all observers.

$\psi_a'' \underset{\text{def}}{\equiv} \sqrt{\gamma_a}\,\psi_a$ could work, and might be elegant,

$\gamma_a = \frac{1}{\sqrt{1-\beta_a^2}}$ where $\beta_a = \frac{p_a}{E_a}$

but for a mix of historical and practical reasons we use

$$\psi_a' \underset{\text{def}}{\equiv} \sqrt{2E_a}\sqrt{V}\,\psi_a \qquad (F)$$

(States in new norm convention are notated with a "prime": ψ_a')

which is called "2E particles per unit volume" since

$$(\#\text{particles in box}) \sim \int_V \psi_a''^{\dagger}\psi_a'\,dV = 2E_a V \int_V \psi_a^{*}\psi_a\,dV = 2E_a V \quad \text{and}$$

$$(\text{vol of box}) = V \qquad \therefore \quad (\#\text{particles per unit vol}) \sim \frac{2E_a V}{V} = 2E_a .$$

Not examinable

Not examinable

Where do we use these newly normalised states?

- Recall definition of T_{fi} in (A2) as a sandwich of $\hat{H}_{int}$ with "old" states as the bread on either side.

  Eg: $$T_{fi} = \underbrace{\langle \psi_1, \psi_2, \ldots|}_{\langle f|} \hat{H}_{int} \underbrace{|\ldots \psi_{N-1}, \psi_N\rangle}_{|i\rangle} + \ldots \quad (G)$$

- For any such T_{fi}, denote by "M_{fi}" the corresponding

  "Lorentz Invariant Matrix Element"

  by replacing $\langle f|$ and $|i\rangle$ with states of the new norm (and by multiplying by one conventional factor of $\frac{1}{V}$).

- i.e. for the T_{fi} in (G) define the corresponding M_{fi} to be:

$$M_{fi} \underset{\text{def}}{\equiv} \frac{1}{V} \langle \psi_1', \psi_2', \ldots | \hat{H}_{int} | \ldots \psi_{N-1}', \psi_N' \rangle + \ldots \quad (H)$$

DEFENITION

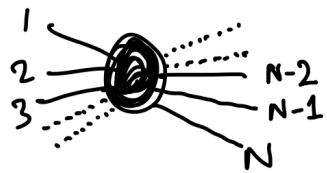
$$\therefore \quad M_{fi} = \frac{1}{V} \sqrt{2E_1}\sqrt{V} \ldots \sqrt{2E_N}\sqrt{V} \langle \psi_1, \psi_2 \ldots | \hat{H}_{int} | \ldots \psi_{N-1}, \psi_N \rangle + \ldots$$

ASIDE: The length of the const-factor $\frac{1}{V}$ will be seen later. Likewise, the appropriateness of the name "Lorentz Invariant Matrix Element" is not yet clear, but will later become so.

COROLLARY:

$$\therefore \quad M_{fi} = V^{N/2-1} \left(\sqrt{2E_1} \cdot \ldots \cdot \sqrt{2E_N}\right) T_{fi} . \quad (I)$$

COROLLARY

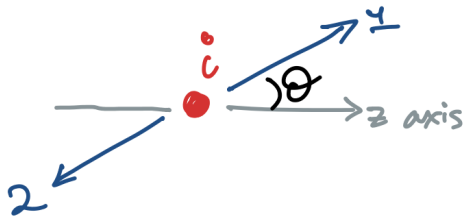


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28

Making (E) nicer still.



Recall that for the process

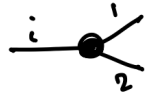
we had the expression:

$i \to 1+2$

$$\Gamma_{i\to f} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \, \delta(E_1+E_2-E_i)\, \delta^3(p_1+p_2-p_i) \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \qquad (E)$$

MATHS | PHYSICS | ENERGY CONS | MOM CONS | DENSITY OF STATES

Does this become nicer with the newly normalised states?

For this process $N=3$ (i.e. three external legs ) and so

$$(I) \Rightarrow \qquad M_{fi} = V^{\frac{3}{2}-1} \left(\sqrt{2E_i}\sqrt{2E_1}\sqrt{2E_2}\right) T_{fi}$$

and thus (E) $\Rightarrow$

$$\Gamma_{i\to f} = (2\pi)^4 \cdot \cancel{V} \cdot \int |M_{fi}|^2 \frac{\cancel{V^{2(1-\frac{3}{2})}}}{(2E_i)(2E_1)(2E_2)} \, \delta(E_1+E_2-E_i)\, \delta^3(p_1+p_2-p_i) \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3}$$

$i \to 1+2$

$$\Rightarrow \Gamma_{i\to f} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \, \delta(E_1+E_2-E_i)\, \delta^3(p_1+p_2-p_i) \frac{d^3p_1}{(2\pi)^3\, 2E_1} \frac{d^3p_2}{(2\pi)^3\, 2E_2} \qquad (J)$$

Not examinable

Not examinable

$i \to 1+2$

$$\Rightarrow \boxed{\Gamma_{i\to f} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_1+E_2-E_i)\,\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_i)\,\frac{d^3p_1}{(2\pi)^3\,2E_1}\,\frac{d^3p_2}{(2\pi)^3\,2E_2}} \quad (J)$$

TIME DILATION FACTOR | PHYSICS | ENERGY CONS | MOM CONS | DENSITY OF STATES

SUPER NICE

LHS **SHOULD** time dilate by factor γ of initial state E_i.
But $E_i = \gamma m_i$, so time dilation is perfectly captured by leading $\frac{1}{2E_i}$ term! Great! ☺

Separately:

- $\dfrac{d^3p_1}{(2\pi)^3 2E_1}$ is Lorentz **INVARIANT** (see Ex. Sht. Q3)
- $\delta^4(p_1^\mu + p_2^\mu - p_i^\mu)$ is Lorentz **INVARIANT**.

(non-trivial! uses $\mathrm{Det}(\Lambda)=1$ and $\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$ in Jacobian)

So BY PROCESS OF ELIMINATION $|M_{fi}|^2$ is also **LORENTZ INVARIANT.** *

* This justifies the name we gave it earlier! ☺

Not examinable

Not examinable

ASIDE

At this point one might wish to compare/contrast the calculation leading to (5) above for $i \to 1+2$ with a similar but subtly different calculation (page 512) for the case $1+2 \to 3+4$. Differences occur because the number of external particles is then $N=4$ instead of $N=3$.

——— However ———

Instead we try to use (5) to calculate an $i \to 1+2$ particle decay rate.

Not examinable

Appendix II: Detailed derivation of the cross section formula

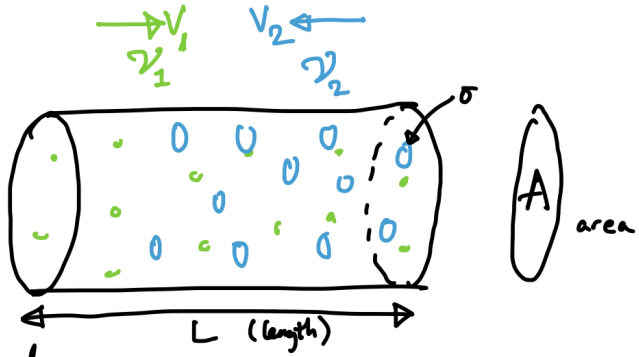
Not examinable

This appendix contains the full, hand-written derivation of the master cross section formula, eq. (7).

Not examinable

Not examinable

Classical link between cross section and effective area of interaction.



Suppose that:

- two beams of particle are in collision:
- "Species 1" (scatterers) are:
  - point-like
  - travelling $\rightarrow$ with speed v_1
  - have number-density n_1 particles-per-unit volume
- "Species 2" (targets) are:
  - present area "σ" to species 1.
  - travelling $\leftarrow$ with speed v_2
  - have number-density n_2 particles-per-unit volume

$\therefore$ In time δt each scatterer "passes" a volume $(v_1+v_2)\delta t A$ of targets.

$\therefore$ .. .. on effective target area of $(v_1+v_2)\delta t A n_2 \sigma$ distributed within a total area A.

The probability of interaction of the scatter is thus $(v_1+v_2)\delta t\, n_2 \sigma$.

There are, however, $AL n_1$ scatterers in the volume, so the expected number of interactions in the whole region is $(v_1+v_2)\delta t\, n_2 \sigma A L n_1$.

Therefore the interaction rate in this cylinder is $(v_1+v_2) n_2 \sigma A L n_1$ (K)

Not examinable

Not examinable

Saw that the interaction rate in this cylinder is: $(v_1+v_2)\nu_2\sigma AL\nu_1$ (K)

Our form of Fermi's Golden Rule computes the interaction rate in the cylinder only for the special case: $\nu_1 = \frac{1}{V} = \nu_2$ and $AL = V$.

$\therefore$ (K) $\Rightarrow$ $$\Gamma_{if} = (v_1 + v_2)\frac{1}{V}\sigma V\frac{1}{V}$$

$\Rightarrow$ $$\boxed{\sigma = \frac{\Gamma_{if} V}{v_1 + v_2}}$$ (M)

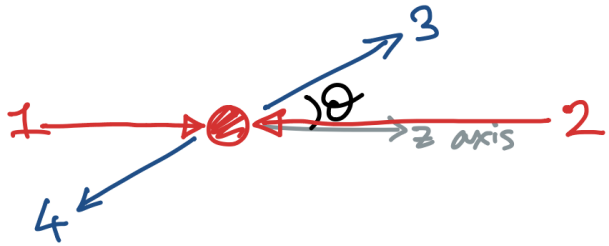
Message:

- F.G.R. can be used to calculate "σ", the effective area presented by scatterers to each other
- This cross section "σ" is really a property of the pair of species meeting each other - rather than of either species separately.

Not examinable

Not examinable

Let's now compute σ for the process $1+2 \to 3+4$:



Aside: this is closely related to what we did on slide 506

We start by adapting (E) with the replacements:

$$E_1 \to E_3$$
$$E_2 \to E_4$$
$$E_i \to E_1 + E_2$$
$$\mathbf{p}_1 \to \mathbf{p}_3$$
$$\mathbf{p}_2 \to \mathbf{p}_4$$

which results in:

$1+2 \to 3+4$

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \, \delta(E_3+E_4-E_1-E_2) \, \delta^3(\mathbf{p}_3+\mathbf{p}_4-\mathbf{p}_1-\mathbf{p}_2) \, \frac{d^3\mathbf{p}_3}{(2\pi)^3} \, \frac{d^3\mathbf{p}_4}{(2\pi)^3} \qquad (N)$$

MATHS | PHYSICS | ENERGY CONS | MOM CONS | DENSITY OF STATES

Not examinable

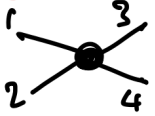
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$1+2 \to 3+4$

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \, \delta(E_3+E_4-E_1-E_2)\, \delta^3(\mathbf{p}_3+\mathbf{p}_4-\mathbf{p}_1-\mathbf{p}_2)\, \frac{d^3p_3}{(2\pi)^3}\, \frac{d^3p_4}{(2\pi)^3} \qquad (N)$$

MATHS | PHYSICS | ENERGY CONS | MOM CONS | DENSITY OF STATES

Does this become nicer with the newly normalised states?

For this process $N=4$ (i.e. three external legs 

) and so

$$(I) \Rightarrow \qquad M_{fi} = V^{\frac{4}{2}-1} \left(\sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_3}\sqrt{2E_4}\right) T_{fi}$$

and thus $(N) \Rightarrow$

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |M_{fi}|^2 \frac{V^{2(1-\frac{4}{2})}}{(2E_1)(2E_2)(2E_3)(2E_4)} \delta(E_3+E_4-E_1-E_2)\, \delta^3(\mathbf{p}_3+\mathbf{p}_4-\mathbf{p}_1-\mathbf{p}_2)\, \frac{d^3p_3}{(2\pi)^3}\, \frac{d^3p_4}{(2\pi)^3}$$

$1+2 \to 3+4$

$$\Rightarrow \Gamma_{if} = \frac{(2\pi)^4}{V(2E_1)(2E_2)} \int |M_{fi}|^2 \delta(E_3+E_4-E_1-E_2)\, \delta^3(\mathbf{p}_3+\mathbf{p}_4-\mathbf{p}_1-\mathbf{p}_2)\, \frac{d^3p_3}{(2\pi)^3\, 2E_3}\, \frac{d^3p_4}{(2\pi)^3\, 2E_4} \qquad (O)$$

Not examinable

Not examinable

$\therefore$ (m) & (o) $\Rightarrow$

$1+2 \to 3+4$

$$\sigma = \frac{(2\pi)^{-2}}{(2E_1)(2E_2)(v_1+v_2)} \int |M_{fi}|^2 \delta(E_3+E_4-E_1-E_2)\delta^3(\mathbf{p}_3+\mathbf{p}_4-\mathbf{p}_1-\mathbf{p}_2)\frac{d^3p_3}{2E_3}\frac{d^3p_4}{2E_4} \quad (P)$$

???? PHYSICS ENERGY CONS MOM CONS DENSITY OF STATES

↑ INVESTIGATE !

σ should** be **INVARIANT** w.r.t. Longitudinal Lorentz Boosts.

*** (No Lorentz contraction ⊥ to Lorentz boost!)

(ENERGY CONS), (MOM CONS) and (PHASE-SPACE) terms are certainly Lorentz **INVARIANT.**

Appendix III (page 515) shows that (provided $\mathbf{p}_1$ & $\mathbf{p}_2$ are collinear, which is a property preserved by Longitudinal boosts) it is always the case that

$$(2E_1)(2E_2)(v_1+v_2) = F \underset{\text{def}}{\equiv} 4\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2}$$

Hence the "????" term is Longitudinally Lorentz **INVARIANT** too!

(And as a corollary we see that consistency will thus force $|M_{fi}|^2$ to be Longitudinally Lorentz **INVARIANT** too.)*

$|M_{fi}|^2$ is actually Lorentz INVARIANT, but the corollary on the left only guarantees this Longitudinally.

SUPER NICE

Not examinable

Appendix III: Logitudinal invariance of 'Lorentz Invariant Flux'

Not examinable

The argument in this appendix aims to show that the so-called 'Lorentz Invariant Flux', F, defined only for collinear collisions $a \xrightarrow[v_a,\vec{p}_a]{} \xleftarrow[v_b,\vec{p}_b]{} b$ by

$$F = 2E_a 2E_b \left(v_a + v_b\right)$$

may be written in a Lorentz Invariant way, constant across all frames for which the collision is colliner.

For all such frames: $p_a \cdot p_b = p_a^\mu p_{b\mu} = E_a E_b - \vec{p}_a \cdot \vec{p}_b = E_a E_b + |\vec{p}_a|\,|\vec{p}_b|$. (It is the last step therein which assumes collinearity!) Thus, for all such frames:

$$\begin{aligned}
F^2/16 - \left(p_a^\mu p_{b\mu}\right)^2 &= \frac{1}{16}\left(2E_a 2E_b\left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b}\right)\right)^2 - \left(p_a \cdot p_b\right)^2 \\
&= \left(|\vec{p}_a|\,E_b + |\vec{p}_b|\,E_a\right)^2 - \left(E_a E_b + |\vec{p}_a|\,|\vec{p}_b|\right)^2 \\
&= |\vec{p}_a|^2\left(E_b^2 - |\vec{p}_b|^2\right) + E_a^2\left(|\vec{p}_b|^2 - E_b^2\right) \\
&= |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 \\
&= -m_a^2 m_b^2
\end{aligned}$$

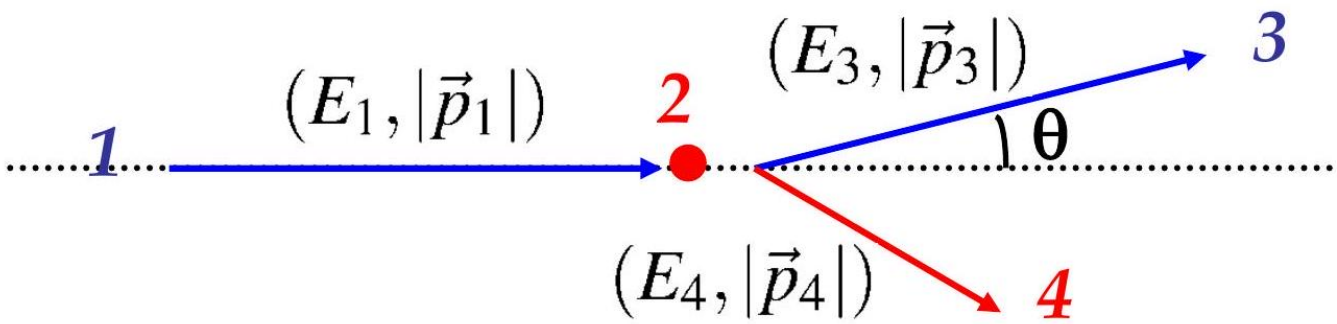
and so

$$F = 4\left[\left(p_a^\mu p_{b\mu}\right)^2 - m_a^2 m_b^2\right]^{1/2} \qquad \square.$$

Not examinable

Appendix IV: General $2 \to 2$ Body Scattering in lab frame I

Not examinable



$p_1 = (E_1, 0, 0, |\vec{p}_1|)\,, p_2 = (M_2, 0, 0, 0)\,, p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta)\,, p_4 = (E_4, \vec{p}_4)$ again

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

But now the invariant quantity t :

$$\begin{aligned} t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2 \cdot p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\ &= m_2^2 + m_4^2 - 2m_2 (E_1 + m_2 - E_3) \\ \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} &= 2m_2 \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \end{aligned}$$

Not examinable

Appendix IV: General $2 \to 2$ Body Scattering in lab frame II

Not examinable

Which gives $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi} \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{\mathrm{d}\sigma}{\mathrm{d}t}$

To determine $\mathrm{d}E_3/\mathrm{d}\left(\cos\theta\right.$, first differentiate $E_3^2 - |\vec{p}_3|^2 = m_3^2$

$$2E_3 \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = 2\,|\vec{p}_3| \frac{\mathrm{d}\,|\vec{p}_3|}{\mathrm{d}(\cos\theta)} \tag{184}$$

Then equate

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2 \qquad \text{to give}$$

$$m_1^2 + m_3^2 - 2\left(E_1 E_3 - |\vec{p}_1|\,|\vec{p}_3|\cos\theta\right) = m_4^2 + m_2^2 - 2m_2\left(E_1 + m_2 - E_3\right)$$

Differentiate wrt. $\cos\theta$

$$\left(E_1 + m_2\right) \frac{\mathrm{d}E_3}{\mathrm{d}\cos\theta} - |\vec{p}_1|\cos\theta \frac{\mathrm{d}\,|\vec{p}_3|}{\mathrm{d}\cos\theta} = |\vec{p}_1|\,|\vec{p}_3|$$

Using (184)

$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{|\vec{p}_1|\,|\vec{p}_3|^2}{|\vec{p}_3|\left(E_1 + m_2\right) - E_3\,|\vec{p}_1|\cos\theta} \tag{185}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi} \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{m_2}{\pi} \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{1}{64\pi s\,|\vec{p}_i^{\,*}|^2} |M_{fi}|^2$$

Not examinable

Appendix IV: General $2 \to 2$ Body Scattering in lab frame III

Not examinable

It is easy to show $|\vec{p}_i^{\,*}|\sqrt{s} = m_2 |\vec{p}_1|$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{m_2}{64\pi^2 m_2^2 |\vec{p}_1|^2} |M_{fi}|^2$$

and using (185) obtain

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{m_2 |\vec{p}_1|} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3| (E_1 + m_2) - E_3 |\vec{p}_1| \cos\theta} \cdot |M_{fi}|^2}.$$

Not examinable

Appendix V: Dimensions of the Dirac Matrices I

Not examinable

In a d-dimensional spacetime there will always be d gamma matrices, as one is associated with each spacetime derivative in the Hamiltonian. That is why in 4-dimensional spacetime we have four gamma matrices: γ_0, γ_1, γ_2 and γ_3.

But why does $d = 4$ force those matrices to be (4×4)-matrices ?

Rather than answer the above question, we instead state (and later prove) the more general result (186) linking the $(n \times n)$ size of gamma matrices to the number d of spacetime dimension with which they are associated:

$$n = 2^{\lfloor \frac{d}{2} \rfloor}. \tag{186}$$

The result (186) is a direct consequence of the gamma matrices having to satisfy (as we already saw in (33)) the defining property of a (so called) 'Clifford Algebra', namely that:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} 1_{n\times n}. \tag{187}$$

Warning: the proof we provide for the above statement relies on Schur's Lemma. This may be a source of dissatisfaction for some persons taking the course because **Schur's Lemma**, although stated in the Groups and Representations section of the Part IB Mathematics course within Natural Sciences Tripos, was stated in that coure without proof. If you find that annoying, you will have to find an alternative proof.

Not examinable

Appendix V: Dimensions of the Dirac Matrices II

Aside on size of Pauli matrices:

Although we are mainly interested in proving (186) to substantiate the claim that each γ^μ is a (4×4)-matrix, we note that the same result can be used to explain why the Pauli matrices are (2×2)-matrices. The reason is that the three ($d = 3$) Pauli matrices satisfy their own equivalent of (187), namely: $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$. Hence $n = 2^{\lfloor 3/2 \rfloor} = 2^1 = 2$.

We wish to prove the result stated in (186) is the relationship between the dimension d of spacetime and the dimension n of the (irreducible) $(n \times n)$ irreducible matrices γ_μ satisfying (187) with $\mu, \nu = 0, 1, \cdots, d-1$. Conveniently, the relationship (186) between n and d which we seek to prove does not depend on the signature of the metric since it is possible to convert a representation designed for one signature (say $g_{\mu\nu} = \text{diag}(+,-,-,-)$) to another (say $g_{\mu\nu} = \text{diag}(+,+,+,+)$) without changing n by multiplying appropriate γ-matrices by $i = \sqrt{-1}$.
Therefore, without loss of generality, we actually take as our start point the simplest possibility, namely:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \cdot 1_{n \times n}. \tag{188}$$

We nonetheless demand that the γ-matrices are irreducible – i.e. that there is not a similarity transformation that would reduce them all to a (non-trivial) block diagonal form. We start by noting that with those assumptions:

Not examinable

Appendix V: Dimensions of the Dirac Matrices III

Not examinable

- **Every γ^μ is invertible**. [To prove this simply set $\mu = \nu$ in (188) and take the determinant of both sides.]
- **For the matrix $\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1}$ we have**

$$\gamma^*\gamma^\mu = (-1)^{d-1}\gamma^\mu\gamma^*. \tag{189}$$

[Proof: When γ^μ commutes with γ^* it must pass $d-1$ dissimilar γ-matrices and a single 'identical' γ-matrix. Given (188) there are therefore $d-1$ anti-commutations and a single commutation. $\square$]

- **The matrix $\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1}$ squares to either +1 or -1 depending on d.** [Proof: it takes $\frac{1}{2}(d-1)d$ flips of adjacent pairs to reverse the order of d objects, and since all the γ-matrices in γ^* are dissimilar and thus anti-commute we can deduce that

$$\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1} = (-1)^{\frac{1}{2}(d-1)d}\cdot\gamma^{d-1}\cdots\gamma^1\gamma^0$$

Not examinable

Appendix V: Dimensions of the Dirac Matrices IV

Not examinable

and so

$$
\begin{aligned}
(\gamma^*)^2 &= (-1)^{\frac{1}{2}(d-1)d} \cdot (\gamma^{d-1} \cdots \gamma^1 \gamma^0) \cdot (\gamma^0 \gamma^1 \cdots \gamma^{d-1}) \\
&= (-1)^{\frac{1}{2}(d-1)d} \prod_{\mu=0}^{d-1} \delta^{\mu\mu} \\
&= (-1)^{\frac{1}{2}(d-1)d} \\
&= s(d)
\end{aligned}
\tag{190}
$$

in which $s(d) \equiv (-1)^{\frac{1}{2}(d-1)d}$ is a d-dependent sign in $\{+1, -1\}$.]

- **If $d > 1$ then n must be even.** [To prove this, consider $\mu \neq \nu$ (which requires $d > 1$) in (188). In this case (188) becomes $\gamma^\mu \gamma^\nu = -\gamma^\mu \gamma^\mu$ which implies that $\det\{\gamma^\mu\} \det\{\gamma^\nu\} = (-1)^n \det\{\gamma^\nu\} \det\{\gamma^\mu\}$ which (since every γ^μ is invertible) implies that $1 = (-1)^n$ and thus that n is even.]

Not examinable

Appendix V: Dimensions of the Dirac Matrices V

Not examinable

- **Theorem A: Any product of any number of γ-matrices may (up to a sign) be written as a product of at most d gamma matrices in strictly ascending order of their indices.** [This is because (188) states that dissimilar γ-matrices anti-commute, and that individual γ-matrices square to ± 1'. Therefore, an arbitrary product of γ-matrices can always have its γ-matrices permuted into numerical order (with a sign change if an odd number of permutations is required) leaving at most one copy of each γ-matrix as repeats will disappear (up to a sign) on account of the squaring property.]

The last result above motivates the following definition.

Definition

If A is any integer whose binary representation modulo 2^d is $\vec{A}$, i.e. if $(A \bmod 2^d) = \sum_{i=0}^{d-1} A_i \cdot 2^i$ with each $A_i \in \{0, 1\}$, then define Γ_A by

$$\Gamma_A = \prod_{i=0}^{d-1} \left\{ \begin{array}{ll} \gamma_i & \text{if } A_i = 1 \\ 1 & \text{otherwise} \end{array} \right\}. \tag{191}$$

For example, this definition would make $\Gamma_{13} = \gamma_0\gamma_2\gamma_3$ since $13 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$.

Appendix V: Dimensions of the Dirac Matrices VI

Not examinable

On account of the modulo 2^d part of the definition, any continuous range of indices of length 2^d would suffice to include every such Γ-matrix. Without loss of generality will always take indices A to be in the set

$$\mathcal{A} = \{1, 2, \cdots, 2^d\},$$

and mapped into that range, if necessary, by an implicit modulo 2^d operation. We therefore define a complete list, L, of Γ-matrices as follows:

$$L = (\Gamma_1, \Gamma_2, \ldots, \Gamma_{2^d}) = (\Gamma_A \mid A \in \mathcal{A}). \tag{192}$$

Note that although we have defined 2^d quantities Γ_A in the list L we have not shown that they are all unique. In other words, we cannot assume '$(A \neq B) \implies (\Gamma_A \neq \Gamma_B)$' or '$(\Gamma_A = \Gamma_B) \implies (A = B)$' unless later proved.

We now state and prove two important properties of the Γ-matrices:

Lemma 1:

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Appendix V: Dimensions of the Dirac Matrices VII

Not examinable

The most general form of this Lemma is

$$\mathrm{Tr}\left[\Gamma_A\right] = \begin{cases} n & \text{if } A = 0 \mod 2^n \\ 0 & \text{if } (A \neq 0 \mod 2^n) \text{ and } (d \text{ is even or } \sum_{i=1}^{d} A_i \text{ is even}) \\ Tr[\Gamma_A] & \text{otherwise.} \end{cases} \tag{193}$$

Alternatively, a narrower form could be stated as follows

$$\text{When } d \text{ is even:} \qquad \mathrm{Tr}\left[\Gamma_A\right] = \begin{cases} n & \text{if } A = 0 \mod 2^n \\ 0 & \text{otherwise.} \end{cases} \tag{194}$$

Proof of Lemma 1:
The trace of Γ_0 is always trivially n as $\Gamma_0 = 1_{n\times n}$. Every other Γ_A is the product of one or more dissimilar γ-matrices. We split the remainder of the proof into two parts: part (i) shows that traces of products are zero where the remaining products contain an **even** number of γ-matrices, while part (ii) shows the same for products containing any **odd** number of γ-matrices. Note the subtle differences between these two parts of of the proof: the first needs to assume that the multiplied gammas are **distinct** but does not need to worry about whether d is even or odd. In contrast the second does not care about distinctness in the gammas but **needs to assume that d is even**.

Not examinable

Appendix V: Dimensions of the Dirac Matrices VIII

Not examinable

Part (i): even products

If k is an integer greater than zero, and if $a_1, a_2, \ldots, a_k$ are k **distinct** integers in $[0, d-1]$ and if $T = \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right]$ then

$$\begin{aligned} T &= \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] \\ &= (-1)^{k-1}\cdot\mathrm{Tr}\left[\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] \\ &\qquad\qquad (\text{after } k-1 \text{ anti-commutations using (188) and } k>0) \\ &= (-1)^{k-1}\cdot\mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] \qquad (\text{trace cyclicity}) \\ &= (-1)^{k-1}\cdot T \end{aligned}$$

therefore:

"The trace of the product of an **even** number of **distinct** γ-matrices ...

...is zero provided the even number is greater than or equal to two". (195)

Part (ii): odd products

Not examinable

Appendix V: Dimensions of the Dirac Matrices IX

Not examinable

If k is an integer greater than zero, and if $a_1, a_2, \ldots, a_k$ are k integers in $[0, d-1]$ and if $T = \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right]$ then

$$
\begin{aligned}
& T = \mathrm{Tr}\left[\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] && \\
\Longrightarrow \quad & s(d)\cdot T = \mathrm{Tr}\left[(\gamma^*\gamma^*)\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\right] && \text{(by (190))} \\
\Longrightarrow \quad & s(d)\cdot T = \mathrm{Tr}\left[\gamma^*\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\gamma_{a_k}\gamma^*\right] && \text{(trace cyclicity)} \\
\Longrightarrow \quad & s(d)\cdot T = ((-1)^{d-1})^k\cdot\mathrm{Tr}\left[\gamma^*\gamma^*\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] && \text{(after } k \text{ uses of (189))} \\
\Longrightarrow \quad & T = (-1)^{k(d-1)}\cdot\mathrm{Tr}\left[\gamma_{a_k}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{k-1}}\right] && \text{(by (190) again)} \\
\Longrightarrow \quad & T = (-1)^{k(d-1)}\cdot T &&
\end{aligned}
$$

therefore:

"when d is even, the trace of the product of an odd number of γ-matrices is zero". (196)

This concludes our proof of Lemma 1. □

Lemma 2:

Not examinable

$$\Gamma_A\Gamma_B = s(A,B)\cdot\Gamma_{A\oplus B} \tag{197}$$

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Not examinable

in which '$\oplus$' represents 'BITWISE EXCLUSIVE OR' and $s(A, B)$ is a function mapping pairs of indices to the set $\{+1, -1\}$.

Proof of Lemma 2:

$$\begin{aligned}\Gamma_A\Gamma_B &= \prod_{i=0}^{d-1}\begin{Bmatrix}\gamma_i & \text{if } A_i = 1\\ 1 & \text{otherwise}\end{Bmatrix}\prod_{i=0}^{d-1}\begin{Bmatrix}\gamma_i & \text{if } B_i = 1\\ 1 & \text{otherwise}\end{Bmatrix}\\ &= s_1(A, B)\prod_{i=0}^{d-1}\left(\begin{Bmatrix}\gamma_i & \text{if } A_i = 1\\ 1 & \text{otherwise}\end{Bmatrix}\begin{Bmatrix}\gamma_i & \text{if } B_i = 1\\ 1 & \text{otherwise}\end{Bmatrix}\right)\end{aligned}$$

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Not examinable

where $s_1(A,B) \in \{+1,-1\}$ is a sign which will depend on how many anti-commutations deriving from (188) were needed to re-order the matrices, and so

$$\Gamma_A \Gamma_B = s_1(A,B) \prod_{i=0}^{d-1} \begin{Bmatrix} (\gamma_i)^2 & \text{if } A_i = B_i = 1 \\ \gamma_i & \text{if } A_i \oplus B_i = 1 \\ 1 & \text{otherwise} \end{Bmatrix}$$

$$= s_1(A,B) \prod_{i=0}^{d-1} \begin{Bmatrix} g_{ii} \quad (\text{no sum } i) & \text{if } A_i = B_i = 1 \\ \gamma_i & \text{if } A_i \oplus B_i = 1 \\ 1 & \text{otherwise} \end{Bmatrix} \quad (\text{by (188)})$$

$$= s(A,B) \prod_{i=0}^{d-1} \begin{Bmatrix} 1 & \text{if } A_i = B_i = 1 \\ \gamma_i & \text{if } A_i \oplus B_i = 1 \\ 1 & \text{otherwise} \end{Bmatrix}$$

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Appendix V: Dimensions of the Dirac Matrices XII

Not examinable

where $s(A,B)$ is a new sign function that accounts for our having replaced g_{ii} with 1, and so

$$\begin{aligned}\Gamma_A\Gamma_B &= s(A,B)\prod_{i=0}^{d-1}\left\{\begin{array}{ll}\gamma_i & \text{if } A_i \oplus B_i = 1\\ 1 & \text{otherwise}\end{array}\right\}\\ &= s(A,B)\Gamma_{A\oplus B}\end{aligned}$$ □.

A corollary of (197) is that every Γ-matrix is invertible. [Proof: setting B equal to A in (197) tells us that $(\Gamma_A)^2 = s(A,A)\cdot\Gamma_0 = s(A,A)\cdot 1_{n\times n} = \pm 1_{n\times n}$ and so

$$(\Gamma_A)^{-1} \text{ is either } \Gamma_A \text{ or } -\Gamma_A. \tag{198}$$

]

Perhaps we can do better. Suppose A has a ones in its binary representation (i.e. $a = \sum_{i=0}^{d-1} A_i$ so that Γ_A is a product of a gamma matrices in ascending order of index). If we then square Γ_A we could attempt to permute adjacent gamma matrices within the product so as to annihilate every identical pairing, leaving behind only a sign. This process would require $a-1$ anticommutations to annihilate the first pair, $a-2$ the

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second, *etc*, and none for the last. This is a total of $\frac{1}{2}(a-1)a$ anticommutations, and so we can make the very specific claim that

$$(\Gamma_A)^2 = (-1)^{\frac{1}{2}(a-1)a} \tag{199}$$

or equivalently

$$(\Gamma_A)^{-1} = (-1)^{\frac{1}{2}(a-1)a} \cdot \Gamma_A. \tag{200}$$

Indeed, we see that the already derived result (190) could be viewed with hindsight as a simple corollary of (199).
Knowing that the Γ-matrices are all invertible we may define a matrix S as follows:

$$S = \sum_{X \in \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot \Gamma_X \tag{201}$$

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Appendix V: Dimensions of the Dirac Matrices XIV

Not examinable

where Y is an arbitrary $(n \times n)$-matrix whose value we will fix later. It is follows that for any integer A (not summed) in the usual range $\mathcal{A}$:

$$
\begin{aligned}
(\Gamma_A)^{-1} \cdot S \cdot \Gamma_A &= \sum_{X \in \mathcal{A}} (\Gamma_X \Gamma_A)^{-1} \cdot Y \cdot (\Gamma_X \Gamma_A) \\
&= \sum_{X \in \mathcal{A}} (s_X \Gamma_{A \oplus X})^{-1} \cdot Y \cdot (s_X \Gamma_{A \oplus X}) \qquad \text{(using (197))} \\
&= \sum_{X \in \mathcal{A}} (\Gamma_{A \oplus X})^{-1} \cdot Y \cdot (\Gamma_{A \oplus X}) \\
&= \sum_{X \in A \oplus \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot (\Gamma_X) \\
&= \sum_{X \in \mathcal{A}} (\Gamma_X)^{-1} \cdot Y \cdot (\Gamma_X) \qquad \text{(since } A \oplus \mathcal{A} \equiv \{A \oplus B, B \in \mathcal{A}\} = \mathcal{A}\text{)} \\
&= S
\end{aligned}
$$

and thus $S \cdot \Gamma_A = \Gamma_A \cdot S$.
Having found a matrix S which commutes with every element Γ_A of a list L of matrices, one might hope to use Schur's Lemma to claim that S is some multiple of $1_{n \times n}$. However, a precondition of the only version of Schur's Lemma which I understand and which also

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allows that conclusion to be drawn requires the elements of L to form an irreducible representation of some group G. Not only have we not yet shown that this precondition is satisfied, it actually looks likely to be false! For example, for the usual γ-matrices in $d = 4$ dimensions we would have $\Gamma_1\Gamma_2 = \gamma_1\gamma_2 = -\gamma_2\gamma_1 = -\Gamma_2\Gamma_1$ and so for L to be closed under multiplication it would need to contain both $+\Gamma_2\Gamma_1$ and $-\Gamma_2\Gamma_1$. This seems unlikely as we did not set up L to contain negated copies of every element. It therefore seems unlikely that L is closed under multiplication and so it seems unlikely that L represents a group.
It could be argued that the source of the problem is the annoying sign $s(A, B)$ in (197). If that pesky sign were not there and the constant '$+1$' were always in its place, products of Γ-matrices would be closed. We cannot arbitrarily dispose of that pesky sign, but it does suggest a resolution: we could double the length of our list L by adding to it another copy of itself but with the sign of every matrix reversed in the second half. The elements of this list will then be closed under multiplication, which is would be a requirement for them to be any kind of representation. We shall call the set containing all those elements G:

$$G = \{+\Gamma_A \mid A \in \mathcal{A}\} \cup \{-\Gamma_A \mid A \in \mathcal{A}\}. \tag{202}$$

This set of matrices is: (i) closed under multiplication, (ii) contains the identity $\Gamma_{2^d} = 1_{n\times n}$, (iii) contains an inverse for every element (see proof in (198)). Finally (iv) matrix multiplication is associative. Therefore G together with the operation of matrix multiplication forms a group. As it is a finite matrix group it is also representation of

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Not examinable

itself. This representation must be irreducible since the representation contains elements which are copies of the original γ-matrices (e.g. $\Gamma_1 = \gamma_0$, $\Gamma_2 = \gamma_1$, ... $\Gamma_{2^d} = \gamma_d$), and those original γ-matrices were taken to be be irreducible at the outset by assumption (see paragraph containing (188)). Although we have increased the number of elements in G relative to L, we can be sure that our old S will commute with every element of the new G because

$$([S, +\Gamma_A] = 0) \iff ([S, -\Gamma_A] = 0).$$

We have thus established all the preconditions necessary to allow us to use Schur's Lemma to state that S is a multiple of the identity, or more specifically:

$$\lambda \cdot 1_{n\times n} = \sum_{X\in\mathcal{A}} (\Gamma_A)^{-1} \cdot Y \cdot \Gamma_A \tag{203}$$

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Not examinable

for some scalar λ that will depend on Y. Taking the trace of both sides of (203) and using the cyclicity of the trace gives us:

$$\begin{aligned} n\lambda &= \sum_{X \in \mathcal{A}} \operatorname{Tr}\left[(\Gamma_A)^{-1} \cdot Y \cdot \Gamma_A\right] \\ &= \sum_{X \in \mathcal{A}} \operatorname{Tr}\left[Y \cdot \Gamma_A \cdot (\Gamma_A)^{-1}\right] \\ &= \sum_{X \in \mathcal{A}} \operatorname{Tr} Y \\ &= 2^d \cdot \operatorname{Tr} Y \end{aligned}$$

and thus

$$\lambda = \frac{2^d}{n} \cdot \operatorname{Tr} Y. \tag{204}$$

Putting this value for λ back into (203) yields

$$\frac{2^d}{n} \cdot \operatorname{Tr} Y \cdot 1_{n \times n} = \sum_{X \in \mathcal{A}} (\Gamma_A)^{-1} \cdot Y \cdot \Gamma_A. \tag{205}$$

Not examinable

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Not examinable

We now exercise our remaining freedom to choose Y to be any $(n \times n)$-matrix we wish, deciding to let

$$[Y]_{ij} = \delta_{is}\delta_{jt}$$

where s and t are integers in $[1, n]$ which we may choose to fix later. With that choice in mind, and with i and j being other arbitrary integers also in $[1, n]$, (205) can be expanded as:

$$\left[\frac{2^d}{n} \cdot \mathrm{Tr}\, Y \cdot 1_{n\times n}\right]_{ij} = \left[\sum_{X\in\mathcal{A}} (\Gamma_A)^{-1} \cdot Y \cdot \Gamma_A\right]_{ij}$$

or equivalently

$$\frac{2^d}{n} \cdot (\delta_{ms}\delta_{mt}) \cdot \delta_{ij} = \sum_{X\in\mathcal{A}} ((\Gamma_A)^{-1})_{im} \cdot (\delta_{ms}\delta_{nt}) \cdot (\Gamma_A)_{nj}$$

which simplifies to

$$\frac{2^d}{n} \cdot \delta_{st} \cdot \delta_{ij} = \sum_{X\in\mathcal{A}} ((\Gamma_A)^{-1})_{is} \cdot (\Gamma_A)_{tj}. \tag{206}$$

Not examinable

Appendix V: Dimensions of the Dirac Matrices XIX

Not examinable

Since (206) is true for any i,j,s,t in $[1,n]$, let us set $s \to i$ and $t \to j$ and then sum over i and j. Making use of the summation convention over i and j we find that:

$$\frac{2^d}{n} \cdot \delta_{ij} \cdot \delta_{ij} = \sum_{A \in \mathcal{A}} ((\Gamma_A)^{-1})_{ii} \cdot (\Gamma_A)_{jj}$$

which simplifies to

$$\frac{2^d}{n} \cdot n = \sum_{A \in \mathcal{A}} \mathrm{Tr}\left[(\Gamma_A)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_A]$$

or

$$2^d = \sum_{A \in \mathcal{A}} \mathrm{Tr}\left[(\Gamma_A)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_A]. \tag{207}$$

d-even

Not examinable

Appendix V: Dimensions of the Dirac Matrices XX

Not examinable

For the case that d is even we may now use (194) to simplify (207) to

$$\begin{aligned} 2^d &= \mathrm{Tr}\left[(\Gamma_0)^{-1}\right] \cdot \mathrm{Tr}[\Gamma_0] \\ &= \mathrm{Tr}\left[(1_{n\times n})^{-1}\right] \cdot \mathrm{Tr}[1_{n\times n}] \\ &= \mathrm{Tr}[1_{n\times n}] \cdot \mathrm{Tr}[1_{n\times n}] \\ &= n \cdot n = n^2 \\ \implies \qquad n &= 2^{d/2} \qquad \text{(but only for } d \text{ even!).} \end{aligned} \tag{208}$$

d-odd

This is a bit of a trick. One may always generate an irreducible representation of the gamma matrices for an **odd** spacetime dimension $d+1$ from an irreducible representation valid for an **even** number of spacetime dimensions d. The way to do this is surprisingly simple: if

$$\left\{\gamma^0, \gamma^1, \ldots, \gamma^{d-1}\right\}$$

is an irrep of (188) for an **even** number of spacetime dimensions d, and if we define

$$\gamma^* \equiv \gamma^0\gamma^1\cdots\gamma^{d-1}$$

Not examinable

Appendix V: Dimensions of the Dirac Matrices XXI

Not examinable

and if we recall the definition of $s(d)$ from (190), then

$$\left\{\gamma^0, \gamma^1, \ldots, \gamma^{d-1}\right\} \cup \{\sqrt{s(d)} \cdot \gamma^*\} \tag{209}$$

will be an irrep of (188) valid for dimension $d+1$ spacetime dimensions. That (209) is the irrep it is claimed to be is a consequence of three things: (i) γ^* was proved in (189) to anticommute with all the other gamma matrices when d is **even** and this anti-commutation is the property enforced/required by (188) whenever $\mu \neq \nu$, (ii) that $\sqrt{s(d)}\gamma^*$ squares to 1 was proved in (190), and this is the property enforced/required by (188) whenever $\mu = \nu$, and (iii) the representation (209) is an irrep as the first d gammas formed an irrep by themselves (i.e. as there was no transformation which could 'reduce' them, there cannot be an irrep that could 'reduce' both then and γ^*). It may be observed that this argument cannot be used to grow irreps without limit, since once an irrep for even d is grown to an irrep for odd d, the 'next' γ^* would fail to anticommute as desired. Nonetheless, the clear message is that the dimension of the gamma matrices for odd spacetime dimension d is always the same as the even dimension $d-1$, and so (208) now informs us that

$$n = 2^{(d-1)/2} \qquad \text{(but only when } d \text{ is odd!)}. \tag{210}$$

Not examinable

Appendix V: Dimensions of the Dirac Matrices XXII

Not examinable

General spacetime dimension d (even or odd)

There result (208) for even d can be merged with the result (210) for odd d into a single expression valid for any d:

$$n = \begin{cases} 2^{d/2} & \text{(when } d \text{ is even)} \\ 2^{(d-1)/2} & \text{(when } d \text{ is odd)} \end{cases}$$

$$\Longrightarrow \qquad n = 2^{\lfloor d/2 \rfloor} \qquad \text{(for any } d\text{).} \tag{211}$$

This concludes the proof of (186) which is also a proof of the lesser claim that Dirac Spinors have four components in the usual 4-dimensional spacetime.

Not examinable

Appendix VI: Magnetic Moment I

Not examinable

- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \vec{A})$ can be obtained by making the minimal substitution $\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi$
- Applying this to (40) and (41)

$$\begin{aligned} (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B &= (E - m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (E + m - q\phi)u_B \end{aligned} \tag{212}$$

Multiplying (212) by $(E + m - q\phi)$

$$\begin{aligned} (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B &= (E - m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (E + m - q\phi)u_B \end{aligned} \tag{213}$$

where kinetic energy $T = E - m$

- In the non-relativistic limit $T \ll m$ (213) becomes

$$\begin{aligned} (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &\approx 2m(T - q\phi)u_A \\ \left[(\vec{\sigma}.\vec{p})^2 - q(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p}) - q(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{A}) + q^2(\vec{\sigma}.\vec{A})^2\right] u_A &\approx 2m(T - q\phi)u_A \end{aligned} \tag{214}$$

Not examinable

Appendix VI: Magnetic Moment II

Not examinable

- Now $\vec{\sigma}.\vec{A} = \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}$; $\vec{\sigma}.\vec{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$; which leads to $(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \vec{A}.\vec{B} + i\vec{\sigma}.(\vec{A} \wedge \vec{B})$ and $(\vec{\sigma}.\vec{A})^2 = |\vec{A}|^2$
- The operator on the LHS of (214):

$$\begin{aligned}
&= \vec{p}^2 - q\left[\vec{A}.\vec{p} + i\vec{\sigma}.\vec{A} \wedge \vec{p} + \vec{p}.\vec{A} + i\vec{\sigma}.\vec{p} \wedge \vec{A}\right] + q^2\vec{A}^2 \\
&= (\vec{p} - q\vec{A})^2 - iq\vec{\sigma}.\left[\vec{A} \wedge \vec{p} + \vec{p} \wedge \vec{A}\right] \\
&= (\vec{p} - q\vec{A})^2 - q^2\vec{\sigma}.\left[\vec{A}.\vec{\nabla} + \vec{\nabla}.\vec{A}\right] \qquad (\text{since } \vec{p} = -i\vec{\nabla}) \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.(\vec{\nabla} \wedge \vec{A}) \qquad (\text{since } (\vec{\nabla} \wedge \vec{A})\psi = \vec{\nabla} \wedge (\vec{A}\psi) + \vec{A} \wedge (\vec{\nabla}\psi)) \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.\vec{B} \qquad (\text{since } \vec{B} = \vec{\nabla} \wedge \vec{A})
\end{aligned}$$

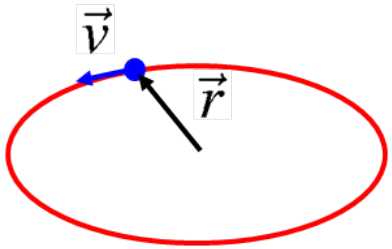
Substituting back into (214) gives the Schrödinger-Pauli equation for the motion of a non-relativisitic spin $\frac{1}{2}$ particle in an EM field:

$$\left[\frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}.\vec{B} + q\phi\right] u_A = Tu_A.$$

Not examinable

Appendix VI: Magnetic Moment III

Not examinable



- Since the energy of a magnetic moment in a field is we can identify the intrinsic magnetic moment of a spin-half particle to be:

$$\vec{\mu} = \frac{q}{2m}\vec{\sigma}$$

In terms of the spin: $\vec{S} = \frac{1}{2}\vec{\sigma}$

$$\vec{\mu} = \frac{q}{m}\vec{S}$$

- Classically, for a charged particle current loop

$$\mu = \frac{q}{2m}\vec{L}$$

- The intrinsic magnetic moment of a spin half Dirac particle is twice that expected from classical physics. This is often expressed in terms of the gyromagnetic ratio is g=2.

$$\vec{\mu} = g\frac{q}{2m}\vec{S}$$

Not examinable

Appendix VII: Generators of Lorentz Transformations I

Not examinable

It will shortly be seen that the quantities

$$(M^{\alpha\beta})^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta} \tag{215}$$

or the equivalent (but less symmetric) quantities

$$(M^{\alpha\beta})^{\mu}{}_{\nu} = g^{\mu\alpha}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}g^{\mu\beta} \tag{216}$$

are generators of Lorentz Transformations. The indices $\alpha\beta$ choose between generators $M^{\alpha\beta}$, while ${}^{\mu}{}_{\nu}$ in $(M^{\alpha\beta})^{\mu}{}_{\nu}$ are there to act on vector indices. Evident antisymmetry in the $\alpha\beta$ of (215) means that there are only six independent non-zero generators. Suppressing

Not examinable

Appendix VII: Generators of Lorentz Transformations II

Not examinable

the vector indices (taken to be ${}^{\mu}{}_{\nu}$) and taking $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$ the six independent generators are:

$$K_1 = M^{01} = -M^{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_2 = M^{02} = -M^{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_3 = M^{03} = -M^{30} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Not examinable

Appendix VII: Generators of Lorentz Transformations III

Not examinable

and

$$J_1 = M^{23} = -M^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}$$

$$J_2 = M^{31} = -M^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = M^{12} = -M^{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or, for short:

$$J_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$$

$$K_i = M^{0i}.$$

Not examinable

Appendix VII: Generators of Lorentz Transformations IV

Not examinable

[Aside: The generators obey commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad [J_i, K_j] = \epsilon_{ijk} K_k, \qquad [K_i, K_j] = -\epsilon_{ijk} J_k.$$

The first of these says that the J's generate rotations in three-dimensional space and fixes the overall sign of the Js. The second says the Ks transform as a vector under rotations. End of aside]

With above definition[1] one could represent and arbitrary Lorentz transformation (boost, rotation or both) as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

with

$$\Lambda^{\mu}{}_{\nu} = \left(\exp\left[\frac{1}{2} w_{\alpha\beta} (M^{\alpha\beta})^{\bullet}{}_{\bullet} \right] \right)^{\mu}{}_{\nu} \tag{217}$$

$$= \delta^{\mu}_{\nu} + \frac{1}{2} \omega_{\alpha\beta} (M^{\alpha\beta})^{\mu}{}_{\nu} + O(\omega^2) \tag{218}$$

using a set of parameters $w_{\alpha\beta}$ which may as well be antisymmetric in $\alpha\beta$ (since any symmetric part would not participate in (218) on account of the $(\alpha \leftrightarrow \beta)$-antisymmetry of $M^{\alpha\beta}$) and so contain six independent degrees of freedom (controlling three boosts and

Not examinable

Appendix VII: Generators of Lorentz Transformations V

Not examinable

three rotations) as required. In most of the proofs which follow we use the infinitesimal transformations to first order in ω since if some properties can be proved for infinitesimal transformations then it is always be possible to generalise that result to the exponential form for a finite transformation.

Not examinable

Appendix VII Why do $(M^{\alpha\beta})^{\mu}{}_{\nu}$ generate Lorentz transformations? I

Not examinable

Lorentz transformations should be continuously connected to the identity (which (218) is, when $\omega_{\alpha\beta} = 0$) and should preserve inner products. The transformation in Eq. (218) preserves inner products because:

$$
\begin{aligned}
x' \cdot y' &= g_{\mu\nu} x'^{\mu} y'^{\nu} \\
&= g_{\mu\nu} (\Lambda^{\mu}{}_{\sigma} x^{\sigma})(\Lambda^{\nu}{}_{\tau} y^{\tau}) \\
&= g_{\mu\nu} (\delta^{\mu}_{\sigma} + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}{}_{\sigma})(\delta^{\nu}_{\tau} + \frac{1}{2}\omega_{\bar{\alpha}\bar{\beta}}(M^{\bar{\alpha}\bar{\beta}})^{\nu}{}_{\tau}) x^{\sigma} y^{\tau} + O(\omega)^2 \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} + \omega_{\bar{\alpha}\bar{\beta}}(M^{\bar{\alpha}\bar{\beta}})_{\sigma\tau} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} + \omega_{\alpha\beta}(M^{\alpha\beta})_{\sigma\tau} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \qquad \text{relabelling} \\
&= \left[g_{\sigma\tau} + \frac{1}{2}\left(\omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} - \omega_{\alpha\beta}(M^{\alpha\beta})_{\tau\sigma} \right) \right] x^{\sigma} y^{\tau} + O(\omega^2) \qquad \text{antisymmetry of } M \\
&= g_{\sigma\tau} x^{\sigma} y^{\tau} + O(\omega^2) \\
&= x \cdot y + O(\omega^2).
\end{aligned}
$$

Not examinable

Appendix VII Why do $(M^{\alpha\beta})^{\mu}{}_{\nu}$ generate Lorentz transformations? II

Not examinable

If the above argument seems too abstract, a more concrete way of checking that we have generators of Lorentz transformations might instead be to compute

$$\exp\{(\eta K_1)\} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{219}$$

as this will be recognised by some as a boost in the positive x-direction with rapidity η (that is with $\cosh\eta = \gamma$ and $\sinh\eta = \beta\gamma$) while

$$\exp\{(\theta J_1)\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{220}$$

will be recognised by most as a rotation by an angle θ about the x-axis.

Not examinable

Appendix VII: Lorentz covariance of the Dirac equation I

Not examinable

If the Dirac Equation:

$$i\gamma^\mu \partial_\mu \psi = m\psi \tag{221}$$

is to be Lorentz covariant, there would have to exist a matrix $S(\Lambda)$ such that $\psi' = S(\Lambda)\psi$ is the solution of the Lorentz transformed Dirac Equation

$$i\gamma^\mu \partial'_\mu \psi' = m\psi'. \tag{222}$$

Equation (222) implies

$$i\gamma_\mu \partial'^\mu \psi' = m\psi' \tag{223}$$

and so

$$i\gamma_\mu \Lambda^\mu{}_\nu \partial^\nu S(\Lambda)\psi = mS(\Lambda)\psi \tag{224}$$

and so since $S(\Lambda)$ is independent of position

$$i\gamma_\mu S(\Lambda)\Lambda^\mu{}_\nu \partial^\nu \psi = S(\Lambda)m\psi \tag{225}$$

which using (221) becomes

Not examinable

$$i\gamma_\mu S(\Lambda)\Lambda^\mu{}_\nu \partial^\nu \psi = S(\Lambda)i\gamma^\mu \partial_\mu \psi$$

Appendix VII: Lorentz covariance of the Dirac equation II

Not examinable

and hence

$$i\gamma^{\mu}S(\Lambda)\Lambda_{\mu}{}^{\nu}\partial_{\nu}\psi = S(\Lambda)i\gamma^{\nu}\partial_{\nu}\psi$$

or

$$i\left[\gamma^{\mu}S(\Lambda)\Lambda_{\mu}{}^{\nu} - S(\Lambda)\gamma^{\nu}\right]\partial_{\nu}\psi = 0. \tag{226}$$

Therefore, if we can show that there exists a matrix $S(\Lambda)$ satisfying

$$\gamma^{\mu}S(\Lambda)\Lambda_{\mu}{}^{\nu} = S(\Lambda)\gamma^{\nu} \tag{227}$$

we will have found a solution to (226) and thus will have found that the Dirac Equation is Lorentz covariant as desired. Thought it would be entirely possible to work directly with (227) it is perhaps nicer to bring both S matrices to the left hand side

$$S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda)\Lambda_{\mu}{}^{\nu} = \gamma^{\nu}$$

and then use the identity

$$\Lambda_{\mu}{}^{\nu}\Lambda^{\sigma}{}_{\nu} \equiv \delta^{\sigma}_{\mu} \tag{228}$$

Not examinable

Appendix VII: Lorentz covariance of the Dirac equation III

Not examinable

so that (227) ends up being written in the more common and (perhaps) more suggestive and useful form:

$$S^{-1}(\Lambda)\gamma^{\sigma}S(\Lambda) = \Lambda^{\sigma}{}_{\nu}\gamma^{\nu}. \tag{229}$$

[Aside: Here is (for infinitesimal Lorentz transformations) a proof of the identity (228):

$$\begin{aligned}
\Lambda_{\mu}{}^{\nu}\Lambda^{\sigma}{}_{\nu} &= \left(g_{\mu}{}^{\nu} + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu}{}^{\nu}\right)\left(g^{\sigma}{}_{\nu} + \frac{1}{2}\omega_{\bar{\alpha}\bar{\beta}}(M^{\bar{\alpha}\bar{\beta}})^{\sigma}{}_{\nu}\right) + O(\omega^2) \\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\left[\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu}{}^{\sigma} + \omega_{\bar{\alpha}\bar{\beta}}(M^{\bar{\alpha}\bar{\beta}})^{\sigma}{}_{\mu}\right] + O(\omega^2) \\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\left[\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu}{}^{\sigma} + \omega_{\alpha\beta}(M^{\alpha\beta})^{\sigma}{}_{\mu}\right] + O(\omega^2) \qquad \text{(relabelling)} \\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})_{\mu}{}^{\sigma} + (M^{\alpha\beta})^{\sigma}{}_{\mu}\right] + O(\omega^2) \qquad \text{(factorising)} \\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})^{\tau\sigma} + (M^{\alpha\beta})^{\sigma\tau}\right] g_{\mu\tau} + O(\omega^2) \qquad \text{(tidying)} \\
&= \delta^{\sigma}_{\mu} + \frac{1}{2}\omega_{\alpha\beta}\left[(M^{\alpha\beta})^{\tau\sigma} - (M^{\alpha\beta})^{\tau\sigma}\right] g_{\mu\tau} + O(\omega^2) \qquad \text{(antisymmetry of } M\text{)} \\
&= \delta^{\sigma}_{\mu} + O(\omega^2).
\end{aligned}$$

Not examinable

Appendix VII: Lorentz covariance of the Dirac equation IV

Not examinable

End of aside]

Lemma

A valid choice of $S(\Lambda)$ (for an infinitesimal Lorentz transformation) is given by:

$$S(\Lambda) = 1 + \frac{1}{4}\omega_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta} + O(\omega^2). \tag{230}$$

Not examinable

Appendix VII: Lorentz covariance of the Dirac equation V

Proof.

$$
\begin{aligned}
S^{-1}(\Lambda)\gamma^\sigma S(\Lambda) &= \left(1-\frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)\gamma^\sigma\left(1+\frac{1}{4}\omega_{\bar\alpha\bar\beta}\gamma^{\bar\alpha}\gamma^{\bar\beta}\right)+O(\omega^2)\\
&= \gamma^\sigma+\frac{1}{4}\left(\omega_{\bar\alpha\bar\beta}\gamma^\sigma\gamma^{\bar\alpha}\gamma^{\bar\beta}-\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\gamma^\sigma\right)+O(\omega^2)\\
&= \gamma^\sigma+\frac{1}{4}\omega_{\alpha\beta}\left(\gamma^\sigma\gamma^\alpha\gamma^\beta-\gamma^\alpha\gamma^\beta\gamma^\sigma\right)+O(\omega^2)\\
&= \gamma^\sigma+\frac{1}{4}\omega_{\alpha\beta}\left((\gamma^\sigma\gamma^\alpha+\gamma^\alpha\gamma^\sigma)\gamma^\beta-\gamma^\alpha(\gamma^\sigma\gamma^\beta+\gamma^\beta\gamma^\sigma)\right)+O(\omega^2)\\
&= \gamma^\sigma+\frac{1}{4}\omega_{\alpha\beta}\left(2g^{\sigma\alpha}\gamma^\beta-\gamma^\alpha 2g^{\sigma\beta}\right)+O(\omega^2) \qquad \text{since } \{\gamma^\mu,\gamma^\nu\}\equiv 2g^{\mu\nu}\\
&= \left(\delta^\sigma_\nu+\frac{1}{2}\omega_{\alpha\beta}\left(g^{\sigma\alpha}\delta^\beta_\nu-\delta^\alpha_\nu g^{\sigma\beta}\right)\right)\gamma^\nu+O(\omega^2)\\
&= \left(\delta^\sigma_\nu+\frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^\sigma{}_\nu\right)\gamma^\nu+O(\omega^2) \qquad \text{using (216)}\\
&= \Lambda^\sigma{}_\nu\gamma^\nu+O(\omega^2) \qquad \text{using (218).}
\end{aligned}
$$

Appendix VII: Lorentz covariance of the Dirac equation VI

Not examinable

[Aside: Since $\gamma^\alpha\gamma^\beta = \frac{1}{2}\{\gamma^\alpha, \gamma^\beta\} + \frac{1}{2}[\gamma^\alpha, \gamma^\beta]$ we can also rewrite (230) in the more frequently seen (conventional) form:

$$S(\Lambda) = 1 + \frac{1}{8}\omega_{\alpha\beta}[\gamma^\alpha, \gamma^\beta] + O(\omega^2). \tag{231}$$

End of aside]

Not examinable

Appendix VII: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. I

Not examinable

Each of the expressions $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$ is of the form $\overline{\phi}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi$. To understand how any of them is affected by a Lorentz transformation it is therefore interesting to consider the following set of manipulations:[2]

$$\begin{aligned}
\overline{\phi'}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi' &= \overline{(S(\Lambda)\phi)}[\gamma^\mu\gamma^\nu\cdots\gamma^\tau](S(\Lambda)\psi) \\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0[\gamma^\mu S(\Lambda)S^{-1}(\Lambda)\gamma^\nu S(\Lambda)\cdots S^{-1}(\Lambda)\gamma^\tau]S(\Lambda)\psi \\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0 S(\Lambda)(S^{-1}(\Lambda)\gamma^\mu S(\Lambda))(S^{-1}(\Lambda)\gamma^\nu S(\Lambda))\cdots(S^{-1}(\Lambda)\gamma^\tau S(\Lambda))\psi \\
&= \phi^\dagger S^\dagger(\Lambda)\gamma^0 S(\Lambda)(\Lambda^\mu{}_\alpha\gamma^\alpha)(\Lambda^\nu{}_\beta\gamma^\gamma)\cdots(\Lambda^\tau{}_\lambda\gamma^\lambda)\psi \qquad \text{using (229)}
\end{aligned}$$

which itself suggests that if we can show that

$$S^\dagger(\Lambda)\gamma^0 S(\Lambda) = \gamma^0 \tag{232}$$

then we will have proved that

$$\overline{\phi'}\gamma^\mu\gamma^\nu\cdots\gamma^\tau\psi' = \overline{\phi}(\Lambda^\mu{}_\alpha\gamma^\alpha)(\Lambda^\nu{}_\beta\gamma^\gamma)\cdots(\Lambda^\tau{}_\lambda\gamma^\lambda)\psi$$

which will itself have showed that each of the expressions under consideration transforms like a tensor of the appropriate rank.

Not examinable

Appendix VII: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. II

Not examinable

We must therefore prove (232). To do so is a two-stage process. First we compute $S^\dagger(\Lambda)$. Then we combine it with $\gamma^0 S(\Lambda)$. Starting with (230):

$$\begin{aligned}
S^\dagger(\Lambda) &= \left[1+\frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\right]^\dagger + O(\omega^2) \\
&= 1+\frac{1}{4}\omega_{\alpha\beta}(\gamma^\alpha\gamma^\beta)^\dagger + O(\omega^2) \qquad (\omega_{\alpha\beta} \text{ are real}) \\
&= 1+\frac{1}{4}\omega_{\alpha\beta}(\gamma^\beta)^\dagger(\gamma^\alpha)^\dagger + O(\omega^2) \\
&= 1+\frac{1}{4}\omega_{\alpha\beta}(\gamma^0\gamma^\beta\gamma^0)(\gamma^0\gamma^\alpha\gamma^0) + O(\omega^2) \\
&= 1+\frac{1}{4}\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0 + O(\omega^2) \qquad (233)
\end{aligned}$$

Not examinable

Appendix VII: Transformation properties of $\overline{\phi}\psi$, $\overline{\phi}\gamma^\mu\psi$ and $\overline{\phi}\gamma^\mu\gamma^\nu\psi$. III

Not examinable

from which we can deduce (using (230)) that

$$
\begin{aligned}
S^\dagger(\Lambda)\gamma^0 S(\Lambda) &= \left(1+\frac{1}{4}\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0\right)\gamma^0\left(1+\frac{1}{4}\omega_{\bar{\alpha}\bar{\beta}}\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}}\right)+O(\omega^2)\\
&= \gamma^0+\frac{1}{4}\left(\omega_{\alpha\beta}\gamma^0\gamma^\beta\gamma^\alpha\gamma^0\gamma^0+\omega_{\bar{\alpha}\bar{\beta}}\gamma^0\gamma^{\bar{\alpha}}\gamma^{\bar{\beta}}\right)+O(\omega^2)\\
&= \gamma^0\left[1+\frac{1}{4}\left(\omega_{\alpha\beta}\gamma^\beta\gamma^\alpha+\omega_{\beta\alpha}\gamma^\beta\gamma^\alpha\right)\right]+O(\omega^2) \qquad ((\bar{\alpha},\bar{\beta})\to(\beta,\alpha))\\
&= \gamma^0\left[1+0\right]\psi+O(\omega^2) \qquad (\omega_{\alpha\beta}=-\omega_{\beta\alpha})\\
&= \gamma^0+O(\omega^2)
\end{aligned}
$$

verifying (232) as required. This completes our proof that:

- $\overline{\phi}\psi$ is Lorentz invariant scalar,
- $\overline{\phi}\gamma^\mu\psi$ transforms as a Lorentz vector, and
- $\overline{\phi}\gamma^\mu\gamma^\nu\psi$ transforms as a second-rank tensor, *etc.*

Not examinable

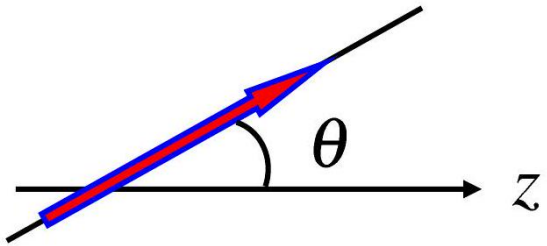
Appendix VIII: Spin-1 Rotation Matrices I

Not examinable

- Consider the spin-1 state with spin $+1$ along the axis defined by unit vector

$$\vec{n} = (\sin\theta, 0, \cos\theta)$$

- Spin state is an eigenstate of $\vec{n} \cdot \vec{S}$ with eigenvalue $+1$



$$(\vec{n}.\vec{S})|\psi\rangle = +1|\psi\rangle \tag{234}$$

- Express in terms of linear combination of spin 1 states which are eigenstates of S_z

$$|\psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

with

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

Not examinable

Appendix VIII: Spin-1 Rotation Matrices II

Not examinable

- (234) becomes:

$$(\sin\theta S_x + \cos\theta S_z)\left(\alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle\right) = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle \tag{235}$$

- Write S_x in terms of ladder operators $S_x = \frac{1}{2}(S_+ + S_-)$ where

$$S_+|1,1\rangle = 0 \quad S_+|1,0\rangle = \sqrt{2}|1,1\rangle \quad S_+|1,-1\rangle = \sqrt{2}|1,0\rangle$$

$$S_-|1,1\rangle = \sqrt{2}|1,0\rangle \quad S_-|1,0\rangle = \sqrt{2}|1,-1\rangle \quad S_-|1,-1\rangle = 0$$

- from which we find $S_x|1,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$
- (235) becomes

$$S_x|1,0\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$

$$S_x|1,-1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$$

$$\sin\theta\left[\frac{\alpha}{\sqrt{2}}|1,0\rangle + \frac{\beta}{\sqrt{2}}|1,-1\rangle + \frac{\beta}{\sqrt{2}}|1,1\rangle + \frac{\gamma}{\sqrt{2}}|1,0\rangle\right] +$$
$$\alpha\cos\theta|1,1\rangle - \gamma\cos\theta|1,-1\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

Not examinable

Appendix VIII: Spin-1 Rotation Matrices III

Not examinable

- which gives

$$\left.\begin{aligned} \beta\tfrac{\sin\theta}{\sqrt{2}} + \alpha\cos\theta &= \alpha \\ (\alpha+\gamma)\tfrac{\sin\theta}{\sqrt{2}} &= \beta \\ \beta\tfrac{\sin\theta}{\sqrt{2}} - \gamma\cos\theta &= \gamma \end{aligned}\right\}.$$

- Using $\alpha^2+\beta^2+\gamma^2=1$ the above equations yield

$$\alpha = \frac{1}{\sqrt{2}}(1+\cos\theta) \quad \beta = \frac{1}{\sqrt{2}}\sin\theta \quad \gamma = \frac{1}{\sqrt{2}}(1-\cos\theta)$$

- hence

$$\psi = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle.$$

- The coefficients α,β,γ are examples of what are known as quantum mechanical rotation matrices. The express how angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction $d^j_{m',m}(\theta)$.
- For spin-1 ($j=1$) we have just shown that

$$d^1_{1,1}(\theta) = \frac{1}{2}(1+\cos\theta) \quad d^1_{0,1}(\theta) = \frac{1}{\sqrt{2}}\sin\theta \quad d^1_{-1,1}(\theta) = \frac{1}{2}(1-\cos\theta).$$

Not examinable

Appendix VIII: Spin-1 Rotation Matrices IV

Not examinable

- For spin-1/2 it is straightforward to show

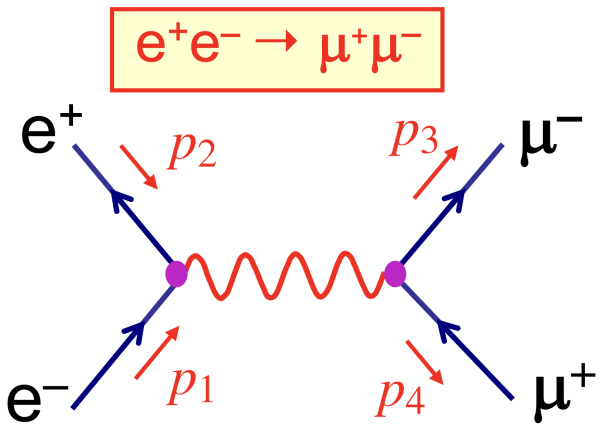
$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \quad d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\frac{\theta}{2}.$$

Not examinable

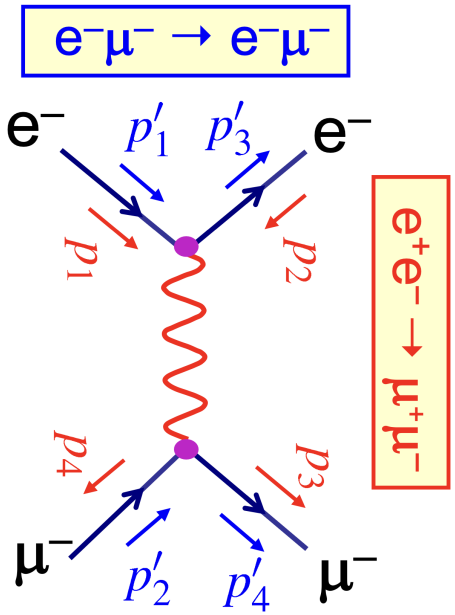
Appendix IX: Crossing Symmetry

Not examinable

- Having derived the Lorentz invariant matrix element for $e^-e^+ \to \mu^-\mu^+$ 'rotate' the diagram to correspond to $e^-\mu^- \to e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element !



rotates to



- The transformation: $p_1 \to p_1'$; $p_2 \to -p_3'$; $p_3 \to p_4'$; $p_4 \to -p_2'$ changes the spin averaged matrix element (see page 119) for

$e^-e^+ \to \mu^-\mu^+$ to that for $e^-\mu^- \to e^-\mu^-$:

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_3)^2 + (p_1.p_4)^2}{(p_1.p_2)^2} \quad \to \quad \langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1'.p_4')^2 + (p_1'.p_2')^2}{(p_1'.p_3')^2}.$$

Not examinable

Appendix X: the $SU(2)$ anti-quark representation

Not examinable

Define an anti-quark doublet

$$\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$$

from which it follows that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q} = \begin{pmatrix} u^* \\ d^* \end{pmatrix}. \tag{236}$$

The quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ transforms as $\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix}$ which complex conjugates to

$$\begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = U^* \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

which using (236) can be re-written as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}' = U^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}.$$

Therefore, multiplying both sides of the last equation by the inverse of its left-most matrix, we see that $\bar{q}$ transforms as follows:

$$\bar{q}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{q}. \tag{237}$$

Not examinable

Appendix X: the $SU(2)$ anti-quark representation (cont.)

Not examinable

An arbitrary 2×2 unitary matrix with unit determinant can always be written in the form

$$U = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix}$$

provided that one chooses c_{11} and c_{12} such that $|c_{11}|^2 + |c_{12}|^2 = 1$. Therefore, (237) can be re-written to express an arbitrary $SU(2)$ trasformation of $\bar{q}$ as:

$$\begin{aligned} \bar{q}' &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ -c_{12} & c_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q} \\ &= \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix} \bar{q} \\ &= U\bar{q} \end{aligned}$$

which proves that the anti-quark doublet $\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$ transforms in the same way as the quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ – thus allowing us to use the same ladder operators on q and $\bar{q}$.

This is a special property of SU(2). For $\mathrm{SU}(3)$ there is no analogous representation of the anti-quarks.

Appendix XI: Electromagnetism

Not examinable

⋆ In Heaviside-Lorentz units $\varepsilon_0 = \mu_0 = c = 1$ Maxwell's equations in the vacuum become

$$\vec{\nabla} \cdot \vec{E} = \rho; \quad \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \quad \vec{\nabla} \cdot \vec{B} = 0; \quad \vec{\nabla} \wedge \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

⋆ The electric and magnetic fields can be expressed in terms of scalar and vector potentials

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi; \quad \vec{B} = \vec{\nabla} \wedge \vec{A}$$

⋆ In terms of the 4-vector potential $A^\mu = (\phi, \vec{A})$ and the 4-vector current $j^\mu = (\rho, \vec{J})$ Maxwell's equations can be expressed in the covariant form:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{238}$$

where $F^{\mu\nu}$ is the anti-symmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{239}$$

Not examinable

-Combining (238) and (239)

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = j^\nu$$

Appendix XI: Electromagnetism (cont.)

Not examinable

which can be written

$$\Box^2 A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu \tag{240}$$

where the D'Alembertian operator

$$\Box^2 = \partial_\nu \partial^\nu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

-Acting on (240) with ∂_V gives

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial_\nu \partial^\nu A^\mu = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{Conservation of Electric Charge}$$

Not examinable

Appendix XII: Gauge Invariance

Not examinable

- Conservation laws are associated with symmetries. Here the symmetry is the GAUGE INVARIANCE of electro-magnetism

⋆ The electric and magnetic fields are unchanged for the gauge transformation:

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi; \quad \phi \to \phi' = \phi - \frac{\partial \chi}{\partial t}$$

where $\chi = \chi(t, \vec{x})$ is any finite differentiable function of position and time

⋆ In 4-vector notation the gauge transformation can be expressed as:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$$

Not examinable

Appendix XII: Gauge Invariance (cont.)

Not examinable

- Using the fact that the physical fields are gauge invariant, choose χ to be a solution of

⋆ In this case we have

$$\partial^\mu A'_\mu = \partial^\mu \left(A_\mu + \partial_\mu \chi\right) = \partial^\mu A_\mu + \Box^2 \chi = 0$$

⋆ Dropping the prime we have a chosen a gauge in which

$$\partial_\mu A^\mu = 0 \qquad \text{The Lorentz Condition}$$

- With the Lorentz condition, equation (240) becomes:

$$\boxed{\Box^2 A^\mu = j^\mu}. \tag{241}$$

- Having imposed the Lorentz condition we still have freedom to make a further gauge transformation, i.e.

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$$

where $\Lambda(t, \vec{x})$ is any function that satisfies

$$\Box^2 \Lambda = 0 \tag{242}$$

⋆ Clearly (241) remains unchanged, in addition the Lorentz condition still holds:

Not examinable

$$\partial^\mu A'_\mu = \partial^\mu \left(A_\mu + \partial_\mu \Lambda\right) = \partial^\mu A_\mu + \Box^2 \Lambda = \partial^\mu A_\mu = 0$$

Appendix XIII: Photon Polarization

Not examinable

- For a free photon (i.e. $j^\mu = 0$) equation (241) becomes

$$\boxed{\Box^2 A^\mu = 0} \tag{243}$$

(note have chosen a gauge where the Lorentz condition is satisfied)

- Equation (242) has solutions (i.e. the wave-function for a free photon)

$$A^\mu = \varepsilon^\mu(q) e^{-iq\cdot x}$$

where ε^μ is the four-component polarization vector and q is the photon four-momentum

$$0 = \Box^2 A^\mu = -q^2 \varepsilon^\mu e^{-iq\cdot x}$$

$$\Rightarrow q^2 = 0$$

- Hence equation (243) describes a massless particle.
- But the solution has four components - might ask how it can describe a spin-1 particle which has three polarization states?
- But for (242) to hold we must satisfy the Lorentz condition:

$$0 = \partial_\mu A^\mu = \partial_\mu \left(\varepsilon^\mu e^{-iq\cdot x}\right) = \varepsilon^\mu \partial_\nu \left(e^{-iq\cdot x}\right) = -i\varepsilon^\mu q_\mu e^{-iq\cdot x}$$

Hence the Lorentz condition gives

$$q_\mu \varepsilon^\mu = 0 \tag{244}$$

Not examinable

Appendix XIII: Photon Polarization (cont.)

Not examinable

⋆ However, in addition to the Lorentz condition still have the addional gauge freedom of $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ with (242) $\Box^2 \Lambda = 0$
-Choosing $\Lambda = iae^{-iq\cdot x}$ which has $\Box^2 \Lambda = q^2 \Lambda = 0$

$$\begin{aligned} A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda &= \varepsilon_\mu e^{-iq\cdot x} + ia\partial_\mu e^{-iq\cdot x} \\ &= \varepsilon_\mu e^{-iq\cdot x} + ia\left(-iq_\mu\right) e^{-iq\cdot x} \\ &= \left(\varepsilon_\mu + aq_\mu\right) e^{-iq\cdot x} \end{aligned}$$

⋆ Hence the electromagnetic field is left unchanged by

$$\varepsilon_\mu \to \varepsilon'_\mu = \varepsilon_\mu + aq_\mu$$

⋆ Hence the two polarization vectors which differ by a mulitple of the photon four-momentum describe the same photon. Choose a such that the time-like component of ε_μ is zero, i.e. $\varepsilon_0 \equiv 0$
⋆ With this choice of gauge, which is known as the COULOMB GAUGE, the Lorentz condition (244) gives

$$\vec{\varepsilon} \cdot \vec{q} = 0$$

i.e. only 2 independent components, both transverse to the photons momentum

Not examinable

Appendix XIII: Photon Polarization (cont.)

Not examinable

⋆ A massless photon has two transverse polarisation states. For a photon travelling in the z direction these can be expressed as the transversly polarized states:

$$\varepsilon_1^\mu = (0,1,0,0); \quad \varepsilon_2^\mu = (0,0,1,0)$$

⋆ Alternatively take linear combinations to get the circularly polarized states

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

- It can be shown that the ε_+state corresponds the state in which the photon spin is directed in the $+\mathbf{z}$ direction, i.e. $S_z = +1$

Not examinable

Appendix XIV: Massive Spin-1 particles

Not examinable

- For a massless photon we had (before imposing the Lorentz condition) we had from equation (240):

$$\Box^2 A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu$$

⋆ The Klein-Gordon equation for a spin-0 particle of mass m is

$$\left(\Box^2 + m^2\right)\phi = 0$$

suggestive that the appropriate equations for a massive spin-1 particle can be obtained by replacing $\Box^2 \to \Box^2 + m^2$

- This is indeed the case, and from QFT it can be shown that for a massive spin 1 particle equation (240): becomes

$$\left(\Box^2 + m^2\right) B^\mu - \partial^\mu \left(\partial_\nu B^\nu\right) = j^\mu$$

- Therefore a free particle must satisfy

$$\left(\Box^2 + m^2\right) B^\mu - \partial^\mu \left(\partial_\nu B^\nu\right) = 0 \tag{245}$$

Not examinable

Appendix XIV: Massive Spin-1 particles (cont.)

Not examinable

- Acting on equation (245) with ∂_ν gives

$$\left(\Box^2 + m^2\right)\partial_\mu B^\mu - \partial_\mu\partial^\mu\left(\partial_\nu B^\nu\right) = 0$$
$$\left(\Box^2 + m^2\right)\partial_\mu B^\mu - \Box^2\left(\partial_\nu B^\nu\right) = 0$$
$$m^2\partial_\mu B^\mu = 0 \tag{246}$$

- Hence, for a massive spin-1 particle, unavoidably have $\partial_\mu B^\mu = 0$; note this is not a relation that reflects to choice of gauge.

-Equation (245) becomes

$$\boxed{\left(\Box^2 + m^2\right)B^\mu = 0}: \tag{247}$$

$\star$ For a free spin-1 particle with 4-momentum, p^μ, equation (247): admits solutions

$$B_\mu = \varepsilon_\mu e^{-ip.x}$$

- Substituting into equation (246) gives

$$p_\mu\varepsilon^\mu = 0$$

$\star$ The four degrees of freedom in ε^μ are reduced to three, but for a massive particle, equation (247) does not allow a choice of gauge and we can not reduce the number of degrees of freedom any further.

Not examinable

Appendix XIV: Massive Spin-1 particles (cont.)

Not examinable

⋆ Hence we need to find three orthogonal polarisation states satisfying

$$p_\mu \varepsilon^\mu = 0 \tag{248}$$

⋆ For a particle travelling in the z direction, can still admit the circularly polarized states.

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0); \quad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0, 1, i, 0)$$

⋆ Writing the third state as

$$\varepsilon_L^\mu = \frac{1}{\sqrt{\alpha^2 + \beta^2}}(\alpha, 0, 0, \beta)$$

equation (248) gives $\alpha E - \beta p_z = 0$

$$\Rightarrow \quad \varepsilon_L^\mu = \frac{1}{m}(p_z, 0, 0, E)$$

- This longitudinal polarisation state is only present for massive spin-1 particles, i.e. there is no analogous state for a free on-shell photon.

Not examinable

Appendix XV: Local Gauge Invariance

Not examinable

⋆ The Dirac equation for a charged particle in an electro-magnetic field can be obtained from the free particle wave-equation by making the minimal substitution

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi \quad (q = \text{ charge })$$

In QM: $i\partial_\mu \to i\partial_\mu - qA_\mu$ and the Dirac equation becomes

$$\gamma^\mu \left(i\partial_\mu - qA_\mu\right)\psi - m\psi = 0$$

- In Appendix XII: saw that the physical EM fields where invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

⋆ Under this transformation the Dirac equation becomes

$$\gamma^\mu \left(i\partial_\mu - qA_\mu + q\partial_\mu \chi\right)\psi - m\psi = 0$$

which is not the same as the original equation. If we require that the Dirac equation is invariant under the Gauge transformation then under the gauge transformation we need to modify the wave-functions

Not examinable

$$\psi \to \psi' = \psi e^{iq\chi}$$

Appendix XV: Local Gauge Invariance (cont.)

Not examinable

⋆ To prove this, applying the gauge transformation :

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi \quad \psi \to \psi' = \psi e^{iq\chi}$$

to the original Dirac equation gives

$$\gamma^\mu \left(i\partial_\mu - qA_\mu + q\partial_\mu \chi\right) \psi e^{iq\chi} - m\psi e^{iq\chi} = 0 \tag{249}$$

⋆ But

$$i\partial_\mu \left(\psi e^{iq\chi}\right) = ie^{iq\chi}\partial_\mu \psi - q\left(\partial_\mu \chi\right) e^{iq\chi}\psi$$

⋆ Equation (249) becomes

$$\begin{aligned}
&\gamma^\mu e^{iq\chi}\left(i\partial_\mu - qA_\mu + q\partial_\mu \chi - q\partial_\mu \chi\right)\psi - m\psi e^{iq\chi} = 0 \\
&\Rightarrow \quad \gamma^\mu e^{iq\chi}\left(i\partial_\mu - qA_\mu\right)\psi - m\psi e^{iq\chi} = 0 \\
&\Longrightarrow \\
&\gamma^\mu \left(i\partial_\mu - qA_\mu\right)\psi - m\psi = 0
\end{aligned}$$

which is the original form of the Dirac equation

Not examinable

Appendix XVI : Local Gauge Invariance 2

Not examinable

⋆ Reverse the argument of Appendix XV. Suppose there is a fundamental symmetry of the universe under local phase transformations

$$\psi(x) \to \psi'(x) = \psi(x)e^{iq\chi(x)}$$

- Note that the local nature of these transformations: the phase transformation depends on the space-time coordinate $x = (t, \vec{x})$

⋆ Under this transformation the free particle Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

becomes $i\gamma^\mu \partial_\mu \left(\psi e^{iq\chi}\right) - m\psi e^{iq\chi} = 0$

$$ie^{iq\chi}\gamma^\mu \left(\partial_\mu \psi + iq\psi\partial_\mu \chi\right) - m\psi e^{iq\chi} = 0$$
$$i\gamma^\mu \left(\partial_\mu + iq\partial_\mu \chi\right)\psi - m\psi = 0$$

Local phase invariance is not possible for a free theory, i.e. one without interactions

- To restore invariance under local phase transformations have to introduce a massless "gauge boson" A^μ which transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

and make the substitution

Not examinable

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu$$

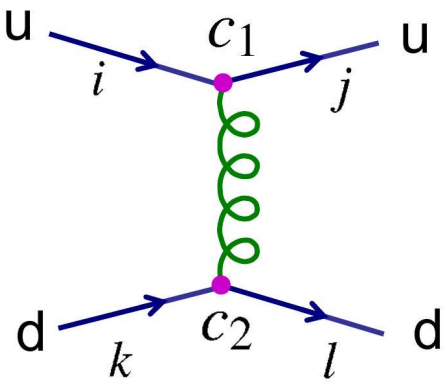
Appendix XVII: Alternative evaluation of colour factors

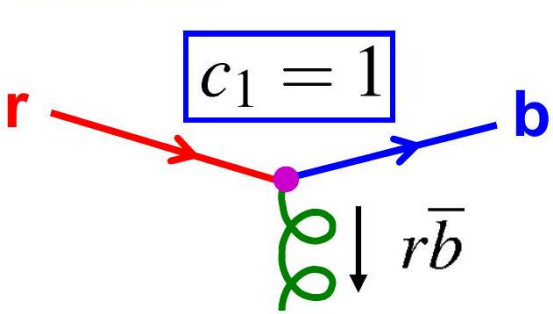
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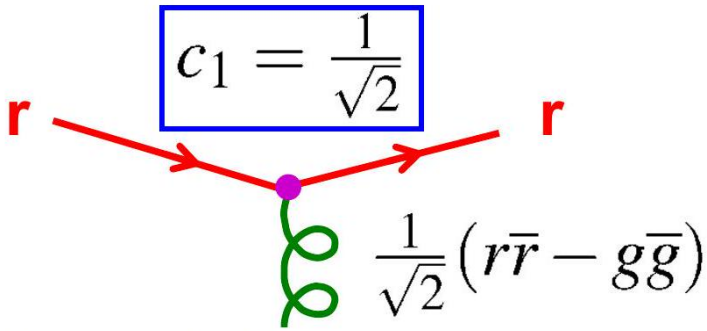
⋆ The colour factors can be obtained (more intuitively) as follows :

-Write $C(ik \to jl) = \frac{1}{2} c_1 c_2$

-Where the colour coefficients at the two vertices depend on the quark and gluon colours







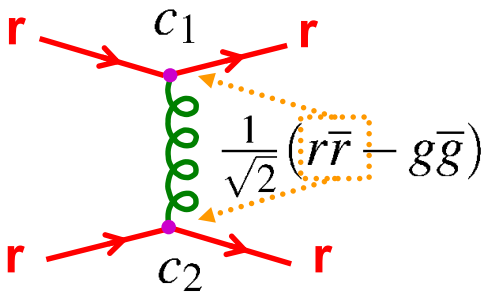
-Sum over all possible exchanged gluons conserving colour at both vertices

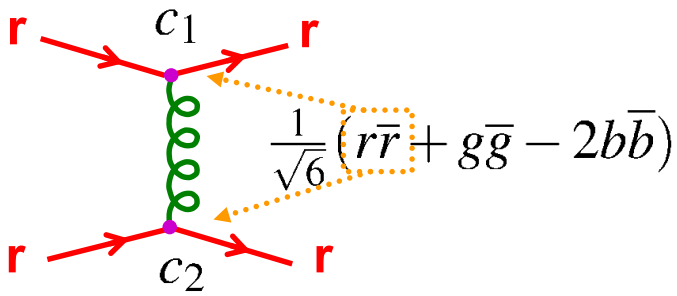
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Appendix XVII: Alternative evaluation of colour factors (cont.)

① Configurations involving a single colour

e.g. $rr \to rr$ **: two possible exchanged gluons**

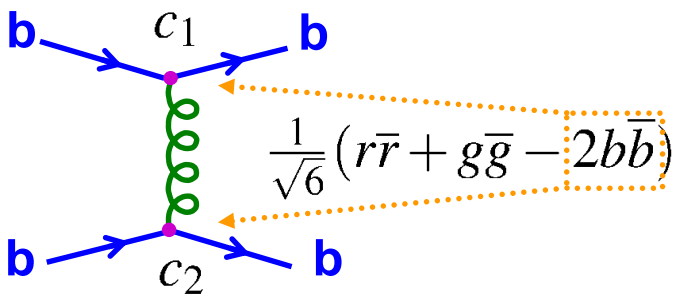




$$c_1 = c_2 = \frac{1}{\sqrt{2}} \qquad c_1 = c_2 = \frac{1}{\sqrt{6}}$$

$$C(rr \to rr) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{3}$$

e.g. $bb \to bb$ **: only one possible exchanged gluon**

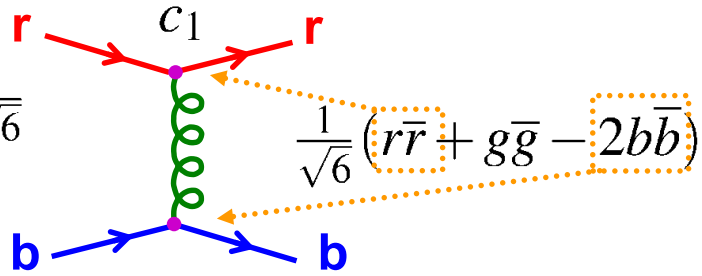


$$c_1 = c_2 = -\frac{2}{\sqrt{6}}$$

$$\Rightarrow \quad C(bb \to bb) = \frac{1}{2}\left(\frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = \frac{1}{3}$$

Appendix XVII: Alternative evaluation of colour factors (cont.)

② Other configurations where quarks don't change colour

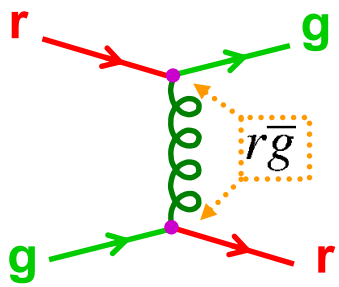


$$c_1 = \frac{1}{\sqrt{6}} \qquad \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$$

$$c_2 = -\frac{2}{\sqrt{6}}$$

$$C(rb \to rb) = \frac{1}{2}\left(-\frac{1}{\sqrt{6}}\frac{2}{\sqrt{6}}\right) = -\frac{1}{6}$$

③ Configurations where quarks swap colours



$$c_1 = c_2 = 1$$

$$C(rg \to gr) = \frac{1}{2}$$

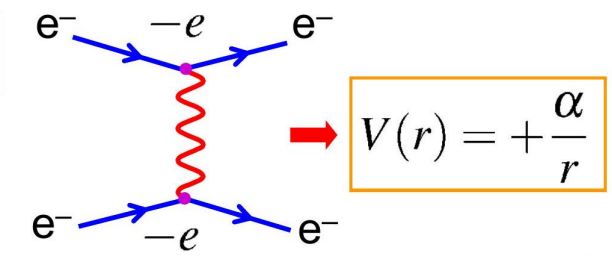
Appendix XVIII: Colour Potentials

-Previously argued that gluon self-interactions lead to a $+\lambda r$ long-range potential and that this is likely to explain colour confinement

- Have yet to consider the short range potential - i.e. for quarks in mesons and baryons does QCD lead to an attractive potential?

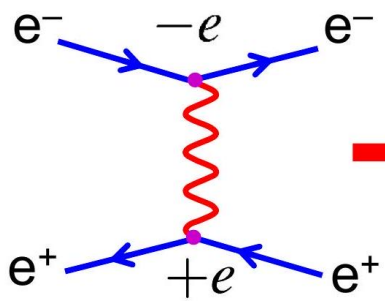
-Analogy with QED: (NOTE this is very far from a formal proof)

QED



$$V(r) = +\frac{\alpha}{r}$$

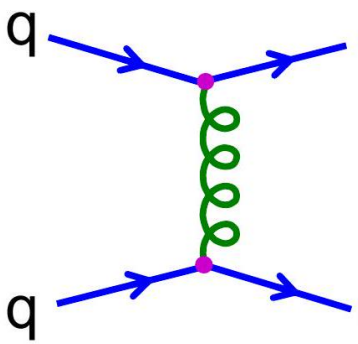
Repulsive Potential



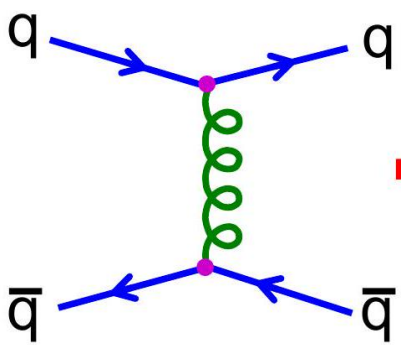
$$V(r) = -\frac{\alpha}{r}$$

Attractive Potential

⋆ by analogy with QED expect potentials of form



$$V(r) = +C\frac{\alpha_S}{r}$$



$$V(r) = -C\frac{\alpha_S}{r}$$

⋆ Whether it is a attractive or repulsive potential depends on sign of colour factor

Appendix XVIII: Colour Potentials (cont.)

Not examinable

⋆ Consider the colour factor for a q $\bar{q}$ system in the colour singlet state:

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

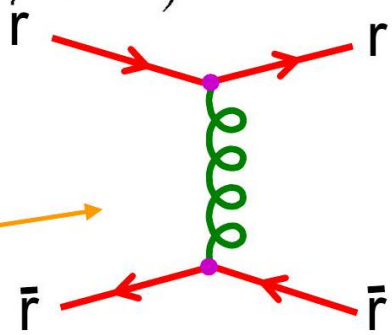
with colour potential $\langle V_{q\bar{q}} \rangle = \langle \psi | V_{\rm QCD} | \psi \rangle$

$$\Rightarrow \quad \langle V_{q\bar{q}} \rangle = \frac{1}{3}\left(\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle + \ldots + \langle r\bar{r}|V_{\rm QCD}|b\bar{b}\rangle + \ldots\right)$$

owing the QED analogy:

$$\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle = -C(r\bar{r} \to r\bar{r})\frac{\alpha_S}{r}$$

hich is the term arising from $r\bar{r} \to r\bar{r}$



-Have 3 terms like $r\bar{r} \to r\bar{r}, b\bar{b} \to b\bar{b}, \ldots$ and 6 like $r\bar{r} \to g\bar{g}, r\bar{r} \to b\bar{b}, \ldots$

$$\langle V_{q\bar{q}} \rangle = -\tfrac{1}{3}\tfrac{\alpha_S}{r}[3 \times C(r\bar{r} \to r\bar{r}) + 6 \times C(r\bar{r} \to g\bar{g})] = -\tfrac{1}{3}\tfrac{\alpha_S}{r}\left[3 \times \tfrac{1}{3} + 6 \times \tfrac{1}{2}\right]$$
$$\longrightarrow \langle V_{q\bar{q}} \rangle = -\tfrac{4}{3}\tfrac{\alpha_S}{r} \quad \text{NEGATIVE} \Rightarrow \text{ATTRACTIVE}$$

-The same calculation for a q $\bar{q}$ colour octet state, e.g. $r\bar{g}$ gives a positive repulsive potential: $C(r\bar{g} \longrightarrow r\bar{g}) = -\frac{1}{6}$

⋆ Whilst not a formal proof, it is comforting to see that in the colour singlet $q\bar{q}$ state the QCD potential is indeed attractive.

Not examinable

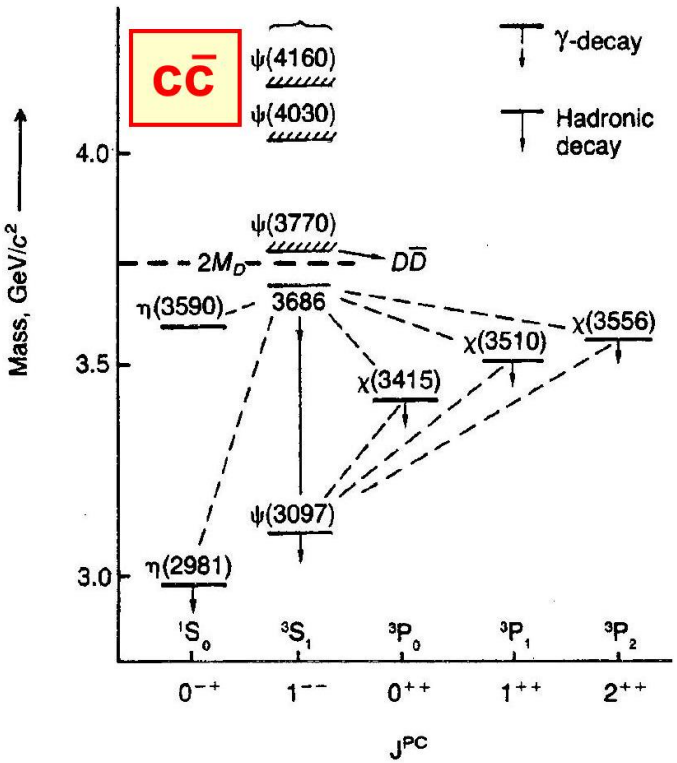
Appendix XVIII: Colour Potentials (cont.)

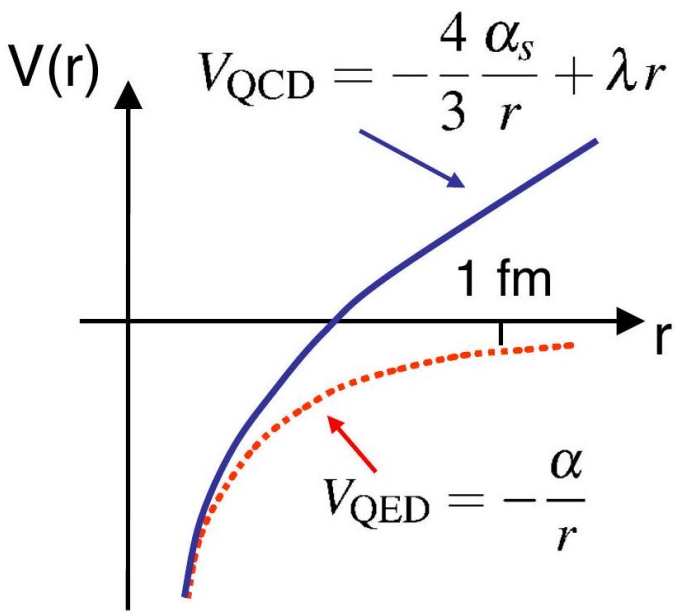
Not examinable

* Combining the short-range QCD potential with the linear long-range term discussed previously:

$$V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s}{r} + \lambda r$$

⋆ This potential is found to give a good description of the observed charmonium (cc) and bottomonium (bb) bound states



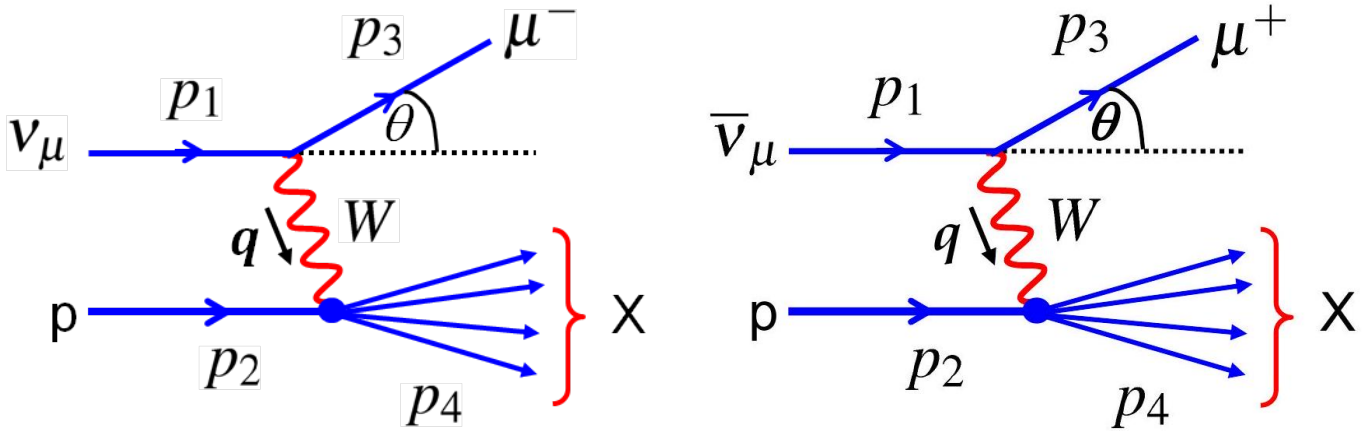


NOTE:
·c, b are heavy quarks -approx. non-relativistic - orbit close toge
- probe $1/\mathrm{r}$ part of $\mathrm{V_{QCD}}$

Agreement of data with prediction provides strong evidence that V_{QCD} has the Expected form

Appendix XIX: Deep-Inelastic Neutrino Scattering

Not examinable



- Two steps:
  - First write down most general cross section in terms of structure functions.
  - Then evaluate expressions in the quark-parton model.
- QED Revisited:
  - In the limit $s \gg M^2$the most general electro-magnetic deep-inelastic cross section (from single photon exchange) can be written (Eq. (101) of Handout 6) as

$$\frac{d^2\sigma_{e^\pm p}}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\,\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right].$$

  - For neutrino scattering typically measure the energy of the produced muon $E_\mu = E_\nu(1-y)$ and differential cross-sections expressed in terms of $dx\,dy$
  - $Q^2 = (s - M^2)xy \approx sxy \;\rightarrow$

$$\frac{d^2\sigma}{dx\,dy} = \left|\frac{dQ^2}{dy}\right| \frac{d^2\sigma}{dx\,dQ^2} = sx\,\frac{d^2\sigma}{dx\,dQ^2}.$$

Not examinable

Appendix XIX: Deep-Inelastic Neutrino Scattering (cont.)

Not examinable

- In the limit $s \gg M^2$ the general Electro-magnetic DIS cross section can be written

$$\frac{d^2\sigma^{e^{\pm}p}}{dx\,dy} = \frac{4\pi\alpha^2 s}{Q^4}\left[(1-y)\,F_2(x,Q^2) + y^2 x F_1(x,Q^2)\right]. \qquad (250)$$

- NOTE: This is the most general Lorentz Invariant parity conserving expression
- For neutrino DIS parity is violated and the general expression includes an additional term to allow for parity violation. New structure function: $F_3(x,Q^2)$:
  $\nu_\mu p \to \mu^- X$

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\nu p}(x,Q^2) + y^2 x F_1^{\nu p}(x,Q^2) + y\left(1-\frac{y}{2}\right)x F_3^{\nu p}(x,Q^2)\right]$$

- For anti-neutrino scattering new structure function enters with opposite sign
  $\bar{\nu}_\mu p \to \mu^+ X$

$$\frac{d^2\sigma^{\bar{\nu} p}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\bar{\nu} p}(x,Q^2) + y^2 x F_1^{\bar{\nu} p}(x,Q^2) - y\left(1-\frac{y}{2}\right)x F_3^{\bar{\nu} p}(x,Q^2)\right]$$

- Similarly for neutrino-neutron scattering
  $\nu_\mu n \to \mu^- X$

$$\frac{d^2\sigma^{\nu n}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\nu n}(x,Q^2) + y^2 x F_1^{\nu n}(x,Q^2) + y\left(1-\frac{y}{2}\right)x F_3^{\nu n}(x,Q^2)\right]$$

  $\bar{\nu}_\mu n \to \mu^+ X$

$$\frac{d^2\sigma^{\bar{\nu} n}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\bar{\nu} n}(x,Q^2) + y^2 x F_1^{\bar{\nu} n}(x,Q^2) - y\left(1-\frac{y}{2}\right)x F_3^{\bar{\nu} n}(x,Q^2)\right]$$

Not examinable

Appendix XIX: Neutrino Interaction Structure Functions

Not examinable

- In terms of the parton distribution functions we found (117):

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2}{\pi} sx\left[d(x) + (1-y)^2\bar{u}(x)\right]$$

- Compare coefficients of y with the general Lorentz Invariant form (250) and assume Bjorken scaling, i.e. $F(x, Q^2) \to F(x)$

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[(1-y)\,F_2^{\nu p}(x) + y^2 x F_1^{\nu p}(x) + y\left(1-\frac{y}{2}\right)xF_3^{\nu p}(x)\right]$$

- Re-writing (117):

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2}{2\pi} s\left[2xd(x) + 2x\bar{u}(x) - 4xy\bar{u}(x) + 2xy^2\bar{u}(x)\right]$$

and equating powers of y

$$\begin{aligned} 2xd + 2x\bar{u} &= F_2 \\ -4x\bar{u} &= -F_2 + xF_3 \\ 2\bar{u} &= F_1 - xF_3/2 \end{aligned}$$

gives:

$$F_2^{\nu p} = 2xF_1^{\nu p} = 2x[d(x) + \bar{u}(x)]$$
$$xF_3^{\nu p} = 2x[d(x) - \bar{u}(x)].$$

Not examinable

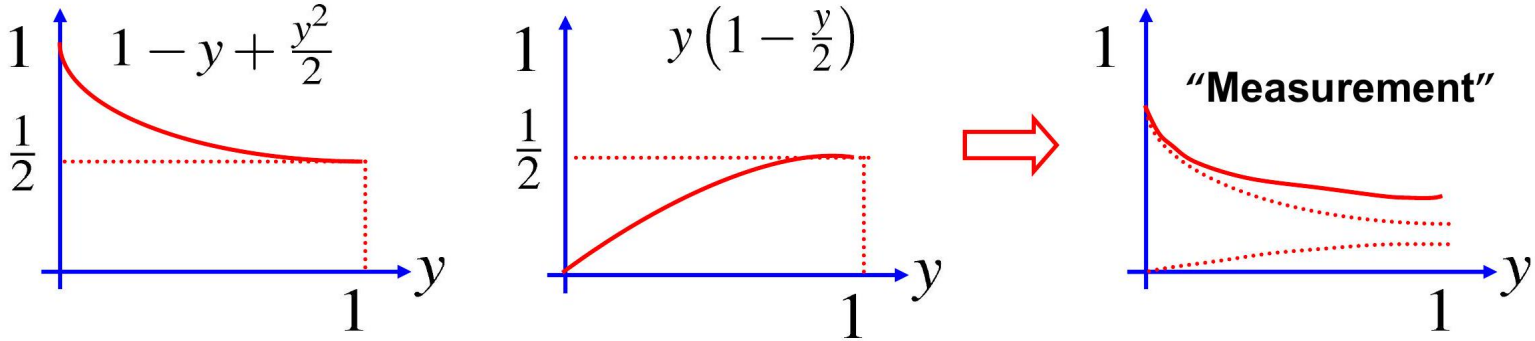
Appendix XIX: Deep-Inelastic Neutrino Scattering (cont.)

Not examinable

- NOTE: again we get the Callan-Gross relation $F_2 = 2xF_1$.
- No surprise, underlying process is scattering from point-like spin-1/2 quarks

$$\frac{d^2\sigma^{\nu p}}{dx\,dy} = \frac{G_F^2 s}{2\pi}\left[\left(1-y+\frac{y^2}{2}\right)F_2^{\nu p}(x) + y\left(1-\frac{y}{2}\right)xF_3^{\nu p}(x)\right]$$

- Experimentally measure F_2 and F_3 from y distributions at fixed x
  - Different y dependencies (from different rest frame angular distributions) allow contributions from the two structure functions to be measured



  - Then use $F_2^{\nu p} = 2x[d(x)+\bar{u}(x)]$ and $F_3^{\nu p} = 2[d(x)-\bar{u}(x)]$ $\rightarrow d(x)$ and $\bar{u}(x)$ separately

Not examinable

Appendix XIX: Deep-Inelastic Neutrino Scattering (cont.)

Not examinable

- Neutrino experiments require large detectors (often iron) i.e. isoscalar target
  $F_2^{\nu N} = 2xF_1^{\nu N} = \frac{1}{2}\left(F_2^{\nu p} + F_2^{\nu n}\right) = x[u(x) + d(x) + \bar{u}(x) + \bar{d}(x)]$
  $xF_3^{\nu N} = \frac{1}{2}\left(xF_3^{\nu p} + xF_3^{\nu n}\right) = x[u(x) + d(x) - \bar{u}(x) - \bar{d}(x)]$
- For electron – nucleon scattering: $F_2^{ep} = 2xF_1^{ep} = x[\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\bar{u}(x) + \frac{1}{9}\bar{d}(x)]$
  $F_2^{en} = 2xF_1^{en} = x[\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\bar{d}(x) + \frac{1}{9}\bar{u}(x)]$ $F_2^{\nu N} = \frac{18}{5}F_2^{eN}$
- For an isoscalar target

$$F_2^{eN} = \frac{1}{2}\left(F_2^{ep} + F_2^{en}\right) = \frac{5}{18}x[u(x) + d(x) + \bar{u}(x) + \bar{d}(x)]$$

- Note that the factor $\frac{5}{18} = \frac{1}{2}\left(q_u^2 + q_d^2\right)$ and by comparing neutrino to electron scattering structure functions measure the sum of quark charges

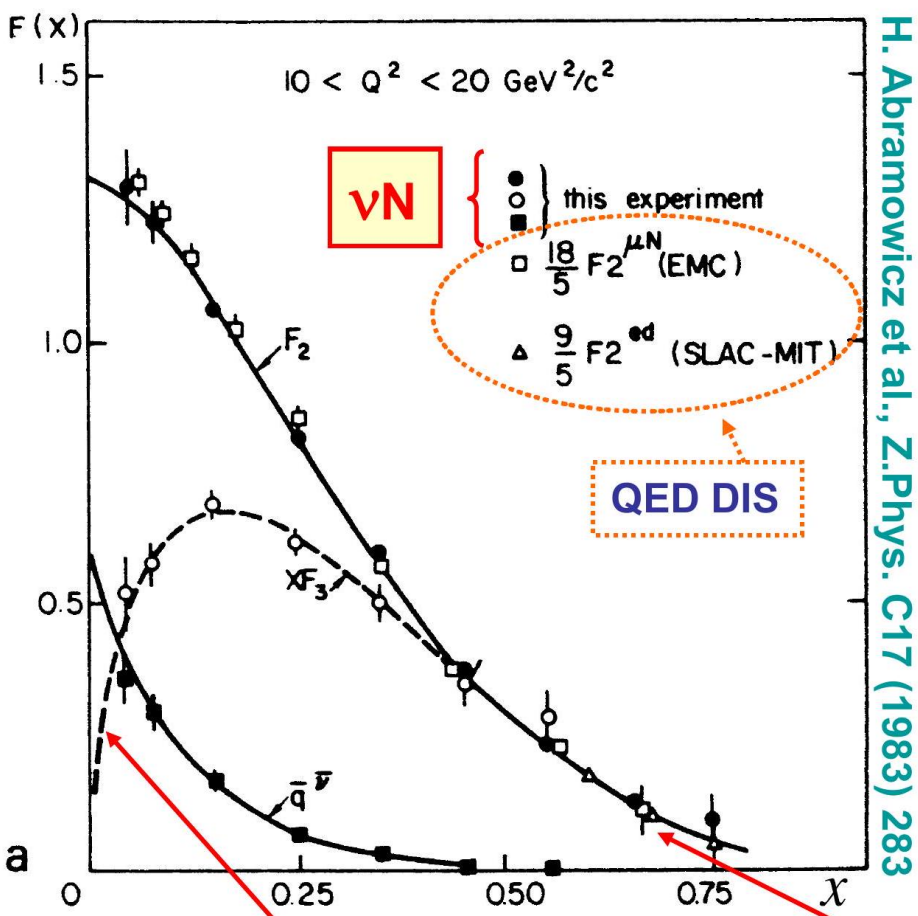
Experiment: 0.29 ± 0.02

Not examinable

Appendix XIX: Measurements of F2(x) and F3(x)

Not examinable

- CHDS Experiment $\nu_\mu + \mathrm{Fe} \to \mu^- + X$



$$F_2^{\nu N} = x[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)]$$
$$xF_3^{\nu N} = x[u(x) + d(x) - \overline{u}(x) - \overline{d}(x)]$$

$$\Rightarrow F_2^{\nu N} - xF_3^{\nu N} = 2x[\overline{u} + \overline{d}]$$

✶ **Difference in neutrino structure functions measures anti-quark (sea) parton distribution functions**

Sea dominates so expect xF_3 to go to zero as q(x) = $\overline{q}$(x)

Sea contribution goes to zero

Appendix XIX: Valence Contribution

Not examinable

- Separate parton density functions into sea and valence components

$$u(x) = u_V(x) + u_S(x) = u_V(x) + S(x)$$

$$d(x) = d_V(x) + d_S(x) = d_V(x) + S(x)$$

$$\bar{u}(x) = \bar{u}_S(x) = S(x)$$

$$\bar{d}(x) = \bar{d}_S(x) = S(x)$$

$$\rightarrow F_3^{\nu N} = [u(x) + d(x) - \bar{u}(x) - \bar{d}(x)] = u_V(x) + d_V(x) \rightarrow$$

$$\int_0^1 F_3^{\nu N}(x)\,dx = \int_0^1 (u_V(x) + d_V(x))\,dx = N_u^V + N_d^V$$

- Area under measured function gives a measurement of the total number of valence quarks in a nucleon! Expect

$$\int_0^1 F_3^{\nu N}(x)\,dx = 3$$

"Gross–Llewellyn-Smith sum rule" Experiment: 3.0 ± 0.2.

- Note: $F_2^{\bar{\nu}p} = F_2^{\nu n}$; $F_2^{\bar{\nu}n} = F_2^{\nu p}$; $F_3^{\bar{\nu}p} = F_3^{\nu n}$; $F_3^{\bar{\nu}n} = F_3^{\nu p}$ and anti-neutrino structure functions contain same pdf information.

Not examinable

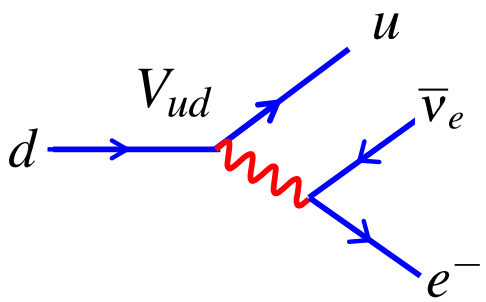
Appendix XX: Determination of the CKM Matrix

Not examinable

- The experimental determination of the CKM matrix elements comes mainly from measurements of leptonic decays (the leptonic part is well understood).
- It is easy to produce/observe meson decays, however theoretical uncertainties associated with the decays of bound states often limits the precision
- Contrast this with the measurements of the PMNS matrix, where there are few theoretical uncertainties and the experimental difficulties in dealing with neutrinos limits the precision.

❶ **$|V_{ud}|$** **from nuclear beta decay**

$$\begin{pmatrix} \times & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$



Super-allowed $0^+ \to 0^+$ beta decays are relatively free from theoretical uncertainties

$$\Gamma \propto |V_{ud}|^2$$

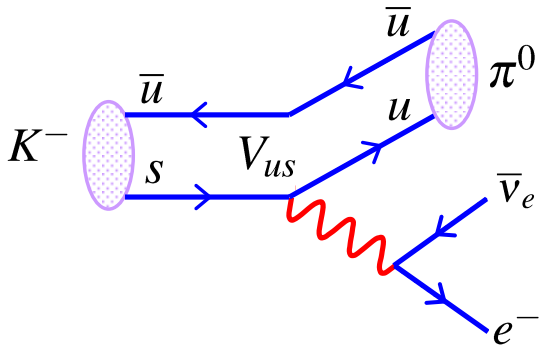
$$|V_{ud}| = 0.97377 \pm 0.00027 \qquad (\approx \cos\theta_c)$$

Not examinable

Appendix XX: Determination of the CKM Matrix (cont.)

Not examinable

❷ **$|V_{us}|$** **from semi-leptonic kaon decays**

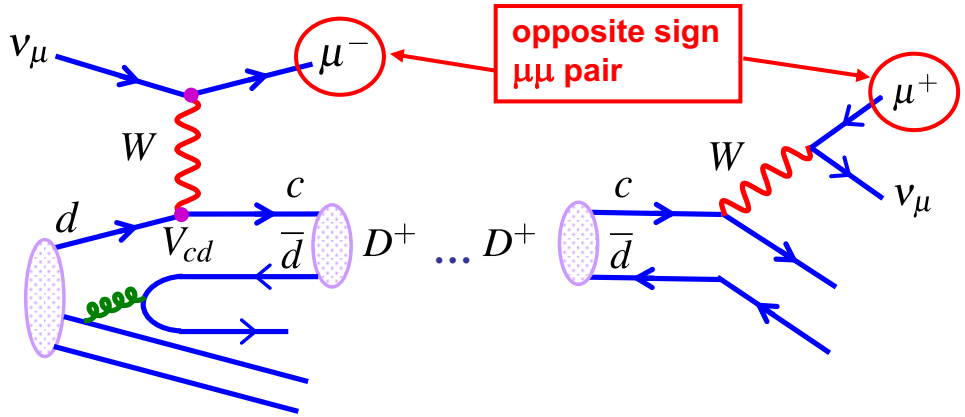


$$\Gamma \propto |V_{us}|^2$$

$$|V_{us}| = 0.2257 \pm 0.0021 \qquad (\approx \sin\theta_c)$$

❸ **$|V_{cd}|$** **from neutrino scattering** $\nu_\mu + N \to \mu^+\mu^- X$

Look for opposite charge di-muon events in ν_μ scattering from production and decay of a $D^+(c\bar{d})$ meson



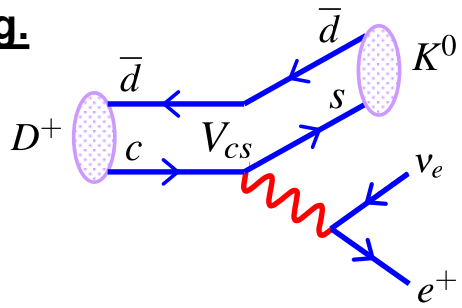
$$\text{Rate} \propto |V_{cd}|^2 \underbrace{\text{Br}(D^+ \to X\mu^+\nu_\mu)}_{\text{Measured in various collider experiments}}$$

$$\Rightarrow \quad |V_{cd}| = 0.230 \pm 0.011$$

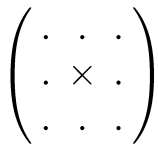
Appendix XX: Determination of the CKM Matrix (cont.)

Not examinable

❹ **$|V_{cs}|$** **from semi-leptonic charmed meson decays**



$$\Gamma \propto |V_{cs}|^2$$

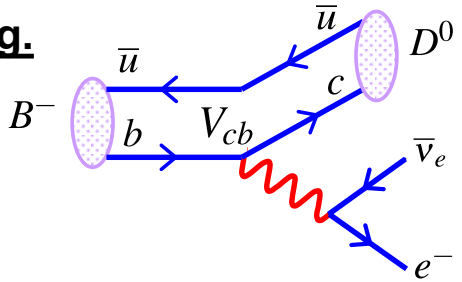


- **Precision limited by theoretical uncertainties**

$$|V_{cs}| = 0.957 \pm 0.017 \pm 0.093$$

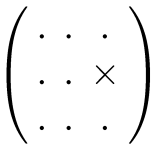
experimental error | theory uncertainty

❺ **$|V_{cb}|$** **from semi-leptonic B hadron decays**

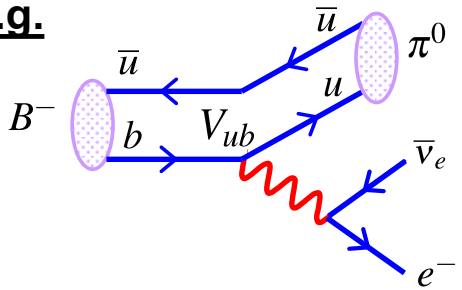


$$\Gamma \propto |V_{cb}|^2$$

$$|V_{cb}| = 0.0416 \pm 0.0006$$

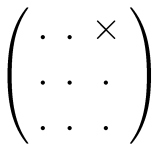


❻ **$|V_{ub}|$** **from semi-leptonic B hadron decays**



$$\Gamma \propto |V_{ub}|^2$$

$$|V_{ub}| = 0.0043 \pm 0.0003$$



Appendix XXI: Particle–AntiParticle Mixing

Not examinable

-The wave-function for a single particle with lifetime $\tau = 1/\Gamma$ evolves with time as:

$$\psi(t) = Ne^{-\Gamma t/2}e^{-iMt}$$

which gives the appropriate exponential decay of

$$\langle \psi(t) \mid \psi(t) \rangle = \langle \psi(0) \mid \psi(0) \rangle e^{-t/\tau}$$

-The wave-function satisfies the time-dependent wave equation:

$$\hat{H}|\psi(t)\rangle = \left(M - \frac{1}{2}i\Gamma\right)|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle$$

-For a bound state such as a K^0 the mass term includes the "mass" from the weak interaction "potential" $\hat{H}_{\text{weak}}$

$$M = m_{K^0} + \left\langle K^0 \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle + \sum_j \frac{\left| \left\langle K^0 \left| \hat{H}_{\text{weak}} \right| j \right\rangle \right|^2}{m_{K^0} - E_j} \leftarrow$$

Sum over intermediate states j

The third term is the 2nd order term in the perturbation expansion corresponding to box diagrams resulting in $K^0 \rightarrow K^0$

Not examinable

Appendix XXI: Particle–AntiParticle Mixing (cont.)

Not examinable

- The total decay rate is the sum over all possible decays $K^0 \to f$

$$\Gamma = 2\pi \sum_f \left| \left\langle f \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle \right|^2 \rho_F \longleftarrow \text{Density of final states}$$

$\star$ Because there are also diagrams which allow $K^0 \leftrightarrow \bar{K}^0$ mixing need to consider the time evolution of a mixed stated

$$\psi(t) = a(t)K^0 + b(t)\bar{K}^0$$

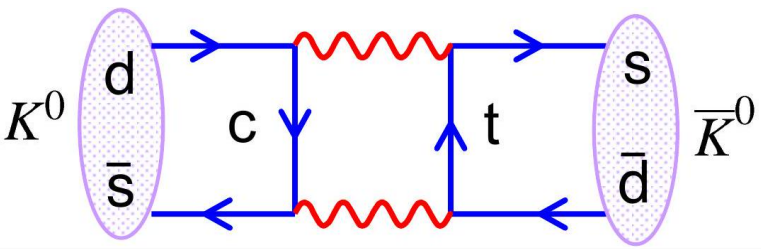
$\star$ The time dependent wave-equation of (A1) becomes

$$\begin{pmatrix} M_{11} - \frac{1}{2}i\Gamma_{11} & M_{12} - \frac{1}{2}i\Gamma_{12} \\ M_{21} - \frac{1}{2}i\Gamma_{21} & M_{22} - \frac{1}{2}i\Gamma_{22} \end{pmatrix} \begin{pmatrix} \left|K^0(t)\right\rangle \\ \left|\bar{K}^0(t)\right\rangle \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} \left|K^0(t)\right\rangle \\ \left|\bar{K}^0(t)\right\rangle \end{pmatrix}$$

the diagonal terms are as before, and the off-diagonal terms are due to mixing.

$$M_{11} = m_{K^0} + \left\langle K^0 \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle + \sum_n \frac{\left| \left\langle K^0 \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle \right|^2}{m_{K^0} - E_n}$$

$$M_{12} = \sum_j \frac{\langle K^0 | \hat{H}_{\text{weak}} | j \rangle^* \langle j | \hat{H}_{\text{weak}} | \overline{K}^0 \rangle}{m_{K^0} - E_j}$$



Appendix XXI: Particle–AntiParticle Mixing (cont.)

Not examinable

-The off-diagonal decay terms include the effects of interference between decays to a common final state

$$\Gamma_{12} = 2\pi \sum_f \left\langle f \left| \hat{H}_{\text{weak}} \right| K^0 \right\rangle^* \left\langle f \left| \hat{H}_{\text{weak}} \right| \bar{K}^0 \right\rangle \rho_F$$

-In terms of the time dependent coefficients for the kaon states, (A3) becomes

$$\left[\mathbf{M} - i\frac{1}{2}\Gamma \right] \begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix}$$

where the Hamiltonian can be written:

$$\mathbf{H} = \mathbf{M} - i\frac{1}{2}\Gamma = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}$$

-Both the mass and decay matrices represent observable quantities and are Hermitian

$$M_{11} = M_{11}^*, \quad M_{22} = M_{22}^*, \quad M_{12} = M_{21}^*$$
$$\Gamma_{11} = \Gamma_{11}^*, \quad \Gamma_{22} = \Gamma_{22}^*, \quad \Gamma_{12} = \Gamma_{21}^*$$

-Furthermore, if CPT is conserved then the masses and decay rates of the $\bar{K}^0$ and K^0 are identical:

$$M_{11} = M_{22} = M; \quad \Gamma_{11} = \Gamma_{22} = \Gamma$$

Not examinable

Appendix XXI: Particle–AntiParticle Mixing (cont.)

Not examinable

-Hence the time evolution of the system can be written:

$$\begin{pmatrix} M - \frac{1}{2}i\Gamma & M_{12} - \frac{1}{2}i\Gamma_{12} \\ M_{12}^* - \frac{1}{2}i\Gamma_{12}^* & M - \frac{1}{2}i\Gamma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix}$$

- To solve the coupled differential equations for $a(t)$ and $b(t)$, first find the eigenstates of the Hamiltonian (the K_L and K_S) and then transform into this basis. The eigenvalue equation is:

$$\begin{pmatrix} M - \frac{1}{2}i\Gamma & M_{12} - \frac{1}{2}i\Gamma_{12} \\ M_{12}^* - \frac{1}{2}i\Gamma_{12}^* & M - \frac{1}{2}i\Gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

-Which has non-trivial solutions for

$$|\mathbf{H} - \lambda I| = 0$$

$$\Rightarrow \left(M - \frac{1}{2}i\Gamma - \lambda\right)^2 - \left(M_{12}^* - \frac{1}{2}i\Gamma_{12}^*\right)\left(M_{12} - \frac{1}{2}i\Gamma_{12}\right) = 0$$

with eigenvalues

$$\lambda = M - \frac{1}{2}i\Gamma \pm \sqrt{\left(M_{12}^* - \frac{1}{2}i\Gamma_{12}^*\right)\left(M_{12} - \frac{1}{2}i\Gamma_{12}\right)}$$

Not examinable

Appendix XXI: Particle–AntiParticle Mixing (cont.)

Not examinable

-The eigenstates can be obtained by substituting back into (A5)

$$\left(M - \tfrac{1}{2}i\Gamma\right)x_1 + \left(M_{12} - \tfrac{1}{2}i\Gamma_{12}\right) = \left(M - \tfrac{1}{2}i\Gamma \pm \sqrt{\left(M_{12}^* - \tfrac{1}{2}i\Gamma_{12}^*\right)\left(M_{12} - \tfrac{1}{2}i\Gamma_{12}\right)}\right)x_1$$

$$\Rightarrow \quad \frac{x_2}{x_1} = \pm\sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}}$$

$\star$ Define

$$\eta = \sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}}$$

- Hence the normalised eigenstates are

$$|K_\pm\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\begin{pmatrix}1\\ \pm\eta\end{pmatrix} = \frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle \pm \eta\left|\bar{K}^0\right\rangle\right)$$

$\star$ Note, in the limit where M_{12}, Γ_{12} are real, the eigenstates correspond to the CP eigenstates K_1 and K_2. Hence we can identify the general eigenstates as as the long and short lived neutral kaons:

Not examinable

$$|K_L\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle + \eta\left|\bar{K}^0\right\rangle\right) \quad |K_S\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle - \eta\left|\bar{K}^0\right\rangle\right)$$

Appendix XXI: Particle–AntiParticle Mixing (cont.)

Not examinable

Substituting these states back into (A2):

$$
\begin{aligned}
|\psi(t)\rangle &= a(t)\left|K^0\right\rangle + b(t)\left|\bar{K}^0\right\rangle \\
&= \sqrt{1+|\eta|^2}\left[\frac{a(t)}{2}\left(K_L+K_S\right)+\frac{b(t)}{2\eta}\left(K_L-K_S\right)\right] \\
&= \sqrt{1+|\eta|^2}\left[\left(\frac{a(t)}{2}+\frac{b(t)}{2\eta}\right)K_L+\left(\frac{a(t)}{2}-\frac{b(t)}{2\eta}\right)K_S\right] \\
&= \frac{\sqrt{1+|\eta|^2}}{2}\left[a_L(t)K_L+a_S(t)K_S\right]
\end{aligned}
$$

with

$$
a_L(t) \equiv a(t)+\frac{b(t)}{\eta} \qquad a_S(t) \equiv a(t)-\frac{b(t)}{\eta}
$$

- Now consider the time evolution of $a_L(t)$

$$
i\frac{\partial a_L}{\partial t} = i\frac{\partial a}{\partial t}+\frac{i}{\eta}\frac{\partial b}{\partial t}
$$

$\star$ Which can be evaluated using (A4) for the time evolution of $a(t)$ and $b(t)$:

Not examinable

Appendix XXI: Particle–AntiParticle Mixing (cont.)

Not examinable

$$
\begin{aligned}
i\frac{\partial a_L}{\partial t} &= \left[\left(M-\frac{1}{2}i\Gamma_{12}\right)a+\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)b\right]+\frac{1}{\eta}\left[\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)a+\left(M-\frac{1}{2}i\Gamma\right)b\right] \\
&= \left(M-\frac{1}{2}i\Gamma\right)\left(a+\frac{b}{\eta}\right)+\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)b+\frac{1}{\eta}\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)a \\
&= \left(M-\frac{1}{2}i\Gamma\right)a_L+\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)b+\left(\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right)a \\
&= \left(M-\frac{1}{2}i\Gamma\right)a_L+\left(\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right)\left(a+\frac{b}{\eta}\right) \\
&= \left(M-\frac{1}{2}i\Gamma\right)a_L+\left(\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right)a_L \\
&= \left(m_L-\frac{1}{2}i\Gamma_L\right)a_L
\end{aligned}
$$

⋆ Hence:

$$
i\frac{\partial a_L}{\partial t} = \left(m_L-\frac{1}{2}i\Gamma_L\right)a_L
$$

with $m_L = M+\mathrm{Re}\left\{\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right\}$

and $\Gamma_L = \Gamma-2\Im\left\{\sqrt{\left(M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*}\right)\left(M_{12}-\frac{1}{2}i\Gamma_{12}\right)}\right\}$

Not examinable

Appendix XXI: Particle–AntiParticle Mixing (cont.)

Not examinable

⋆ Following the same procedure obtain:

$$i\frac{\partial a_S}{\partial t} = \left(m_S - \frac{1}{2}i\Gamma_S\right)a_S$$

with $m_S = M - \Re\left\{\sqrt{\left(M_{12}^* - \frac{1}{2}i\Gamma_{12}^*\right)\left(M_{12} - \frac{1}{2}i\Gamma_{12}\right)}\right\}$

and $\Gamma_S = \Gamma + 2\Im\left\{\sqrt{\left(M_{12}^* - \frac{1}{2}i\Gamma_{12}^*\right)\left(M_{12} - \frac{1}{2}i\Gamma_{12}\right)}\right\}$

⋆ In matrix notation we have

⋆ Solving we obtain

$$\begin{pmatrix} M_L - \frac{1}{2}i\Gamma_L & 0 \\ 0 & M_S - \frac{1}{2}i\Gamma_S \end{pmatrix}\begin{pmatrix} a_L \\ a_S \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} a_L \\ a_S \end{pmatrix}$$

$$a_L(t) \propto e^{-im_L t - \Gamma_L t/2} \quad a_S(t) \propto e^{-im_S t - \Gamma_S t/2}$$

⋆ Hence in terms of the $\mathrm{K_L}$ and $\mathrm{K_S}$ basis the states propagate as independent particles with definite masses and lifetimes (the mass eigenstates). The time evolution of the neutral kaon system can be written

$$|\psi(t)\rangle = A_L e^{-im_L t - \Gamma_L t/2}|K_L\rangle + A_S e^{-im_S t - \Gamma_S t/2}|K_S\rangle$$

where A_L and A_S are constants

Not examinable

Appendix XXII: CP Violation : $\pi\pi$ decays

Not examinable

⋆ Consider the development of the $K^0 - \bar{K}^0$ system now including CP violation

⋆ Repeat previous derivation using

$$|K_S\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_1\rangle + \varepsilon\,|K_2\rangle\right] \quad |K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left[|K_2\rangle + \varepsilon\,|K_1\rangle\right]$$

-Writing the CP eigenstates in terms of $K^0, \bar{K}^0$

$$|K_L\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)\,|K_0\rangle + (1-\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

$$|K_S\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)\,|K_0\rangle - (1-\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

- Inverting these expressions obtain

$$\left|K^0\right\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1+\varepsilon}\left(|K_L\rangle + |K_S\rangle\right) \quad \left|\bar{K}^0\right\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1-\varepsilon}\left(|K_L\rangle - |K_S\rangle\right)|$$

-Hence a state that was produced as a K^0 evolves with time as:

$$|\psi(t)\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1+\varepsilon}\left(\theta_L(t)\,|K_L\rangle + \theta_S(t)\,|K_S\rangle\right)$$

where as before $\theta_S(t) = e^{-\left(im_S + \frac{\Gamma_S}{2}\right)t}$ and $\theta_L(t) = e^{-\left(im_L + \frac{\Gamma_L}{2}\right)t}$

Not examinable

Appendix XXII: CP Violation : $\pi\pi$ decays (cont.)

Not examinable

-If we are considering the decay rate to $\pi\pi$ need to express the wave-function in terms of the CP eigenstates (remember we are neglecting CP violation in the decay)

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\frac{1}{1+\varepsilon}\left[\left(|K_2\rangle + \varepsilon\,|K_1\rangle\right)\theta_L(t) + \left(|K_1\rangle + \varepsilon\,|K_2\rangle\right)\theta_S(t)\right]$$
$$= \frac{1}{\sqrt{2}}\frac{1}{1+\varepsilon}\left[\left(\theta_S + \varepsilon\theta_L\right)|K_1\rangle + \left(\theta_L + \varepsilon\theta_S\right)|K_2\rangle\right]$$

CP Eigenstates

-Two pion decays occur with $\mathrm{CP} = +1$ and therefore arise from decay of the $\mathrm{CP} = +1$ kaon eigenstate, i.e. K_1

$$\Gamma\left(K^0_{t=0} \to \pi\pi\right) \propto |\langle K_1 \mid \psi(t)\rangle|^2 = \frac{1}{2}\left|\frac{1}{1+\varepsilon}\right|^2 |\theta_S + \varepsilon\theta_L|^2$$

- Since $|\varepsilon| \ll 1$

$$\left|\frac{1}{1+\varepsilon}\right|^2 = \frac{1}{(1+\varepsilon^*)(1+\varepsilon)} \approx \frac{1}{1+2\Re\{\varepsilon\}} \approx 1 - 2\Re\{\varepsilon\}$$

- Now evaluate the $|\theta_S + \varepsilon\theta_L|^2$ term again using

$$|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\Re\left(z_1 z_2^*\right)$$

Not examinable

Appendix XXII: CP Violation : $\pi\pi$ decays (cont.)

Not examinable

$$|\theta_S + \varepsilon\theta_L|^2 = \left| e^{-im_S t - \frac{\Gamma_S}{2}t} + \varepsilon e^{-im_L t - \frac{\Gamma_L}{2}t} \right|^2$$
$$= e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2\,\mathrm{Re}\left\{ e^{-im_S t - \frac{\Gamma_S}{2}t} \cdot \varepsilon^* e^{+im_L t - \frac{\Gamma_L}{2}t} \right\}$$

-Writing $\varepsilon = |\varepsilon| e^{i\phi}$

$$|\theta_S + \varepsilon\theta_L|^2 = e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2}\,\mathrm{Re}\left\{ e^{i(m_L - m_S)t - \phi} \right\}$$
$$= e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2} \cos(\Delta m.t - \phi)$$

-Putting this together we obtain:

$$\Gamma\left(K^0_{t=0} \to \pi\pi\right) = \frac{1}{2}(1 - 2\,\mathrm{Re}\{\varepsilon\}) N_{\pi\pi} \left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2} \cos(\Delta m.t - \phi) \right]$$

Short lifetime

component

$\mathrm{K_S} \to \pi\pi$

CP violating long lifetime component KL¦p
Interference term
-In exactly the same manner obtain for a beam which was produced as $\bar{K}^0$

$$\Gamma\left(\bar{K}^0_{t=0} \to \pi\pi\right) = \frac{1}{2}(1 + 2\,\mathrm{Re}\{\varepsilon\}) N_{\pi\pi} \left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} - 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2} \cos(\Delta m.t - \phi) \right]$$

Interference term changes sign

Not examinable

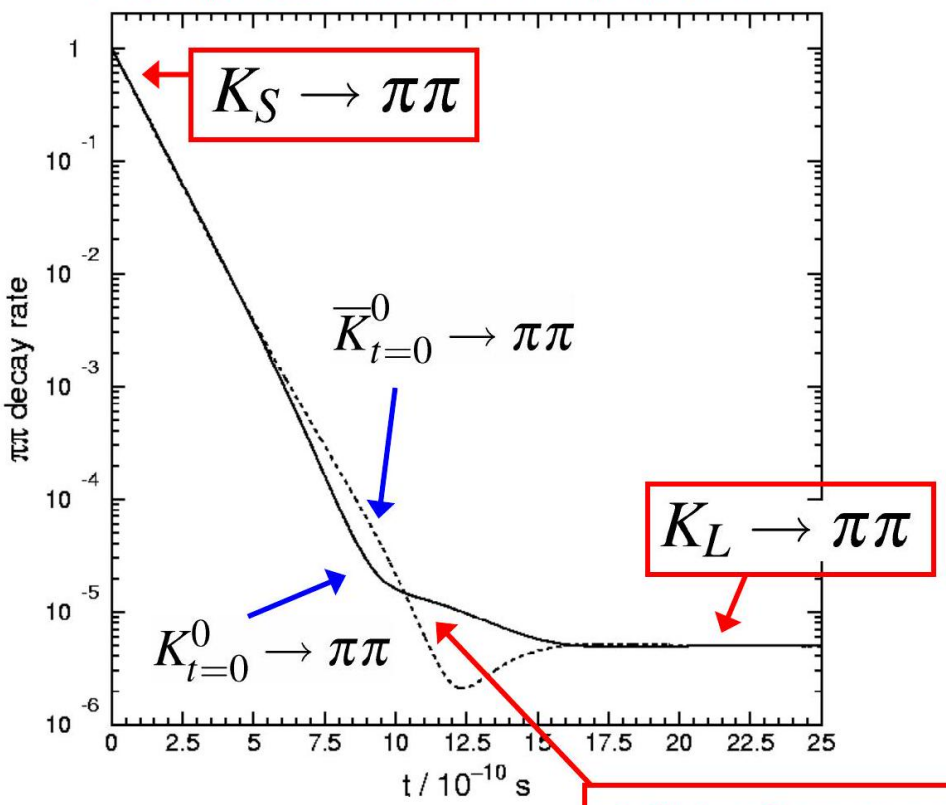
Appendix XXII: CP Violation : $\pi\pi$ decays (cont.)

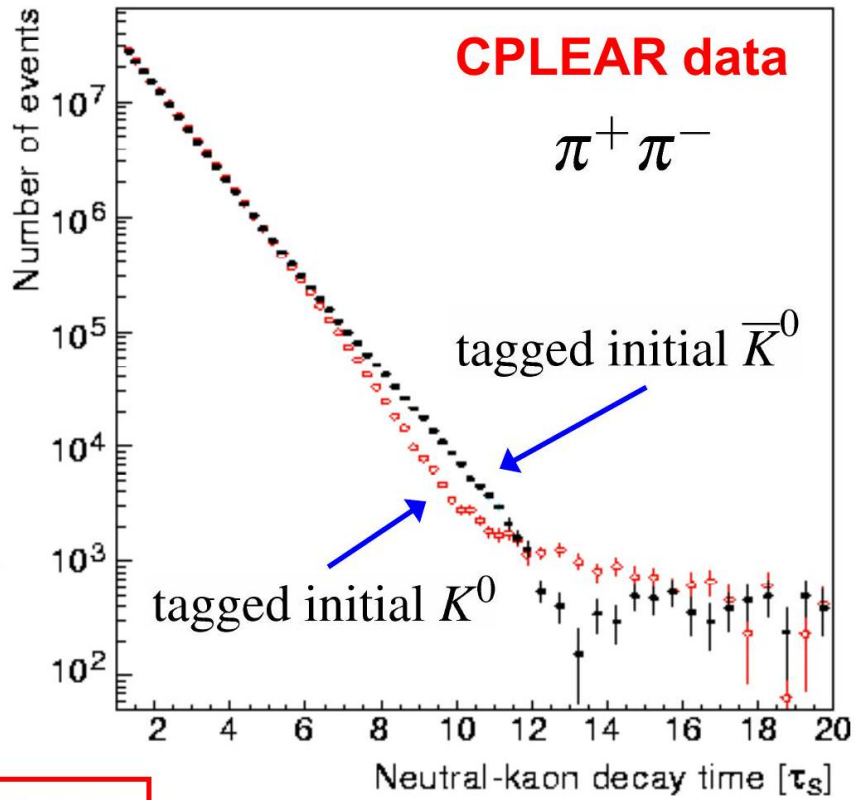
Not examinable

$$\Gamma\left(K^0_{t=0} \to \pi\pi\right) \to \frac{1}{2}(1 - 2\,\mathrm{Re}\{\varepsilon\})N_{\pi\pi} \cdot |\varepsilon|^2 e^{-\Gamma_L t}$$

i.e. CP violating $K_L \to \pi\pi$ decays

$\star$ Since CPLEAR can identify whether a K^0 or $\bar{K}^0$ was produced, able to measure $\Gamma\left(K^0_{t=0} \to \pi\pi\right)$ and $\Gamma\left(\bar{K}^0_{t=0} \to \pi\pi\right)$





Appendix XXII: CP Violation : $\pi\pi$ decays (cont.)

Not examinable

$\star$ The CPLEAR data shown previously can be used to measure $\varepsilon = |\varepsilon|e^{i\phi}$ -Define the asymmetry: $A_{+-} = \frac{\Gamma\left(\bar{K}^0_{t=0}\to\pi\pi\right)-\Gamma\left(K^0_{t=0}\to\pi\pi\right)}{\Gamma\left(\bar{K}^0_{t=0}\to\pi\pi\right)+\Gamma\left(K^0_{t=0}\to\pi\pi\right)}$

-Using expressions on page 606

$$A_{+-} = \frac{4\Re\{\varepsilon\}\left[e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}\right]-4|\varepsilon|e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t-\phi)}{2\left[e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}\right]-\underbrace{8\Re\{\varepsilon\}|\varepsilon|e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t-\phi)}_{\propto |\varepsilon|\Re\{\varepsilon\} \text{ i.e. two small quantities can safely be neglected}}}$$

$$A_{+-} \approx \frac{2\Re\{\varepsilon\}\left[e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}\right]-2|\varepsilon|e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t-\phi)}{e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}}$$

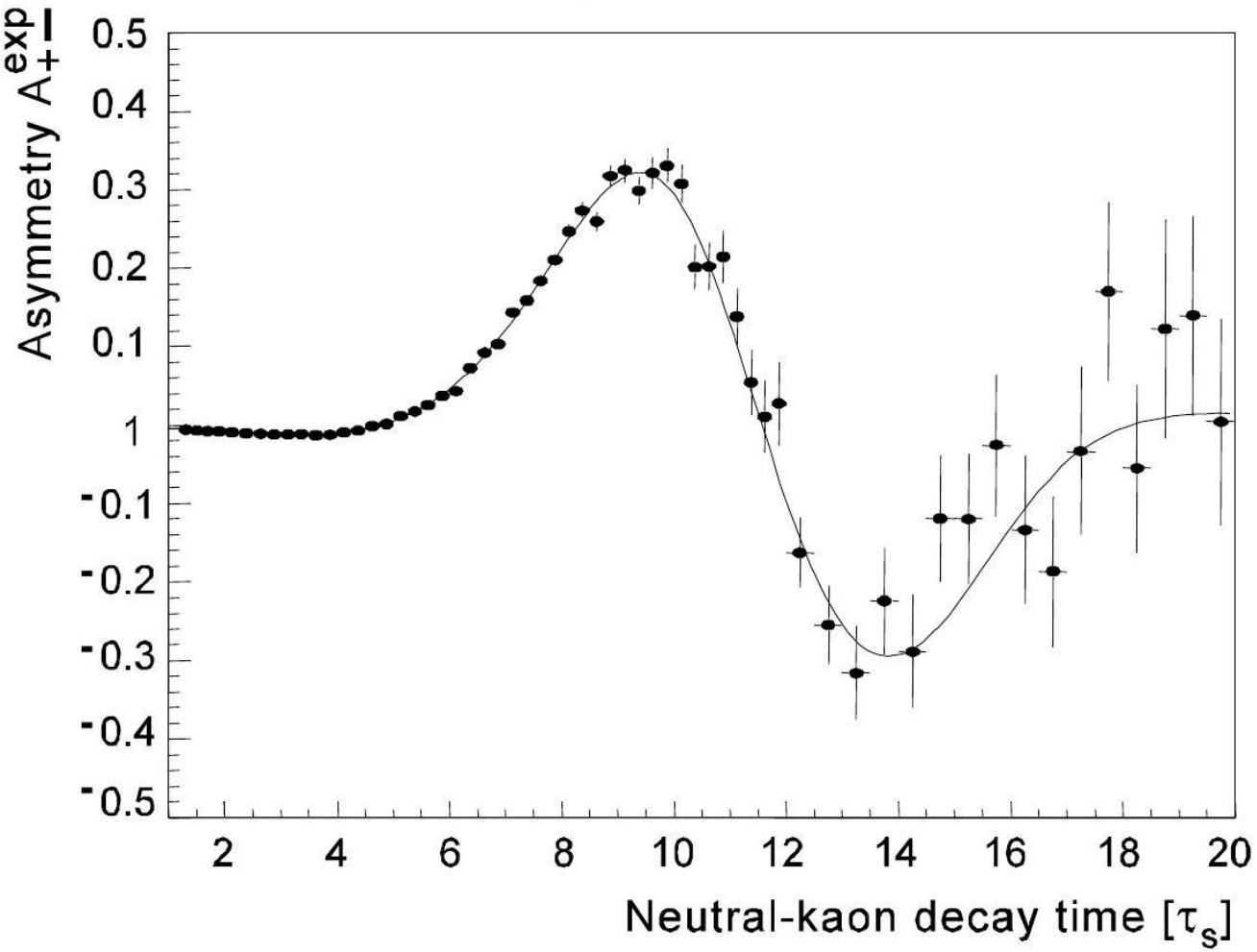
$$= 2\Re\{\varepsilon\} - \frac{2|\varepsilon|e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t-\phi)}{e^{-\Gamma_S t}+|\varepsilon|^2 e^{-\Gamma_L t}}$$

$$= 2\Re\{\varepsilon\} - \frac{2|\varepsilon|e^{(\Gamma_S-\Gamma_L)t/2}\cos(\Delta m.t-\phi)}{1+|\varepsilon|^2 e^{(\Gamma_S-\Gamma_L)t}}$$

Not examinable

Appendix XXII: CP Violation : $\pi\pi$ decays (cont.)

Not examinable



Best fit to the data:

$$|\varepsilon| = (2.264 \pm 0.035) \times 10^{-3}$$

$$\phi = (43.19 \pm 0.73)^{\circ}$$

Not examinable

Appendix XXIII: CP Violation via Mixing

Not examinable

- A full description of the SM origin of CP violation in the kaon system is beyond the level of this course, nevertheless, the relation to the box diagrams is illustrated below

$\star$ The K-long and K-short wave-functions depend on η

$$|K_L\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle + \eta\left|\bar{K}^0\right\rangle\right) \quad |K_S\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\left(\left|K^0\right\rangle - \eta\left|\bar{K}^0\right\rangle\right)$$

$$\text{with} \qquad \eta = \sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}}$$

$\star$ If $M_{12}^* = M_{12}$; $\Gamma_{12}^* = \Gamma_{12}$ then the K-long and K-short correspond to the CP eigenstates K_1 and K_2

-CP violation is therefore associated with imaginary off-diagonal mass and decay elements for the neutral kaon system

-Experimentally, CP violation is small and $\eta \approx 1$

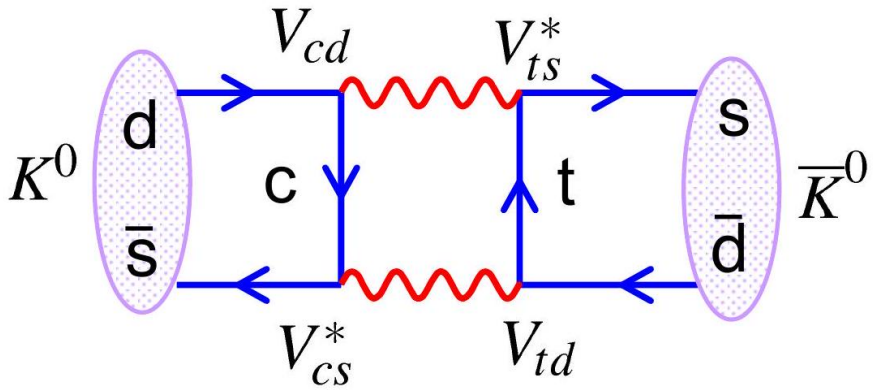
-Define: $\varepsilon = \frac{1-\eta}{1+\eta} \quad \Rightarrow \quad \eta = \frac{1-\varepsilon}{1+\varepsilon}$

Not examinable

Appendix XXIII: CP Violation via Mixing (cont.)

Not examinable

- Consider the mixing term M_{12} which arises from the sum over all possible intermediate states in the mixing box diagrams
e.g.



- In the Standard Model, CP violation is associated

with the imaginary components of the CKM matrix, and it can be shown that mixing leads to *CP* violation with

$$|\varepsilon| \propto \Im\{M_{12}\}$$

-The differences in masses of the mass eigenstates can be shown to be:

$$\Delta m_K = m_{K_L} - m_{K_S} \approx \sum_{q,q'} \frac{G_F^2}{3\pi^2} f_K^2 m_K \left| V_{qd} V_{qs}^* V_{q'd} V_{q's}^* \right| m_q m_{q'}$$

Not examinable

where q and q' are the quarks in the loops and f_K is a constant

Appendix XXIII: CP Violation via Mixing (cont.)

Not examinable

- In terms of the small parameter ε

$$|K_L\rangle = \frac{1}{2\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)\left|K^0\right\rangle + (1-\varepsilon)\left|\bar{K}^0\right\rangle\right]$$
$$|K_S\rangle = \frac{1}{2\sqrt{1+|\varepsilon|^2}}\left[(1-\varepsilon)\left|K^0\right\rangle + (1+\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

- If epsilon is non-zero we have CP violation in the neutral kaon system

$$\text{Writing} \quad \eta = \sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}} = \sqrt{\frac{z^*}{z}} \quad \text{and} \quad z = ae^{i\phi}$$
$$\text{gives} \quad \eta = e^{-i\phi}$$

- From which we can find an expression for ε

$$\varepsilon \cdot \varepsilon^* = \frac{1-e^{-i\phi}}{1+e^{-i\phi}} \cdot \frac{1-e^{+i\phi}}{1+e^{i\phi}} = \frac{2-\cos\phi}{2+\cos\phi} = \tan^2\frac{\phi}{2}$$
$$|\varepsilon| = \left|\tan\frac{\phi}{2}\right|$$

Experimentally we know ε is small, hence ϕ is small

Not examinable

$$|\varepsilon| \approx \frac{1}{2}\phi = \frac{1}{2}\arg z \approx \frac{1}{2}\frac{\Im\left\{M_{12} - \frac{1}{2}i\Gamma_{12}\right\}}{\left|M_{12} - \frac{1}{2}i\Gamma_{12}\right|}$$

Appendix XXIV: Time Reversal Violation

Not examinable

-Previously in equations (153) and (154) we obtained expressions for strangeness oscillations in the absence of CP violation, e.g.:

$$\Gamma\left(K^0_{t=0} \to K^0\right) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

-This analysis can be extended to include the effects of CP violation to give the following rates (see Question 24):

$$\Gamma\left(K^0_{t=0} \to K^0\right) \propto \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma\left(\bar{K}^0_{t=0} \to \bar{K}^0\right) \propto \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma\left(\bar{K}^0_{t=0} \to K^0\right) \propto \frac{1}{4}(1+4\,\mathrm{Re}\{\varepsilon\})\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma\left(K^0_{t=0} \to \bar{K}^0\right) \propto \frac{1}{4}(1-4\,\mathrm{Re}\{\varepsilon\})\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$\star$ Including the effects of CP violation find that

$$\Gamma\left(\bar{K}^0_{t=0} \to K^0\right) \neq \Gamma\left(K^0_{t=0} \to \bar{K}^0\right) \qquad \text{Violation of time reversal symmetry !}$$

Not examinable

- No surprise, as CPT is conserved, CP violation implies T violation

Appendix XXV: Non-relativistic Breit-Wigner

Not examinable

For energies close to the peak of the resonance, can write $\sqrt{s} = m_Z + \Delta$

$$s = m_Z^2 + 2m_Z\Delta + \Delta^2 \approx m_Z^2 + 2m_Z\Delta \quad \text{for } \Delta \ll m_Z$$

and with this approximation

$$\left(s - m_Z^2\right)^2 + m_Z^2\Gamma_Z^2 \approx (2m_Z\Delta)^2 + m_Z^2\Gamma_Z^2 = 4m_Z^2\left(\Delta + \frac{1}{4}\Gamma_Z^2\right) = 4m_Z^2\left[\left(\sqrt{s} - m_Z\right)^2 + \frac{1}{4}\Gamma_Z^2\right]$$

so that the relativistic Breith-Wigner formula of (177) can be approximated $\sigma\left(e^+e^- \to Z \to f\bar{f}\right) \approx \frac{3\pi}{m_Z^4}\frac{s}{\left(\sqrt{s}-m_Z\right)^2+\frac{1}{4}\Gamma_Z^2}\Gamma_e\Gamma_f$ which can be written:

$$\boxed{\sigma(E) = \frac{g\lambda_e^2}{4\pi}\frac{\Gamma_i\Gamma_f}{\left(E - E_0\right)^2 + \frac{1}{4}\Gamma^2}}.$$

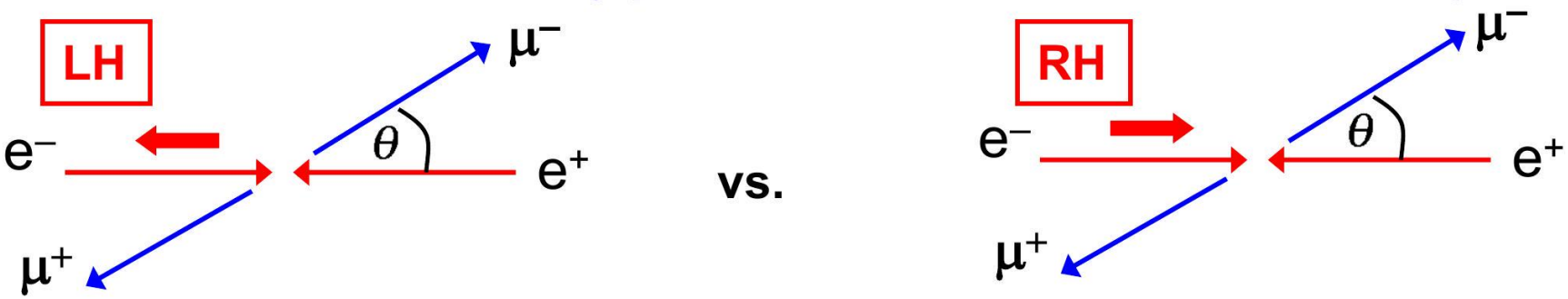
Γ_i and Γ_f are the partial decay widths of the initial and final state particles.
E and E_0 are the centre-of-mass energy and the energy of the resonance.
$g = \frac{(2J_Z+1)}{(2S_e+1)(2S_e+1)}$ is the spin counting factor $g = \frac{3}{2\times 2}$.
$\lambda_e = \frac{2\pi}{E}$ is the Compton wavelength (natural units) in the C.o.M of either initial particle.
The boxed equation is the non-relativistic form of the Breit-Wigner distribution first encountered in the Part II Particle and Nuclear Physics course (e.g. page 36 in Handout 2, "Kinematics, Decays and Recations", of the Part II course given in 2023).

Not examinable

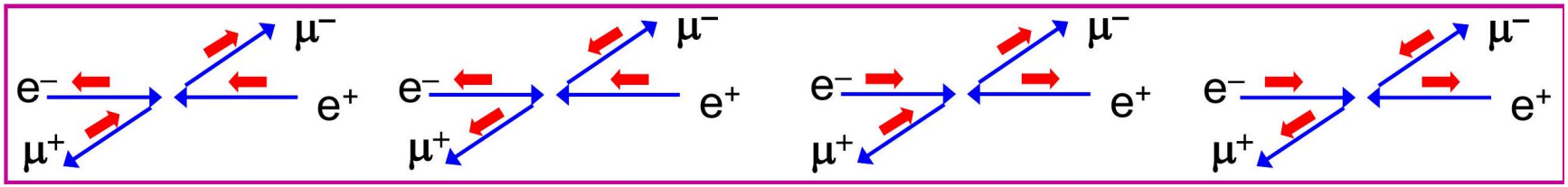
Appendix XXVI: Left-Right Asymmetry, A_{LR}

Not examinable

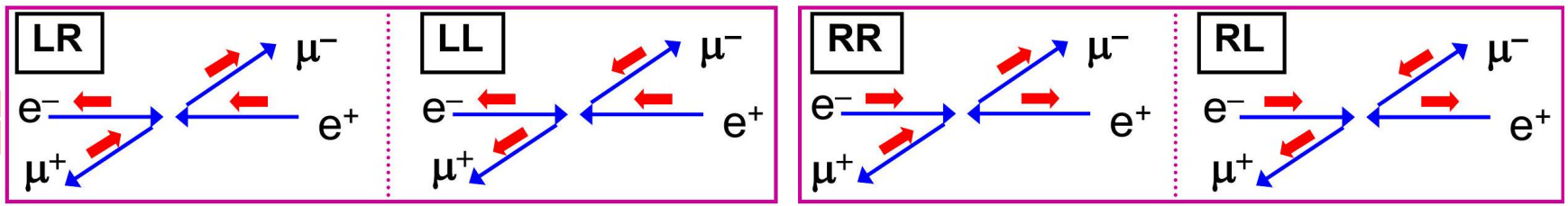
- At an e^+e^- linear collider it is possible to produce polarized electron beams. E.g. Stanford Linear Collider (SLC; California), 1989-2000.
- At such a collider one could measure cross section for any process for LH and RH electrons separately



- At LEP one usually measured the total cross section: a sum of 4 helicity combinations:



- In contrast, at the SLC, tuning the polarization of the electron beams made it possble to measure cross sections separately for LH / RH electrons



Appendix XXVI: Left-Right Asymmetry, A_{LR} (cont.)

Not examinable

- Define cross section asymmetry:

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}$$

where $\sigma_L = \sigma_{LL} + \sigma_{LR}$ and where $\sigma_R = \sigma_{RL} + \sigma_{RR}$ using the notation of page 432.

- Integrating the expressions for the differential cross sections on the same page gives:

$$\sigma_{LL} \propto (c_L^e)^2 (c_L^\mu)^2 \quad \sigma_{LR} \propto (c_L^e)^2 (c_R^\mu)^2 \quad \sigma_{RL} \propto (c_R^e)^2 (c_L^\mu)^2 \quad \sigma_{RR} \propto (c_R^e)^2 (c_R^\mu)^2$$

and so

$$\sigma_L \propto (c_L^e)^2 \left[(c_L^\mu)^2 + (c_R^\mu)^2 \right] \quad \sigma_R \propto (c_R^e)^2 \left[(c_L^\mu)^2 + (c_R^\mu)^2 \right]$$

and

$$A_{LR} = \frac{(c_L^e)^2 - (c_R^e)^2}{(c_L^e)^2 + (c_R^e)^2} = A_e$$

- Hence the Left-Right asymmetry for any cross section depends only on the couplings of the initial state electrons.
- Compare this to the Forward Backward asymmetry (see page 445) which depends on the couplings of the initial state electrons and the final state particles (muons, *etc*).

Not examinable